New classes of uniformly convex functions of fractional power on Banach space

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Abstract. The aim of this paper is to define new certain subclasses of analytic functions of fractional parameters in the well-known unit disk $\mathbb{U}$. Then introduce and study a new integral operator type fractional in the sense of Noor integral on Banach space. In addition, some of its applications are discussed by utilizing a Owa-Hadamard product.
Keywords: analytic functions, Owa-Hadamard product, uniformly convex functions, Banach space.

1. Introduction

In general, one of the most significant problems facing many analytical applications of geometric functions is how to introduce and study operators type fractional of analytic and univalent functions on complex Banach spaces for example (see [1, 2, 3, 4, 5, 6]). Specifically, the theory of analytic functions includes the following format:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \tag{1.1}$$

which are analytic and univalent in $\mathcal{S} \in \mathbb{U} := \{z : |z| < 1\}$, normalized by $f'(0) = 1$ and $f(0) = 0$. For instance the functions of class $\mathcal{S}$ are convex if

$$f(z) = \frac{z}{(1-z)} = z + z^2 + z^3 + \dots,$$

and are starlike if

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots.$$

The aim of this paper is to define a new analytic family type of fractional power $\mathcal{A}_\upsilon$ by

$$\mathbb{F}(z) = z + \sum_{m=2}^{\infty} a_m z^{\upsilon m}, \qquad (|z| < 1), \tag{1.2}$$

where $\upsilon := \frac{m+k-1}{k}$, $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Specifically, when $m = k = 1$ in (1.2) we get the *Koebe* function in $\mathbb{U}$. Then, introduce subclasses of analytic and univalent functions defined by applying fractional integral operators involving the well-known integral operator such as the Noor integral operator. For two functions $\mathbb{F}$ given by (1.2) and $\mathbb{G}(z) = z + \sum_{m=2}^{\infty} h_m z^{\upsilon m}$, the product functions $\mathbb{F} * \mathbb{G}$ is known as the convolution (or Hadamard product) and defined by [7]

$$(\mathbb{F} * \mathbb{G})(z) = z + \sum_{m=2}^{\infty} a_m h_m z^{\upsilon m} \tag{1.3}$$

and

$$(\mathbb{F} * \mathbb{G})'(z) = \mathbb{F} * \mathbb{G}'(z), \quad (|z| < 1).$$

By utilising (1.3) we introduce a new integral operator type of fractional power denote by $\mathcal{I}_{\rho,\upsilon} : \mathcal{A}_\upsilon \to \mathcal{A}_\upsilon$. In this effort, let defined the analytical fractional function $\mathbb{F}_\rho$ by

$$\mathbb{F}_\rho = \frac{z^\upsilon}{(1 - z^\upsilon)^{\rho+1}}, \quad (z \in \mathbb{U}, \upsilon \geq 1, \rho \geq 1)$$

such that

$$\mathbb{F}_\rho(z) * \mathbb{F}_\rho^{-1}(z) = \frac{z^\upsilon}{1 - z^\upsilon}. \tag{1.4}$$

Consequently, we receive the integral operator $\mathcal{I}_{\rho,\upsilon}$ defined by

$$\begin{aligned} \mathcal{I}_{\rho,\upsilon}\mathbb{F}(z) &= \left(\frac{z^\upsilon}{(1 - z^\upsilon)^{\rho+1}} \right)^{-1} * F(z) \\ &= z + \sum_{m=2}^{\infty} \frac{(m-1)!}{(\rho+1)_{m-1}} a_m z^{\upsilon m}. \end{aligned} \tag{1.5}$$

For $\rho = 1$ and $\upsilon \geq 1$, then the integral operator $\mathcal{I}_{1,\upsilon}$ is closed to the Noor integral (see [8]) of the m-th order of function $\mathbb{F} \in \mathcal{A}_\upsilon$. Corresponding to (1.5), we have the following conclusion:

$$z(\mathcal{I}_{\rho,\upsilon} F(z))' = z + \sum_{m=2}^{\infty} \frac{\Gamma(m+1)\Gamma(\rho+1)}{\Gamma(m+\rho)} \upsilon a_m z^{\upsilon m}. \tag{1.6}$$

In the following section, we study some properties of the integral operator $\mathcal{I}_{\rho,\upsilon}$ given by (1.5) in the class of uniformly convex functions type of fractional power on Banach spaces.

