Rough approximate operators based on fuzzy soft relation

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Abstract. Fuzzy soft set is a mapping from a parameter set to the collection of fuzzy subset of universal set. In this paper fuzzy soft relation is presented based on the cartesian product of fuzzy soft sets and the notion of fuzzy soft equivalence relation is introduced. We prove that every fuzzy soft equivalence relation on an arbitrary fuzzy soft set partition the given fuzzy soft set into equivalence classes and thus induces a new relation on the parameter set. Basic properties of the induced relation are studied. A pair of rough approximate operators are investigated and their related properties are given. Relationship between a fuzzy soft topological space and rough approximate operators based on fuzzy soft relation is further established.
Keywords: fuzzy soft set, fuzzy soft relation, fuzzy soft topology.

1. Introduction

Theory of fuzzy sets and fuzzy relation first developed by Zadeh [1] has been applied to many branches of mathematics. Fuzzy equivalence relation introduced by Zadeh as a generalization of the concept of an equivalence relation has been widely studied in [2], [3], [4], [5], [6], [7] as a way to measure the degree of distinguishability or similarity between the objects of a given universe of discourse. And it have been shown to be useful in different context such as fuzzy control [8], approximate reasoning [9], fuzzy cluster analysis [10]. Depending on the authors and the context in which they appeared, it have received other names such as similarity relations, indistinguishability operators [11], many valued equivalence relations[12], etc. Later V. Murali [13] studied the cuts of fuzzy equivalence relation and lattice theoretic properties of fuzzy equivalence relation. In 1999 Molodtsov [14] proposed the novel concept of soft theory which provides a completely new approach for modelling vagueness and uncertainty. Theory of soft

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set has gained popularity among the researchers working in diverse areas. It is getting richer with new developments. Application of soft set theory can be seen in [15], [16], [17], [18], [19], [20], [21], [22], [23].

Relations in soft set have been studied in [24], [25], [26]. Structures of soft set have been studied by many authors [27], [28]. Recently Ali et al. [29] have shown that a collection of soft set with reference to so called new operations give rise to many algebraic structures and form certain complete modular lattice structures. The theory of fuzzy soft set [30], fuzzification of the notion of soft set has the ability of hybridization. In this reagard fuzzy soft set and their applications has been investigated by many authors [31], [32], [33], [34], [35], [36], [37]. Fuzzy soft set is the parametrized collection of fuzzy sets. Fuzzy soft sets can be used to crunch the volume of data. Collection of fuzzy soft set form a complete modular lattice structures with respect to certain binary operations defined on them [38]. Topological structure of fuzzy soft sets can be seen in [39].

Based on these concepts fuzzy soft relation is introduced and it provides both a general and flexible method for the designing of fuzzy logic controller and more generally for the modelling of any decision making process. Fuzzy soft relation stores data in terms of relation between parameters which we define by membership function.

Theory of rough sets proposed by Pawalk [40]is considered as an alternative tool for imperfect data analysis. The rough set approach has fundamental importance in the area of knowledge acquisition, machine learning, decision analysis and many other fields [41], [42]. Approximation space is the basic structure of Rough set theory. Lower and upper approximation induced from an approximation space can be used to reveal and express the knowledge hidden in information systems in the form of decision rules. Various fuzzy generalizations of rough approximations have been proposed in [43], [44]. The most common fuzzy rough set, obtained by replacing the crisp relations with fuzzy relations on the universe and crisp subsets with fuzzy sets, have been used to solve practical problems such as data mining [45], approximate reasoning [46], and medical time series. An interesting topic in rough set theory is to study the relationship between rough sets and topologies. Many authors studied topological properties of rough sets [47], [48]. Using the concept of fuzzy soft relation, R rough set is introduced. Fuzzy soft equivalence relation is the key notion used in R rough set model. The equivalence classes generated by the fuzzy soft equivalence relation are the building blocks for the construction of these approximations.

