Numerical simulation of nonlinear fractional integrodifferential equations of Volterra type via power series expansion

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Abstract. In this article, an effective recent analytical treatment is presented to solve a certain class of nonlinear fractional integrodifferential equations of Volterra type based on the residual error functions. The solution methodology of the fractional power series (FPS) approach is to replace the n-term truncated solution by generalized fractional power series to minimize the residual error function through the derivation of those functions under the Caputo concept. Anyhow, the approximate solution is obtained

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directly in a rapidly convergent fractional power series without needed to linearization, perturbation, or discretization. Numerical examples are performed to show the validity and reliability of the FPS method. Numerical analysis of the results indicates that the RPS approach is simple, efficient and systematic tool in solving fractional nonlinear issues arising in applied mathematics, physics and engineering.

Keywords: fractional derivative, residual power series method, nonlinear fractional models, integro-differential equations.

1. Introduction

The fractional differentiation and integration theory is indeed a generalization of ordinary calculus theory that deals with differentiation and integration to an arbitrary order, which is utilized to describe various real-world phenomena arising in natural sciences, applied mathematics, and engineering fields with great applications for these tools, for instance, fractional fluid-dynamic traffic, economics, solid mechanics, viscoelasticity, the nonlinear oscillation of earthquakes, control theory [1-8]. The major cause behind this is that- modeling of a specific phenomenon doesn't depend only at the time instant but also the historical state, so the fractional differential and integral operators superb tool to describe the hereditary and memory properties for different engineering and physical phenomena. However, several mathematical forms of above-mentioned issues contain nonlinear fractional integro-differential equations (FIDEs) [9-11]. Since most fractional differential and integro-differential equations cannot be solved analytically, thus it is necessary to find an accurate numerical and analytical methods to deal with the complexity of fractional operators involving such equations. Anyhow, in recent times, many experts have devoted their interest in finding solutions of the fractional integro-differential equations and other nonlinear differential equations utilizing different analytic-numeric methods [12-17]. The Adomian decomposition method, variational iteration method, homotopy perturbation method, Taylor expansion method, multistep approach, and reproducing kernel method are powerful and reliable numerical tools for handling many real-world problems [18-23].

The basic goal of the present work is to introduce a recent analytic-numeric method based on the use of residual power series technique for obtaining the approximate solution for a class of nonlinear fractional Volterra integro-differential equations in the form

$$\mathcal{D}^{\beta}_{a^+}\varphi(t) + \int_0^t h(t,s)(\varphi(s))^r ds = f(t), \quad 0 < \beta \leq 1, r \geq 2, \tag{1}$$

with the initial condition

$$\varphi(0) = \varphi_0, \tag{2}$$

where $\mathcal{D}^{\beta}_{a^+}$ denotes the Caputo fractional derivative, $f(t)$ and $h(t,s)$ are smooth functions. Here, $\varphi(t)$ is unknown analytic function to be determined.

The fractional power series (FPS) method is a recent analytic-numeric treatment method based on power series expansion, which is easy and applicable to find the series solutions for several types of the non-linear differential equation and integro-differential equations without being linearized, discretized, or exposed to perturbation [24-27]. The RPS method has been successfully applied to solve linear and non-linear ordinary, partial and fuzzy differential equations for more details, see [28-30].

The rest of the current paper is as follow: In next section, we introduce some essential preliminaries related to fractional calculus and fractional power series representations. In Section 3, we illustrate the solution methodology by using the RPS technique. In Section 4, illustrative problems are provided to demonstrate the simplicity, accuracy, and performance of the present method. Finally, we give a concluding remark in the final section.

2. Preliminaries

In this section, we recall some definitions and basic results concerning fractional calculus and fractional power series representations.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order β, over the interval $[a,b]$ for a function $\varphi \in L_1[a,b]$ is defined by*

$$\mathcal{J}_{a^+}^{\beta}\varphi(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\beta}}, & 0<\tau<t, \beta>0 \\ \varphi(t), & \beta = 0 \end{cases}$$

For $\beta_1, \beta_2 \geq 0$, and $q \geq -1$, the operator $\mathcal{J}_{a^+}^{\beta}$ has the following basic properties:

1) $\mathcal{J}_{a^+}^{\beta}(t-a)^q = \frac{\Gamma(q+1)}{\Gamma(q+1+\beta)}(t-a)^{q-\beta}$.