2. Class of uniformly convex functions

Let $\mathbb{X}$ be a Banach space and $\mathbb{X}^\dagger$ its dual. For any $A \in \mathbb{X}^\dagger$, we interest the set $\mathcal{W}(A) := \{w \in \mathbb{X} : A(w) \neq 0\}$ and let the set $\gamma(A) := \{w \in \mathbb{X} : \mathbb{X} \setminus \mathcal{W}(A)\}$.

If $A \neq 0$ then $\mathcal{W}(A)$ is dense in $\mathbb{X}$ and $\mathcal{W}(A) \cap \hat{\mathcal{B}}$ is dense in $\hat{\mathcal{B}}$, where $\hat{\mathcal{B}} := \{w \in \mathbb{X} : ||w|| = 1\}$. Let define $\mathcal{B}$ be a complex Banach space and $\mathcal{H}(\mathcal{B}, \mathbb{C})$ be a family of all functions $f : \mathcal{B} \to \mathbb{C}$, such that $f(w)|_{w=0} = 0$, this means that these functions are holomorphic in $\mathcal{B}$ and have the Fréchet derivative $f'(w)$ for all points $w \in \mathcal{B}$.

Recall that : Let Υ and Ξ be two Banach spaces, such that $\Omega \subset \Upsilon$ an open subset in V. A function $\phi : \Omega \to \Xi$ is known as Fréchet differentiable at $y \in \Omega$ if there exists a bounded linear operator $\Lambda : \Upsilon \to \Xi$ such that [9]

$$\lim_{h \to 0} \left[\frac{\|\phi(y+h) - \phi(y) - \Lambda h\|_{\Xi}}{\|h\|_{\Upsilon}} \right] = 0.$$

If $f \in \mathcal{H}(\mathcal{B}, \mathbb{C})$, then

$$f(w) = \sum_{m=1}^{\infty} \mathcal{P}_m(w). \tag{2.1}$$

Remark 1. We note that, the series $\mathcal{P}_m : \mathbb{X} \to \mathbb{C}$ are

1- Uniformly convergent on some neighborhood V of the origin.

2- Continuous and homogeneous polynomials of degree m.

In unit disk $\mathbb{U}$, let denote the family CV of functions by

$$\mathbb{F}(z) = z + \sum_{m=2}^{\infty} a_m z^{\upsilon m}, \tag{2.2}$$

are convex in $\mathbb{U}$. In geometrically sense Goodman [10] considered the class $\mathbb{U}CV \subset CV$ of uniformly convex functions in $\mathbb{U}$ and stated that, if f normalized and every (positive oriented) circular arc γ with center ζ in $\mathbb{U}$ such that the image arc $f(\gamma)$ is a convex arc, then $f \in CV$. Moreover, proved the function f given by (1.1) analytic in $\mathbb{U}CV$ if and only if satisfied

$$R\{(z-\zeta)\frac{f''(z)}{f'(z)} + 1\} \geq 0, \quad (z, \zeta) \in \mathbb{U} \times \mathbb{U}. \tag{2.3}$$

Lemma 1 ([10]). *If $f \in \mathbb{U}CV$, then*

$$|a_n| \leq \frac{1}{n}, \quad n \geq 2.$$

Rønning [11] declared that, the function f analytic in $\mathbb{U}CV$ if and only if

$$R\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} \geq \left| \frac{zf''(z)}{f'(z)} \right|, \quad |z| < 1. \tag{2.4}$$