This paper is organized in the following manner. In Section 2 basic definitions related to fuzzy soft sets are given. These basic concepts are required in later sections. In Section 3 fuzzy soft relation is defined and its properties are studied. Also this Section is devoted to the study of composition of fuzzy soft relation and fuzzy soft equivalence relation. We define a new relation on parameter set induced by the fuzzy soft relation with example and its theoretical aspects are studied. Also we prove that the new relation induced by a fuzzy soft eqivalence relation partition the given fuzzy soft set. In Section 4 a pair of rough approximate operators has been defined and fuzzy soft reflexive, fuzzy soft symmetric and fuzzy soft transitive relation have been characterized by these rough approximate operators. In Section 5 relation between a fuzzy soft topological space and rough approximate operators based on fuzzy soft relation is further established. The last section concludes the paper and points out further research work.

2. Preliminaries

Through out this paper X refers to an initial universe, $\wp$ is a set of parameters in relation to objects in X. Parameters are often attributes, characteristics or properties of objects. I^{X} denote the set of all fuzzy subsets of X and P,Q $\subset \wp$.

Definition 2.1 ([30]). The pair (f,P) is called a fuzzy soft set over X if f is a mapping given by f:P$\rightarrow I^X$. Each element in the fuzzy soft set (f,P) corresponding to the parameter p$\in$ P can be denoted by f_p, where f_p is a function from X to [0,1].

The fuzzy soft set(f,P) is said to be a null fuzzy soft set, denoted by $\tilde{0}$, if $f_p(x)=0, \forall$ p $\in$ P and $\forall$ x $\in$ X

Definition 2.2 ([30]). Let (f,P) and (g,Q) be two fuzzy soft set over X. Then (f,P) is called fuzzy soft subset of (g,Q) denoted by (f,P) $\subseteq$(g,Q) if P $\subseteq$ Q and $f_p(x) \leq g_p(x), \forall$ p $\in$ P.

Collection of all fuzzy soft subsets of(f,P) be denoted as S_f(X,P)

Definition 2.3 ([30]). Let (f,P) and (g,Q) be two fuzzy soft sets over X. Then (f,P)-(g,Q) is the fuzzy soft set (h,C) where C= P-Q and $h_c(x)=f_c(x), \forall$ c $\in$ C

Definition 2.4 ([30]). Union of two fuzzy soft sets (f,P) and (g,Q) over X is defined as the fuzzy soft set (h,C)=(f,P) $\cup$ (g,Q) where C=P$\cup$Q and for all c$\in$C

$$h_c(x) = \begin{cases} \mathrm{f}_c(\mathrm{x}), & \text{if c} \in P-Q \\ \mathrm{g}_c(\mathrm{x}), & \text{if c} \in Q-P \\ \mathrm{f}_c(\mathrm{x}) \vee \mathrm{g}_c(\mathrm{x}) \;, & \text{if c} \in P \cap Q. \end{cases}$$

Definition 2.5 ([30]). Intersection of two fuzzy soft sets (f,P) and (g,Q) over X is defined as the fuzzy soft set (h,C)=(f,P) $\cap$ (g,Q) where C=P$\cap$Q and for all c$\in$ C, $h_c(x)=f_c(x) \wedge g_c(x)$

Definition 2.6 ([30]). Let (f,P) and (g,Q) be two fuzzy soft sets over a universe X. Then cartesian product of (f,P) and (g,Q) is defined as (f,P) x (g,Q) = (h, P x Q) where h: P x Q$\rightarrow I^X$ and $h_{(p,q)}(x) = \min(f_p(x), g_q(x)), \forall$ (p,q) $\in$ P x Q.

Example 2.7. Consider the various investment avenues as x_1-bank deposit, x_2-Insurance, x_3-postal savings, x_4- shares and stocks, x_5-mutual funds, x_6-gold, x_7-real estate as the universal state X, and factors influencing investment decision such as e_1-safety of funds, e_2-liquidity of funds, e_3-high returns, e_4-maximum profit in minimum time period, e_5-stable returns, e_6-easy accessibility, e_7-tax concession, e_8-minimum risk of parameters.