2) $\mathcal{J}_{a^+}^{\beta_1}\mathcal{J}_{a^+}^{\beta_2}\varphi(t) = \mathcal{J}_{a^+}^{\beta_1+\beta_2}\varphi(t)$.

3) $\mathcal{J}_{a^+}^{\beta_1}\mathcal{J}_{a^+}^{\beta_2}\varphi(t) = \mathcal{J}_{a^+}^{\beta_2}\mathcal{J}_{a^+}^{\beta_1}\varphi(t)$.

Definition 2.2. *For $\beta > 0$, $a, t, \beta \in \mathbb{R}$ The following fractional differential operator of order β*

$$\mathcal{D}_{a^+}^{\beta}\varphi(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t \frac{\varphi^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}}d\tau,$$

$n-1<\beta<n$ for $n \in \mathbb{N}$, is called the Caputo fractional derivative of order β. In case $\beta = n$, then $\mathcal{D}_{a^+}^{\beta}\varphi(t) = \frac{d^n}{dt^n}\varphi(t)$.

The following are some interesting properties of the operator $\mathcal{D}_{a^+}^{\beta}\varphi(t)$:

1) For any constant $c \in \mathbb{R}$, then $\mathcal{D}_{a^+}^{\beta}c = 0$.

2) $\mathcal{J}_{a^+}^{\beta}\mathcal{D}_{a^+}^{\beta}\varphi(t) = \varphi(t)$.

3) $\mathcal{J}_{a^+}^{\beta}\mathcal{D}_{a^+}^{\beta}\varphi(t) = \varphi(t) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(a^+)}{k!}(t-a)^k$.

Definition 2.3. *The representation of the fractional power series (FPS) about $t = t_0$ is given by $\sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta} = c_0 + c_1(t-t_0)^{\beta} + c_1(t-t_0)^{2\beta} + ...$, where $0 \leq n-1 < \beta \leq n$ and $t \geq t_0$ and c_m's are the coefficients of the series.*

Remark 2.1. Let R be the radius of convergence for the FPS $\sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta}$, then the following are only the possibilities for the FPS

1) For all $t = t_0$, the series $\sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta}$ converges with R.

2) The series converges for all $t \geqslant t_0$ whenever the radius of convergence is equal to ∞.

3) The series converges for $t \in [t_0, t_0 + \mathrm{R})$, for some positive R and diverges for $t > t_0 + \mathrm{R}$.

Theorem 2.1. *Assuming that the FPS expansion $\sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta}$ $0 \leq n-1 < \alpha \leq n$, has radius of convergence $\mathrm{R} > 0$. If function $\varphi(t)$ is defined by $f(t) = \sum_{m=0}^{\infty} c_m(t-t_0)^{m\beta}$, then for $t_0 < t < t_0 + \mathrm{R}$, we have the following:*

1) $\mathcal{D}_{t_0}^{\beta} f(t) = \sum_{m=0}^{\infty} c_m \frac{\Gamma(m\beta+1)}{\Gamma((m-1)\beta+1)}(t-t_0)^{(m-1)\beta}$;

2) $\mathcal{J}_{t_0}^{\beta} f(t) = \sum_{m=0}^{\infty} c_m \frac{\Gamma(m\beta+1)}{\Gamma((m+1)\beta+1)}(t-t_0)^{(m+1)\beta}$