Now, let $A \in \mathbb{X}^{\dagger}$, $A \neq 0$. For any $f \in \mathcal{H}(\mathcal{B}, \mathbb{C})$ of the form

$$\mathbb{F}(w) = A(w) + \sum_{m=2}^{\infty} \mathcal{P}_m(w), \quad w \in \mathcal{B} \tag{2.5}$$

and for any $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ we put

$$\mathbb{F}_a(z) = \frac{\mathbb{F}(za)}{A(a)}, \quad \upsilon \geq 1, z \in \mathbb{U}. \tag{2.6}$$

It is clear that, for all $|z| < 1$

$$\mathbb{F}_a(z) = z + \sum_{m=2}^{\infty} \frac{P_m(a)}{A(a)} z^{\upsilon m} \tag{2.7}$$

and

$$\mathbb{F}_a^{(m)}(z) = \frac{F_a^{(m)}(za)(a, \cdots, a)}{A(a)}, \quad m \in \mathbb{N}. \tag{2.8}$$

Let $\mathbb{U}CV_{\mathcal{A}_\upsilon}$ denote the family of all functions $\mathbb{F} \in \mathcal{H}(\mathcal{B}, \mathbb{C})$ of the form (2.5) such that, for any $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ the function $\mathbb{F}_a$ belongs to the class $\mathbb{U}CV$. From the following results, we investigate some properties of the functions $\mathbb{F}$ in the class $\mathbb{U}CV$.

Theorem 1 (Bounded coefficient)**.** *If the function $\mathbb{F}$ is belong in $\mathbb{U}CV_{\mathcal{A}_\upsilon}$ and $a \in \hat{\mathcal{B}}$. Then*

$$|\mathcal{P}_m(a)| \leq \frac{1}{m} |A(a)|, \quad m \geq 2.$$

Proof. Assume that, the function $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_\upsilon}$, if $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$, then $\mathbb{F}_a \in \mathbb{U}CV$. In another side, if $a \in \gamma(A) \cap \hat{\mathcal{B}}$, clearly that $a = \lim_{n\to\infty} a_n$, where $a_n \in \mathcal{W}(A), n \in \mathbb{N}$. There exists $r_n \in \mathbb{R}^+$ such that $\frac{a_n}{r_n} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}, n \in \mathbb{N}$, it is clear that $(r_n, n > 0)$ is bounded for the origin is an interior point of $\mathcal{B}$. For $\frac{a_n}{r_n} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$, $n \in \mathbb{N}$, we obtain

$$|\mathcal{P}_m(\frac{a_n}{r_n})| \leq \frac{1}{m} |A(\frac{a_n}{r_n})|, \quad m \geq 2$$

consequence

$$|\mathcal{P}_m(a_n)| \leq \frac{r_n^{m-1}}{m} |A(a_n)|, \quad m \geq 2$$

by letting $n \to \infty$, we get $\mathcal{P}_m(a) = 0$. □

Corollary 1. All the functions $\mathbb{F}$ in $\mathbb{U}CV_{\mathcal{A}_\upsilon}$ are vanish on $\gamma(A) \cap \mathcal{B}$.

Corollary 2. If $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_\upsilon}$, then

$$||\mathcal{P}_m|| \leq \frac{1}{m} ||A||, \quad m \geq 2.$$

Theorem 2 (Sufficient condition). *If $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_v}$ and $\mathbb{F}'(w) \neq 0$, for all $w \in \mathcal{B}$, then*

$$R\left\{1+\frac{\mathbb{F}''(w)(w,w)}{\mathbb{F}'(w)(w)}\right\} \geq \left|\frac{\mathbb{F}''(w)(w,w)}{\mathbb{F}'(w)(w)}\right|, \quad w \in \mathcal{W}(A) \cap \mathcal{B}. \tag{2.9}$$

Proof. Let $w \in \mathcal{W}(A) \cap \mathcal{B}$, $w \neq 0$, then $a = \frac{w}{||w||} \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ and thus the $\mathbb{F}_a \in \mathbb{U}CV$. By using (2.4), we get

$$R\left\{1+\frac{z\mathbb{F}''_a(z)}{\mathbb{F}'_a(z)}\right\} \geq \left|\frac{z\mathbb{F}''_a(z)}{\mathbb{F}'_a(z)}\right|, \quad z \in \mathbb{U}.$$