Decision maker P is good at the parameters e_1 and e_5. Decision maker Q is good at the parameters e_3 and e_4. This information can be expressed by two fuzzy soft sets (f,P) and (g,Q) respectively.

$$(f,P) = \left\{ \begin{array}{l} e_1 = \{\frac{x_1}{1}, \frac{x_2}{0.9}, \frac{x_3}{1}, \frac{x_4}{0.2}, \frac{x_5}{0.3}, \frac{x_6}{0.8}, \frac{x_7}{0.4}\} \\ e_5 = \{\frac{x_1}{1}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{0.1}, \frac{x_5}{0.1}, \frac{x_6}{0.3}, \frac{x_7}{0.7}\} \end{array} \right\}$$

and

$$(g,Q) = \left\{ \begin{array}{l} e_3 = \{\frac{x_1}{0.5}, \frac{x_2}{0.5}, \frac{x_3}{0.5}, \frac{x_4}{0.7}, \frac{x_5}{0.6}, \frac{x_6}{0.8}, \frac{x_7}{0.9}\} \\ e_4 = \{\frac{x_1}{0.4}, \frac{x_2}{0.2}, \frac{x_3}{0.4}, \frac{x_4}{0.8}, \frac{x_5}{0.6}, \frac{x_6}{0.8}, \frac{x_7}{0.9}\} \end{array} \right\}$$

A typical element of (h,P x Q) will look like

$$h(e_1, e_3) = \left\{ \frac{x_1}{0.5}, \frac{x_2}{0.5}, \frac{x_3}{0.5}, \frac{x_4}{0.2}, \frac{x_5}{0.3}, \frac{x_6}{0.8}, \frac{x_7}{0.4} \right\}$$

3. Fuzzy soft relations and partition on fuzzy soft set

Fuzzy soft Relation is a suitable tool for describing correspondence between the parameters in a fuzzy soft set, which makes the theory of fuzzy soft set a hot subject for research. It plays an important role in modeling and decision making of systems. In this section we discuss a variety of different properties of a fuzzy soft relation may possess.

Definition 3.1. Fuzzy Soft Relation R from (f,P) to (g,Q) is a fuzzy soft subset of (f,P) x (g,Q). If R is a fuzzy soft subset of (f,P) x (f,P) then it is called a Fuzzy Soft Relation on (f,P).

If R is a Fuzzy Soft Relation on (f,P) then $R^{-1}_{pq} = R_{qp}$, $\forall$ (p,q) $\in$ PxQ. If R is a fuzzy soft relation from (f,P) to (g,Q) then R^{-1} is a fuzzy soft relation from (g,Q) to (f,P).

Definition 3.2. Let R_1 and R_2 be two Fuzzy Soft Relations from (f,P) to (g,Q) and (g,Q) to (h,S) respectively. Composition of R_1 and R_2 denoted by $R_1 \circ R_2$ is a Fuzzy Soft Relation from (f,P) to (h,S) defined as $(R_1 \circ R_2)_{ps} = \bigvee_{q \in Q}((R_1)_{pq} \wedge (R_2)_{qs})$ where (p,q) $\in$ PxQ and (q,s) $\in$ QxS.

Theorem 3.3. *Let Q, R, S be fuzzy soft relation on (f,P) then:*
1) $(R^{-1})^{-1} = R$;
2) $R \subseteq S \implies R^{-1} \subseteq S^{-1}$;
3) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$;
4) $R \subseteq S \implies R \circ Q \subseteq S \circ Q$;

5)(Q∘R)∘S = Q∘(R∘S).

Definition 3.4. Let (f,P) be a fuzzy soft set over the universal set X and R be a fuzzy soft relation on (f,P) then R is said to be:

1) Fuzzy soft reflexive if $\forall$ p,q $\in$ P with p $\neq$ q and $\forall$ x $\in$ X, $R_{pq}(x) \leq R_{pp}(x)$ and $R_{qp}(x) \leq R_{pp}(x)$;

2) Fuzzy soft symmetric relation if $R = R^{-1}$;

3) Fuzzy soft transitive relation if $R \circ R \subseteq R$;

4) Fuzzy soft equivalence relation if it is fuzzy soft reflexive, fuzzy soft symmetric and fuzzy soft transitive.

Definition 3.5. The relation R on fuzzy soft set (f,P) induces a new relation R^{λ} on (f,P) as follows:

Let $\lambda \in [0,1]$. Define the relation R^{λ} on (f,P) such that f_p R^{λ} f_q if and only if $R_{pq}(x) \geq \lambda$, $\forall$ x $\in$ X.