Proof. For the first part, by using the definition of the Caputo fractional derivative and certain properties of the operator $\mathcal{D}_{t_0}^{\beta}$, we conclude that

$$\begin{aligned}
\mathcal{D}_{t_0}^{\beta}\varphi(t) &= \frac{1}{\Gamma(n-\beta)} \int_{t_0}^{t} \frac{\varphi^{(n)}(\varepsilon)}{(t-\varepsilon)^{\beta-n+1}} d\varepsilon \\
&= \frac{1}{\Gamma(n-\beta)} \int_{t_0}^{t} (t-\varepsilon)^{n-\beta-1} \left(\frac{d^n}{dt^n} \sum_{m=0}^{\infty} c_m(\varepsilon-t_0)^{m\beta} \right) d\varepsilon \\
&= \sum_{m=1}^{\infty} c_m \frac{1}{\Gamma(n-\beta)} \int_{t_0}^{t} (t-\varepsilon)^{n-\beta-1} \left(\frac{d^n}{dt^n} (\varepsilon-t_0)^{m\beta} \right) d\varepsilon \\
&= \sum_{m=0}^{\infty} c_m \frac{\Gamma(m\beta+1)}{\Gamma((m-1)\beta+1)} (t-t_0)^{(m-1)\beta}.
\end{aligned} \tag{3}$$

For the second part, apply the Riemann-Liouville fractional integral operator $\mathcal{J}_{t_0}^{\beta}$ and by using the property $\mathcal{J}_{a^+}^{\beta}(t-a)^q = \frac{\Gamma(q+1)}{\Gamma(q+1+\beta)}(t-a)^{q-\beta}$, we conclude that

$$\mathcal{J}_{t_0}^{\beta}\varphi(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} \frac{\varphi(\varepsilon)}{(t-\varepsilon)^{1-\beta}} d\varepsilon = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-\varepsilon)^{\beta-1} \left(\sum_{m=0}^{\infty} c_m(\varepsilon-t_0)^{m\beta} \right) d\varepsilon$$

$$(4)\qquad = \sum_{m=0}^{\infty} c_m \left(\frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-\varepsilon)^{\beta-1} (\varepsilon - t_0)^{m\beta} \right) d\varepsilon = \sum_{m=1}^{\infty} c_m \mathcal{J}_{t_0}^{\beta} (t-t_0)^{m\beta}$$

$$= \mathcal{J}_{t_0}^{\beta} f(t) = \sum_{m=0}^{\infty} c_m \frac{\Gamma(m\beta+1)}{\Gamma((m+1)\beta+1)} (t-t_0)^{(m+1)\beta}.$$

□

Theorem 2.2. *Assuming that* $\mathcal{D}_{t_0}^{m\beta} = \mathcal{D}_{t_0}^{\beta}.\mathcal{D}_{t_0}^{\beta}...\mathcal{D}_{t_0}^{\beta}$ *(m-times)and* $\varphi(t)$ *has the following representation of the FPS at* $t = t_0$

$$\varphi(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\beta}$$

where $0 \le n-1 < \beta \le n, t_0 < t < t_0 + \mathrm{R}, \varphi(t) \in \mathcal{C}[t_0, t_0+\mathrm{R})$ *and* $\mathcal{D}_{t_0}^{m\beta}\varphi(t) \in \mathcal{C}[t_0, t_0+\mathrm{R})$, *for* $m = 0, 1, 2, ...$ *then the coefficients* c_m *is given as follows* $c_m = \frac{\mathcal{D}_{t_0}^{m\beta}\varphi(t_0)}{\Gamma(m\beta+1)}$.

3. Construction solution by RPS algorithm

The purpose of this section is to construct FPS solution for non-linear fractional Volterra integro-differential equations (1) and (2) by substitute its FPS expansion among its truncated residual function.

The RPS algorithm proposed the solution of Eqs. (1) and (2) about $a = 0$ has the following FPS expansion:

$$(5)\qquad \varphi(t) = \sum_{m=0}^{\infty} \frac{c_m}{\Gamma(m\beta+1)} t^{m\beta}.$$

For obtaining the approximate values of (5), consider the following k-th FPS approximate solution

$$(6)\qquad \varphi_k(t) = \sum_{m=0}^{k} \frac{c_m}{\Gamma(m\beta+1)} t^{m\beta}.$$