By recall the equality

$$\frac{z\mathbb{F}''_a(z)}{\mathbb{F}'_a(z)} = \frac{\mathbb{F}''(za)(za,za)}{\mathbb{F}'(za)(za)}$$

then, we have

$$\left|\frac{z\mathbb{F}''_a(z)}{\mathbb{F}'_a(z)}\right| = \left|\frac{z\mathbb{F}''(za)(za,za)}{\mathbb{F}'(za)(za)}\right| \leq \left|1+\frac{\mathbb{F}''(za)(za,za)}{\mathbb{F}'(za)(za)}\right|,$$

by putting $za = ||w||$, we obtain (2.9). □

Corollary 3. For $\mathbb{F} \in \mathcal{H}(\mathcal{B},\mathcal{C})$, $\mathbb{F}'(w)|_{w=0} = A$ and $\mathbb{F}'(w) \neq 0$, for all $w \in \mathcal{B}$. If

$$R\left\{1+\frac{z\mathbb{F}''(w)(w,w)}{\mathbb{F}'(w)(w)}\right\} \geq \left|\frac{z\mathbb{F}''(w)(w,w)}{\mathbb{F}'(w)(w)}\right|, \quad w \in \mathcal{W}(A) \cap \mathcal{B} \tag{2.10}$$

then $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_v}$

Proof. Let $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$. Then $\mathbb{F}'_a(z) = \mathbb{F}'(za)(a) \neq 0$, $|z| < 1$ and

$$\frac{z\mathbb{F}''_a(z)}{\mathbb{F}'_a(z)} = \frac{\mathbb{F}''(za)(za,za)}{\mathbb{F}'(za)}, \quad |z| < 1.$$

□

From (2.10), we get $\mathbb{F}_a \in \mathbb{U}CV$, for all $a \in \mathcal{W}(A) \cap \hat{\mathcal{B}}$ hence $\mathbb{F} \in \mathbb{U}CV_{\mathcal{A}_v}$.

3. Owa-Hadamard product

In this section, we set up some certain results which dealing with the Owa-Hadamard product functions $\mathbb{F}(w)$ of form (2.5). First, let define

$$\mathbb{F}_j(w) = A(w) + \sum_{m=2}^{\infty} \mathcal{P}_{m,j}(w) z^{\upsilon m}, \quad j = \{1,2,\cdots l\}, \tag{3.1}$$

and

$$\mathbb{F}_{a_j}(z) = z + \sum_{m=2}^{\infty} \frac{\mathcal{P}_m(a_j)}{A(a_j)} z^{\upsilon m}, \quad j = \{1, 2, \cdots l\}, z \in \mathbb{U}. \tag{3.2}$$

Let define the Owa-Hadamard product of two functions $\mathbb{F}(w)$ and $\mathbb{G}(w)$ in class $\mathbb{U}CV_{\mathcal{A}_\upsilon}$ by

$$(\mathbb{F} * \mathbb{G})(w) := A(w) + \sum_{m=2}^{\infty} \mathcal{P}(w)\Phi(w) z^{\upsilon m},$$

where $G(w) := A(w) + \sum_{m=2}^{\infty} \Phi(w) z^{\upsilon m} \quad w \in \mathcal{B}$.

Theorem 3. Let $\mathbb{F}_j$ given by (3.1) be in the class $\in \mathbb{U}CV_{\mathcal{A}_\upsilon}$ for every $j = 1, 2, \cdots l$; and let the function $\mathbb{G}_i$ defined by

$$\mathbb{G}_i(w) = A(w) + \sum_{m=2}^{\infty} \Phi_{m,i}(w) z^{\upsilon m}, i = 1, 2, \cdots, s$$

then the Owa-Hadamard product of more two functions $\mathbb{F}_1 * \mathbb{F}_2 * \cdots * \mathbb{F}_l * \mathbb{G}_1 * \mathbb{G}_2, \cdots * \mathbb{G}_s(z)$ belongs to the class $\mathbb{U}CV^{l+s}_{\mathcal{A}_\upsilon}$.