Theorem 3.6. *If R is a fuzzy soft equivalence relation on (f,P) and* $\alpha = \bigwedge_{x\in X} \bigwedge_{p\in P} R_{pp}(x)$ *then for each* $\lambda \in [0,\alpha]$*,* R^{λ} *is an equivalence relation on (f,P).*

Proof. For each p $\in$ P and $\forall$ x $\in$ X, $R_{pp}(x) \geq \alpha \geq \lambda \implies f_pR^{\lambda}f_p \implies R^{\lambda}$ is reflexive.

Let $f_pR^{\lambda}f_q$ i.e. $R_{pq}(x) \geq \lambda$.

Since R is a fuzzy soft symmetric relation, $R^{-1}_{pq}(x) = R_{pq}(x) \implies R_{qp}(x) = R_{pq}(x) \geq \lambda \implies f_qR^{\lambda}f_p$. This implies R^{λ} is a symmetric relation.

Finally let $f_pR^{\lambda}f_r$ and $f_rR^{\lambda}f_q \implies R_{pr}(x) \geq \lambda$ and $R_{rq}(x) \geq \lambda$, $\forall$ x $\in$ X. Since R is a fuzzy soft transitive relation, $R_{pq}(x) \geq (R \circ R)_{pq}(x) = \bigvee_{r\in P}(R_{pr}(x) \wedge R_{rq}(x)) \geq (R_{pr}(x) \wedge R_{rq}(x)) \geq \lambda$. Hence R^{λ} is a transitive relation.

Definition 3.7. Equivalence class of f_p denoted by $[f_p]$ is defined as $[f_p] = \{f_q: f_pR^{\lambda}f_q\}$.

Example 3.8. Let the fuzzy soft set (f,P) over universal set x=$\{x_1, x_2, x_3, x_4\}$ and parameter set P =$\{p, q, r\}$ be given by.

$$(f,P) = \left\{ \begin{array}{c} f_p = \{\frac{x_1}{.76}, \frac{x_2}{0.5}, \frac{x_3}{.82}, \frac{x_4}{0.64}\} \\ \\ f_q = \{\frac{x_1}{.58}, \frac{x_2}{.075}, \frac{x_3}{.6}, \frac{x_4}{0.56}\} \\ \\ f_r = \{\frac{x_1}{.66}, \frac{x_2}{.5}, \frac{x_3}{.7}, \frac{x_4}{0.75}\} \end{array} \right\}$$

Consider the fuzzy soft equivalence relations R on (f,P) as follows

R	x_1	x_2	x_3	x_4
R_{pp}	0.76	0.5	0.82	0.64
R_{pq}	0.58	0.075	0.6	0.56
R_{qp}	0.58	0.075	0.6	0.56
R_{qq}	0.6	0.7	0.82	0.9

Since $R_{rr}(x) = 0$, $\forall x \in X$ we have $\lambda = \bigwedge_{x\in X}\bigwedge_{p\in P} R_{pp}(x) = 0$. Then $R^0 = \{(f_p,f_p), (f_q,f_q), (f_p,f_q), (f_q,f_p),(f_r,f_r)\}$ is an equivalence relation on (f,P).

If $\lambda = 0.55$, $R^{0.55} = \{(f_q,f_q), (f_p,f_q), (f_q,f_p)\}$ is not an equivalence relation on fuzzy soft set (f,P).

Lemma 3.9. *Let R be a fuzzy soft equivalence relation on (f,P) and* $\alpha = \bigwedge_{x\in X}\bigwedge_{p\in P} R_{pp}(x)$. *For* $p,q \in P$ *and* $\lambda \in [0,\alpha]$, $[f_p] = [f_q]$ *if and only if* $f_p R^\lambda f_q$ *if and only if* $[f_p] \cap [f_q] \neq \phi$.

Proof. Suppose $[f_p] = [f_q]$.

Since R is a fuzzy soft equivalence relation on (f,P), R^λ is an equivalence relation $\Longrightarrow$ R^λ is a reflexive relation $\Longrightarrow$ $f_p R^\lambda f_p$.

Hence $f_p \in [f_p] = [f_q]$ i.e. $f_q R^\lambda f_p$. By Theorem 3.6 we have R^λ is a symmetric relation. Hence $f_p R^\lambda f_q$. Conversely suppose that $f_p R^\lambda f_q$.

Let $f_r \in [f_p]$, then $f_p R^\lambda f_r$. Using symmetric and transitive property of R^λ we have $f_q R^\lambda f_r \Longrightarrow f_r \in [f_q]$. Hence $[f_p] \subseteq [f_q]$.