Clearly, if $m = 0$, $\varphi(0) = \varphi_0$. So, the expansion (6) can be written as

$$(7)\qquad \varphi_k(t) = \varphi_0 + \sum_{m=1}^{k} \frac{c_m}{\Gamma(m\beta+1)} t^{m\beta}.$$

Define the so-called the residual function for equations (1) and (2) as follows:

$$(8)\qquad Res(t) = \mathcal{D}_{0+}^{\beta}\varphi(t) + \int_0^t h(t,s)(\varphi(s))^r ds - f(t),$$

and the following k-th residual function

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\varphi_k(t) + \int_0^t h(t,s)(\varphi_k(s))^r ds - f(t). \tag{9}$$

As in [31-36], some useful properties of residual function

1) $\lim_{k\to\infty} Res_k(t) = Res(t) = 0$, for each $t \in (0,1)$.

2) $\mathcal{D}^{m\beta}_{0^+} Res(0) = \mathcal{D}^{m\beta}_{0^+} Res_k(0)$, for each $m = 0, 1, 2, ...k$.

For obtaining the coefficients $c_m, m = 0, 1, 2, , k$, solve the solution of the following relation:

$$\mathcal{D}^{(k-1)\beta}_{0^+} Res_k(0) = 0, \quad k = 1, 2, 3, ... \tag{10}$$

Lemma 3.1 Assuming that $\varphi(t) \in \mathcal{C}[t_0, t_0 + \mathrm{R}), \mathrm{R} > 0, \mathcal{D}^{j\beta}_{t^0}\varphi(t) \in \mathcal{C}[t_0, t_0 + \mathrm{R}), 0 < \beta \le 1$. Then for any $j \in \mathbb{N}$, we have

$$\left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{(j+1)\beta}_{t_0}\mathcal{D}^{(j+1)\beta}_{t_0}\right)\varphi(t) = \frac{\mathcal{D}^{j\beta}_{t_0}\varphi(t)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}.$$

Proof. From the properties of the fractional integral operator, it follows that

$$\begin{aligned} &\left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{(j+1)\beta}_{t_0}\mathcal{D}^{(j+1)\beta}_{t_0}\right)\varphi(t)\\ &= \left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{\beta}_{t_0}\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\mathcal{D}^{\beta}_{t_0}\right)\varphi(t)\\ &= \left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{\beta}_{t_0}(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0})\mathcal{D}^{\beta}_{t_0}\right)\varphi(t)\\ &= \mathcal{J}^{j\beta}_{t_0}\left[\left(\mathcal{D}^{j\beta}_{t_0}\varphi(t)\right)\right] - \left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\left(\mathcal{D}^{\beta}_{t_0}\right)\varphi(t), \end{aligned} \tag{11}$$

Hence, for $\left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\left(\mathcal{D}^{\beta}_{t_0}\right)\varphi(t)$, one can obtain

$$\begin{aligned} &\left(\mathcal{J}^{j\beta}_{t_0}\mathcal{D}^{j\beta}_{t_0}\right)\varphi(t) - \left(\mathcal{J}^{(j+1)\beta}_{t_0}\mathcal{D}^{(j+1)\beta}_{t_0}\right)\varphi(t)\\ &= \mathcal{J}^{j\beta}_{t_0}\left[\left(\mathcal{D}^{j\beta}_{t_0}\varphi(t)\right) - \left(\mathcal{D}^{j\beta}_{t_0}\varphi(t)\right) + \mathcal{D}^{j\beta}_{t_0}\varphi(t)\right]\\ &= \mathcal{J}^{j\beta}_{t_0}\left[\mathcal{D}^{j\beta}_{t_0}\varphi(t)\right] = \frac{\mathcal{D}^{j\beta}_{t_0}\varphi(t_0)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta} \end{aligned} \tag{12}$$

with $c = \mathcal{D}^{j\beta}_{t_0}\varphi(t_0)$ □

Theorem 3.1. *If $\varphi(t)$ has the FPS of (8) with $\mathrm{R} > 0$, such that $\varphi(t) \in \mathcal{C}[t_0, t_0 + \mathrm{R} > 0), \mathcal{D}^{j\beta}_{t_0}\varphi(t) \in \mathcal{C}(t_0, t_0 + \mathrm{R})$ for $j = 0, 1, 2, ..., N+1$. Then,*