Proof. Let

$$H(w) = A(w) + \sum_{m=2}^{\infty} \left\{ \prod_{j=1}^{l} \mathcal{P}_{m,j}(w) \prod_{i=1}^{s} \Phi_{m,i}(w) \right\} z^{\upsilon m}.$$

We aim to show that

$$\sum_{m=2}^{\infty} m^{l+s} \left\{ \prod_{j=1}^{l} \mathcal{P}_m(a_j) \prod_{i=1}^{s} \Phi_m(a_i) \right\} \leq \prod_{j=1}^{l} A(a_j) \prod_{i=1}^{s} A(a_i).$$

Since $\mathbb{F}_j \in \mathbb{U}CV_{\mathcal{A}_\upsilon}$, then from Theorem 1, we obtain (3.3) and (3.4)

$$\sum_{m=2}^{\infty} m\mathcal{P}_m(a_j) \leq A(a_j),$$

for every $j = 1, 2, \cdots, l$. then we have

$$\mathcal{P}_m(a_j) \leq \frac{A(a_j)}{m} \tag{3.3}$$

for every $j = 1, , 2, \cdots, s$. In a similar way, for $G_i \in \mathbb{U}CV_{A_\upsilon}$ we get

$$\sum_{m=2}^{\infty} m\Phi_m(a_i) \leq A(a_i).$$

Therefore

$$\Phi_m(a_i) \leq \frac{A(a_i)}{m}, \tag{3.4}$$

for every $i = 1, 2, \cdots, s$. By (3.3) and (3.4), for $j = 1, 2, \cdots, l$ and $i = 1, 2, \cdots, s$, we attain

$$\sum_{m=2}^{\infty} \left[m^{l+s} \left\{ \prod_{j=1}^{l} \mathcal{P}_m(a_j) \prod_{i=1}^{s} \Phi_m(a_i) \right\} \right]$$
$$\leq \left[m^{l+s} \left\{ m^{-s} m^{-l} \prod_{j=1}^{l} A(a_j) \prod_{i=1}^{s} A(a_i) \right\} \right] \leq \left\{ \prod_{j=1}^{l} A(a_j) \prod_{i=1}^{s} A(a_i) \right\}.$$

Hence $H(w) \in \mathbb{U}CV^{l+s}_{A_\upsilon}$. □

Corollary 4. Let the function $\mathbb{F}_j(w) = A(w) + \sum_{m=2}^{\infty} \mathcal{P}_{m,j}(w) z^{\upsilon m}$ given by (3.1) be in the class $\in \mathbb{U}CV_{\mathcal{A}_\upsilon}$ for every $j = 1, 2, \cdots l$. Then the Hadamard product $\mathbb{F}_1 * \mathbb{F}_2, \cdots * \mathbb{F}_l(z)$ belongs to the class $\mathbb{U}CV^{l}_{\mathcal{A}_\upsilon}$.

Corollary 5. Let the function $\mathbb{G}_i(w) = A(w) + \sum_{m=2}^{\infty} \Phi_{m,i}(w) z^{\upsilon m}$, defined by (3.1) be in the class $\in \mathbb{U}CV_{\mathcal{A}_\upsilon}$ for every $i = 1, 2, \cdots s$. Then the Hadamard product $\mathbb{G}_1 * \mathbb{G}_2, \cdots * \mathbb{G}_s(z)$ belongs to the class $\mathbb{U}CV^{s}_{\mathcal{A}_\upsilon}$.

4. Conclusion

We generalized a class of analytic functions (Koebe type), by utilizing the concept of fractional calculus. Moreover, by utilising the above class, we defined fractional operator type of integral in the sense of Noor integral operator. Some geometrical properties are illustrated in Banach space. The generalized product (Owa-Hadamard product) is discussed in some subclasses.

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Accepted: 18.04.2019