Using a similar argument we can show that $[f_p] \subseteq [f_q]$. Hence $[f_p] =[f_q]$.

Now, let $[f_p] \cap [f_q] \neq \phi \iff f_r \in [f_p] \cap [f_q] \iff f_p R^\lambda f_r$ and $f_q R^\lambda f_r \iff f_p R^\lambda f_q$.

Definition 3.10. A collection of fuzzy soft subset $\{(f_i, P_i)$: $i\in I\}$ of a fuzzy soft set (f,P) is called a partition of (f,P) if:

1) $(f,P) = \bigcup_i (f_i,P_i)$;

2) $P_i \cap P_j = \phi$ when ever $i \neq j$.

Theorem 3.11. *Corresponding to every fuzzy soft equivalence relation defined on the fuzzy soft set (f,P) there exist a partition of (f,P) and this partition precisely consist of the equivalence class of* R^λ *where* $\lambda = \bigwedge_{x\in X}\bigwedge_{p\in P} R_{pp}(x)$.

Proof. Let $[f_p]$ be an equivalence class corresponding to the relation R^λ on (f,P) and $P_p = \{ q \in P$: $f_p R^\lambda f_q\}$. Also denote $[f_p]$ as (f,P_p).

We have to show that $\{(f,P_p)$: $p \in P\}$ of such distinct set forms a partition of (f,P), i.e. we have to prove that:

1) $(f,P) = \bigcup_{p\in P}(f, P_p)$;

2) if P_p, P_q are not identical then $P_p \cap P_q = \phi$.

Since R^λ is a reflexive relation $f_p R^\lambda f_p$ $\forall$ $p \in P$, so that 1) can be easily proved.

Let $r \in P_p \cap P_q$. Then, $f_r \in (f,P_p)$ and $f_r \in (f,P_q) \Longrightarrow f_r R^\lambda f_p$ and $f_r R^\lambda f_q$. Using transitive property of R^λ we have $f_p R^\lambda f_q$ hence by Lemma 3.9 $[f_p]$=$[f_q]$ $\Longrightarrow P_p = P_q$.

Definition 3.12. Let R^λ be the relation induced by the fuzzy soft relation R on (f,P). $\forall$ (g,P_g) $\in S_f$(X,P),Define P_g(f,R)= {$f_p \in$ (f,P):[f_p] $\subseteq$ (g,P_g)} and $P^g(f,R) = \{f_p \in (f,P) : [f_p] \cap (g,P_g) \neq \tilde{0}\}$.

Example 3.13. In Example 3.8 let (g,P_g)$\in S_f$(X,P) be given by

$$(g,P_g) = \left\{ g_p = \{\frac{x_1}{.76}, \frac{x_2}{0.5}, \frac{x_3}{.82}, \frac{x_4}{0.64}\} \right\}.$$

Consider the relation $R^{0.5}$ induced by the fuzzy soft equivalence relation R. [f_p] = [f_q]={f_p,f_q} not a subset of (g,P_g). Hence P_g(f,R) is an empty set with repect to the equivalence classes [f_p] and [f_q].

Since P_g contains only one parameter p, we have [f_r]=$\tilde{0} \nsubseteq$(g,P_g) $\implies$ P_g(f,R)= ϕ with repect to the equivalence classes [f_r]. Next, we can compute P^g(f,R).

[f_p] $\cap$ (g,P_g)$\neq\tilde{0}$ $\implies$ P^g(f,R)={f_p} with respect to the equivalence classes [f_p]. Using a similar argument we can prove that P^g(f,R)={f_q} w.r.t to the equivalence classes [f_q]. [f_r] $\cap$ (g,P_g)=$\tilde{0}$ $\implies$ P^g(f,R)=ϕ w.r.t to the equivalence class [f_r].