$$\varphi(t) = \varphi_N(t) + R_N(\zeta), \tag{13}$$

where $\varphi_N(t) = \sum_{j=0}^{N} \frac{\mathcal{D}^{j\beta}_{t_0}\varphi(t_0)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}$ and $R_N(\zeta) = \sum_{j=0}^{N} \frac{\mathcal{D}^{(N+1)\beta}_{t_0}\varphi(t_0)}{\Gamma((N+1)\beta+1)}(t-t_0)^{j\beta}$, for some $\zeta \in (t_0, t)$

Proof. From the properties of fractional operators, we have

$$\varphi(t)=\sum_{j=0}^{N}\left[\left(\mathcal{J}_{t_0}^{j\beta}\mathcal{D}_{t_0}^{j\beta}\right)\varphi(t)-\left(\mathcal{J}_{t_0}^{(j+1)\beta}\mathcal{D}_{t_0}^{(j+1)\beta}\right)\varphi(t)\right].$$

By using Lemma 3.1, it follows

$$\varphi(t)-\left(\mathcal{J}_{t_0}^{(N+1)\beta}\mathcal{D}_{t_0}^{(N+1)\beta}\right)\varphi(t)=\sum_{j=0}^{N}\frac{\mathcal{D}_{t_0}^{j\beta}\varphi(t_0)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}. \tag{14}$$

So, $\varphi(t)=\sum_{j=0}^{N}\frac{\mathcal{D}_{t_0}^{j\beta}\varphi(t_0)}{\Gamma(j\beta+1)}(t-t_0)^{j\beta}+\left(\mathcal{J}_{t_0}^{(N+1)\beta}\mathcal{D}_{t_0}^{(N+1)\beta}\right)\varphi(t)$. But

$$\begin{aligned}\left(\mathcal{J}_{t_0}^{(N+1)\beta}\mathcal{D}_{t_0}^{(N+1)\beta}\right)\varphi(t)&=\mathcal{J}_{t_0}^{(N+1)\beta}\left(\mathcal{D}_{t_0}^{(N+1)\beta}\right)\varphi(t)\\&=\frac{1}{\Gamma((N+1)\beta)}\int_0^t\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(\tau)(t-\tau)^{(N+1)\beta-1}d\tau\\&=\frac{\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(t_0)}{\Gamma((N+1)\beta)}\int_0^t(t-\tau)^{(N+1)\beta-1}d\tau,\end{aligned} \tag{15}$$

by the MVT of integrals

$$\frac{\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(t_0)}{\Gamma((N+1)\beta)}\frac{(t-t_0)_{t_0}^{(N+1)\beta}}{((N+1)\beta)}=\frac{\mathcal{D}_{t_0}^{(N+1)\beta}\varphi(t_0)}{\Gamma((N+1)\beta)+1}(t-t_0)^{(N+1)\beta}. \tag{16}$$

□

Remark 3.1. The representation of $\varphi_N(t)$ in (13) gives an approximation of $\varphi(t)$, and $R_N(\zeta)$ is remainder term. Furthermore, if $\mid \mathcal{D}_{t_0}^{(N+1)\beta}\varphi(t_0)\mid < M$ on$[t_0,t_0+\mathrm{R})$, then the upper bound of the error can be computed by

$$|R_N(\zeta)|=\sup_{t\in[t_0,t_0+\mathrm{R}]}\frac{M(t-t_0)^{(N+1)\beta}}{\Gamma(N+1)\beta+1}.$$

Remark 3.2. To solve the fractional IVP in (1) and (2) by the FPS method, let

$$\varphi_N(t)=\sum_{n=0}^{N}c_n\frac{t^{n\beta}}{\Gamma(n\beta+1)}, \tag{17}$$

with radius of convergence $\mathrm{R}_0>0$. If $\varphi(t)\in\mathcal{C}[0,\mathrm{R}_0), \mathcal{D}_{t_0}^{j\beta}\varphi(t)\in\mathcal{C}(0,\mathrm{R}_0)$ then $\varphi(t)=\varphi_N(t)+R_N(\zeta)$.