Lemma 3.14. *Let R be a fuzzy soft relation defined on (f,P) then $\forall$ (g,P_g),(h,P_h) $\in S_f$(X,P):*

1. P_f(f,R)=(f,P);
2. *If R^λ is a reflexive relation on (f,P) then P_g(f,R) $\subseteq$ (g,P_g) $\subseteq$ P^g(f,R);*
3. a) (g,P_g)$\subseteq$ (h,P_h) $\implies$ P_g(f,R) $\subseteq$ P_h(f,R);
b) (g,P_g)$\subseteq$ (h,P_h) $\implies$ P^g(f,R) $\subseteq$ P^h(f,R);
4. a) P^l(f,R)= P^g(f,R) $\cup$ P^h(f,R) where (l,P_l)=(g,P_g)$\cup$ (h,P_h);
b)P_l(f,R)= P_g(f,R) $\cap$ P_h(f,R) where (l,P_l)=(g,P_g)$\cap$ (h,P_h).

Proof. 1. This is obvious.

2. Let $f_p \in P_g$(f,R).

Since R^λ is a reflexive relation on (f,P), $f_p \in$[f_p] $\subseteq$ (g,P_g) $\implies$ [f_p] $\cap$ (g,P_g)$\neq$0 $\implies$ $f_p \in P^g$(f,R). Hence the proof.

3. a) $f_p \in P_g$(f,R) $\implies$ [f_p] $\subseteq$ (g,P_g) $\implies$ [f_p] $\subseteq$ (h,P_h) $\implies$ $f_p \in P_h$(f,R).

Hence P_g(f,R) $\subseteq$ P_h(f,R).

b) Proof is similar to a).

4. a) Let $f_p \in P^l$(f,R) $\implies$ [f_p] $\cap$ (l,P_l)$\neq$0.Since (l,P_l)=(g,P_g)$\cap$ (h,P_h),either [f_p] $\cap$ (g,P_g)$\neq$0 or [f_p] $\cap$ (h,P_h)$\neq$0 $\implies$ $f_p \in P^g$(f,R) or $f_p \in P^h$(f,R) $\implies$ f_p $\in P^g$(f,R) $\cup$ P^h(f,R). Conversely this is obvious.

b) Proof is similar to a).

4. Rough approximate operators of fuzzy soft relation

In this section we propose two rough approximate operators for a given fuzzy soft relation.

Definition 4.1. Define the following operations- $\underline{apr}$, $\overline{apr}$: $S_f(X,P) \rightarrow S_f(X,P)$ by $\underline{apr}(g,P_g) = \cup P_g(f,R)$ and $\overline{apr}(g,P_g) = \cup P^g(f,R)$.

$\underline{apr}$ and $\overline{apr}$ are called the R lower approximation operator and R upper approximation operator respectively. The soft subset (g,P) is called R definable if $\underline{apr}(g,P_g) = \overline{apr}(g,P_g)$. (g,P_g) is called R rough set if $\underline{apr}(g,P_g) \neq \overline{apr}(g,P_g)$.

Theorem 4.2. *Let R be a fuzzy soft relation defined on (f,P). Then* $\forall$ $(g,P_g),(h,P_h) \in S_f(X,P)$.

1. *If* $(g,P_g) \subseteq (h,P_h)$, *then:*
   a) $\underline{apr}(g,P_g) \subseteq \underline{apr}(h,P_h)$;
   b) $\overline{apr}(g,P_g) \subseteq \overline{apr}(h,P_h)$.
2. a) $\underline{apr}((g,P_g) \cap (h,P_h)) = \underline{apr}(g,P_g) \cap \underline{apr}(h,P_h)$;
   b) $\overline{apr}((g,P_g) \cup (h,P_h)) = \overline{apr}(g,P_g) \cup \overline{apr}(h,P_h)$.

Proof. 1. This obviously hold by Lemma 3.14.

2. a) Let $(g,P_g) \cap (h,P_h) = (l,P_l)$ then by Lemma 3.10,$P_l(f,R) = P_g(f,R) \cap P_h(f,R)$.

Let $f_p \in \underline{apr}(l,P_l) \implies f_p \in \cup P_l(f,R) \implies f_p \in P_l(f,R) \implies f_p \in P_g(f,R)$ and $f_p \in P_h(f,R) \implies f_p \in \underline{apr}(g,P_g) \cap \underline{apr}(h,P_h)$.

Hence $\underline{apr}(l,P_l) \subseteq \underline{apr}(g,P_g) \cap \underline{apr}(h,P_h)$.

Similarly we can prove the Converse part $\underline{apr}(g,P_g) \cap \underline{apr}(h,P_h) \subseteq \underline{apr}(l,P_l)$.

b) This can be proved similarly as above.