Algorithm 3.1. To find the coefficients $c_m, m=1,2,3,k$, in (7), do the following steps:

Step 1: Substitute the expansion (6) function $\varphi_k(t)$ into the k-th residual residual function (7) such that

$$Res_k(t) = \mathcal{D}^{\beta}_{a^+}\left(\varphi_0 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right) + \int_0^t h(t,s)\left(\varphi_0 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right)^r ds - f(t). \tag{18}$$

Step 2: Find the relation of fractional formula $\mathcal{D}^{(k-1)\beta}_{t_0}$ of $Res_k(t)$ at $t = t_0$.
Step 3: Do the following:

For $k = 1$, obtain $Res_1(t)\Big|_{t=0} = 0$.

For $k = 2$, obtain $\mathcal{D}^{\beta}_{0^+} Res_2(t)\Big|_{t=0} = 0$.

$\vdots$

For $k = m$, obtain $\mathcal{D}^{m\beta}_{0^+} Res_m(t)\Big|_{t=0} = 0$.

Step 4: Solve the obtained system $\mathcal{D}^{(k-1)\beta}_{0^+} Res_k(0)$, $k = 1, 2, 3, ...$
Step 5: Substitute the values of c_m back into Eq. (4) and then STOP.

4. Numerical examples

In this section, we demonstrate the efficiency, accuracy of the RPS approach by applying to two nonlinear fractional VIDEs. All numerical calculations are performed using Mathematica 10.

Example 4.1. Consider the following nonlinear fractional VIDE

$$\mathcal{D}^{\beta}_{0^+}\varphi(t) = e^t + \frac{t}{3}(1 - e^{3t}) + \int_0^t e^{t-s}(\varphi(s))^3 ds, \quad 0 < \beta \le 1, \tag{19}$$

with the initial condition

$$\varphi(0) = 1. \tag{20}$$

Here, the exact solution at $\beta = 1$ is given by $\varphi(t) = e^t$.

Using the FPS algorithm, The k-th residual function $Res_k(t)$ is given by

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\varphi(t) - \int_0^t e^{t-s}(\varphi_k(s))^3 ds - \left(e^t + \frac{t}{3}(1 - e^{3t})\right), \tag{21}$$

where $\varphi_k(t)$ has the form

$$\varphi_k(t) = 1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)},$$

Consequently,

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right)$$

$$- \int_0^t e^{t-s}\left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right)^3 ds - \left(e^t + \frac{t}{3}(1-e^{3t})\right), \quad (22)$$

The absolute errors are listed in Table 1. The results obtained by the RPS method show that the exact solutions are in good agreement with approximate solutions at $\beta = 1, n = 6$ and step size 0.2. While Table 2 show approximate solutions at different values of β such that $\beta \in 1, 0.9, 0.8, 0.7$ with step size 0.16. From the table, one can be found that the RPS method provides us with an accurate approximate solution, which is in good agreement with the exact solutions for all values of t in $[0, 1]$.

Table 1. The numerical results of Absolute error for Example 4.1 at $\beta = 1$.

t	Exact solution	Numerical solution	Absolute error
0.2	1.221402758160169	1.2214027555555556	2.60461×10^{-9}
0.4	1.491824697641270	1.4918243555555555	3.42085×10^{-7}
0.6	1.822118800390509	1.8221128000000000	6.00039×10^{-6}
0.8	2.225540928492468	2.2254947555555558	4.61729×10^{-5}

Table 2. Numerical results for Example 4.1 for different values of β.

t	6^{th} FPS solution			
	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$
0.16	1.1735108704	1.2236588706	1.2896293585	1.3781327965
0.32	1.3771276933	1.4620068483	1.5701199217	1.7112052001
0.48	1.6160731635	1.7354009578	1.8854925107	2.0791905574
0.64	1.8964714019	2.0527277406	2.2480109650	2.4982975959
0.80	2.2254947555	2.4227191207	2.6681456160	2.9806767189
0.96	2.6115273760	2.8549680620	3.1566853797	3.5379276118

Example 4.2. Consider the following nonlinear fractional VIDE

$$\mathcal{D}^{\beta}_{0^+}\varphi(t) = \int_0^t cost - s(\varphi(s))^2 ds - \frac{2}{3}sin(t)(2 + cos(t)), \quad 0 < \beta \leq 1, \quad (23)$$

with the initial condition

$$\varphi(0) = 1. \quad (24)$$

Here, the exact solution is $\varphi(t) = cos(t)$ for $\beta = 1$.

Using the FPS algorithm, The k-th residual function $Res_k(t)$ is given by

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\varphi_k(t) - \int_0^t cos(t-s)(\varphi_k(s))^2 ds + \frac{2}{3} sin(t)(2 + cos(t)), \tag{25}$$

where $\varphi_k(t)$ has the form

$$\varphi_k(t) = 1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)},$$

Consequently,

$$Res_k(t) = \mathcal{D}^{\beta}_{0^+}\left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right) - \int_0^t \cos(t-s)\left(1 + \sum_{m=1}^{k} c_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}\right)^2 ds + \frac{2}{3}\sin(t)(2+\cos(t)),$$

The absolute errors are given in Table 3. The results obtained by the FPS method show that the exact solutions are in good agreement with approximate solutions at $\beta = 1, n = 8$ and step size 0.25. While Table 4 show approximate solutions at different values of β such that $\beta = 1, 0.95, 0.85$, and $\beta = 0.75$ with step size 0.2. In Figure 1, the behavior of the 8th FPS-approximation is plotted for different values of β in $[0, 1]$, where $\beta = 1, 0.95, 0.85$, and $\beta = 0.75$. From these results, it can be observed that the behavior of the approximate solutions for different values of β is in good agreement with each other that depends on the fractional order β.

Table 3. The numerical results of Absolute error for Example 4.2 at $\beta = 1$.

t	Exact solution	Numerical solution	Absolute error
0.25	0.9689124217106447	0.9689124217109074	2.62679×10^{-13}
0.50	0.8775825618903728	0.8775825621589781	2.68605×10^{-10}
0.75	0.7316888688738209	0.7316888843263899	1.54526×10^{-8}
1.0	0.5403023058681397	0.5403025793650793	2.73497×10^{-7}

Table 4. Numerical results for Example 4.1 for different values of β.

t	6^{th} FPS solution			
	$\beta = 1$	$\beta = 0.95$	$\beta = 0.85$	$\beta = 0.75$
0.2	0.98006657784	0.97441161411	0.95844541415	0.93403621799
0.4	0.921060994032	0.90575062873	0.86795945799	0.82005645947
0.6	0.82533561657	0.80058551593	0.74520770139	0.68453141148
0.8	0.69670673879	0.66520694561	0.60100540330	0.54042995454

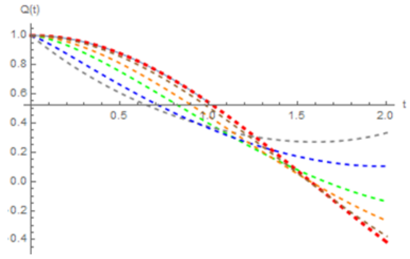

Figure 1: FPS-solution plots of Example 4.2 for different values of β.

5. Concluding remarks

The present paper aims to solve a class of nonlinear fractional Volterra integro-differential equations of order β, $0 < \beta \leq 1$, based on the use of RPS algorithm. The solution methodology depends on the constructing of the residual function and applying the generalized Taylor formula under the Caputo fractional derivative. The proposed algorithm provides the solutions in the form of rapidly convergent series with no need linearization, limitation on the problems nature, sort of classification or perturbation. Graphical and numerical results are performed by Mathematica 10. The results demonstrate the accuracy, efficiency and the capability of the present method. Therefore, the RPS algorithm is reliable, effective, simple, straightforward tool for handling a wide range of nonlinear fractional integro-differential equations.

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Accepted: 22.05.2019