Theorem 4.3. *If R is any arbitrary fuzzy soft relation defined on fuzzy soft set (f,P) then* R^λ, $\lambda \in [0,1]$ *is reflexive iff* $\forall$ $(g,P_g) \in S_f(X,P)$, $\underline{apr}(g,P_g) \subseteq (g,P_g) \subseteq \overline{apr}(g,P_g)$.

Proof. First part of above statement follows from Lemma 3.14.

Now, suppose that $\forall$ $(g,P_g) \in S_f(X,P)$, $\underline{apr}(g,P_g) \subseteq (g,P_g) \subseteq \overline{apr}(g,P_g)$.

Let $f_p \in (f,P)$ and $(g,P_g) = f_p$. By our assumption $f_p \in \overline{apr}(g,P_g) \implies f_p \in P^g(f,R) \implies [f_p] \cap (g,P_g) \neq \tilde{0} \implies f_p \in [f_p]$, $\forall$ $f_p \in (f,P) \implies R^\lambda$ is reflexive.

Theorem 4.4. *Let R be any arbitrary fuzzy soft relation defined on fuzzy soft set (f,P). If* $R^\lambda,\lambda \in [0,1]$ *is reflexive then:*

1) $\underline{apr}(f,P) = \overline{apr}(f,P) = (f,P)$;

2) $\underline{apr}\tilde{0} = \overline{apr}\tilde{0} = \tilde{0}$.

Proof. 1) By Theorem 4.3 we have $\underline{apr}(f,P) \subseteq (f,P) \subseteq \overline{apr}(f,P)$. Conversely since $P^f(f,R) \subseteq (f,P) \implies \overline{apr}(f,P) \subseteq (f,P) \implies \overline{apr}(f,P) = (f,P)$. By Lemma 3.14 $P_f(f,R) = (f,R) \implies \underline{apr}(f,P) = (f,P)$. Hence the result.

2) This is obvious.

Theorem 4.5. *If R is any arbitrary fuzzy soft relation defined on fuzzy soft set (f,P) and* $\alpha = \bigwedge_{x\in X}\bigwedge_{p\in P} R_{pp}(x)$ *then* $R^\lambda,\lambda \in [0,\alpha]$ *is symmetric iff* $\forall$ $(g,P_g) \in S_f(X,P)$, $\overline{apr}(\underline{apr}(g,P_g)) \subseteq (g,P_g) \subseteq \underline{apr}(\overline{apr}(g,P_g))$.

Proof. Let (g,P_g) $\in S_f$(X,P) and denote (k,P_k)=$\underline{apr}$(g,P_g),(h,P_h)=$\overline{apr}$(g,P_g), (w,P_w)=$\overline{apr}$(k,P_k), (l,P_l)=$\underline{apr}$(h,P_h).

Suppose R^λ is symmetric and (w,P_w)-(g,P_g)$\neq \phi$. Pick $f_p \in$ (w,P_w)-(g,P_g) $\Longrightarrow$ [f_p]$\cap$ (k,P_k)$\neq\tilde{0}$.

Let $f_k \in$ [f_p]$\cap$ (k,P_k) $\Longrightarrow f_k \in$(k,P_k)=$\cup P_g$(f,R) $\Longrightarrow f_k \in P_g$(f,R) $\Longrightarrow$ [f_p]$\subseteq$(g,P_g). Since R^λ is symmetric,$f_k \in$ [f_p] $\Longrightarrow f_p \in$ [f_k]. Hence $f_p \in$ [f_k]$\subseteq$(g,P_g), a contradiction. Hence $\overline{apr}(\underline{apr}$(g,$P_g$))$\subseteq$(g,$P_g$). Next suppose that (g,$P_g$)-(l,$P_l$)$\neq \phi$. Pick $f_p \in$(g,P_g)-(l,P_l) $\Longrightarrow f_p \in$(g,P_g) and $f_p \notin$(l,P_l).

$f_p \notin$(l,P_l) $\Longrightarrow f_p \notin\underline{apr}$(h,$P_h$) $\Longrightarrow f_p \notin P_h$(f,R) $\Longrightarrow$ [f_p] $\not\subseteq$(h,P_h). Pick $f_q \in$[f_p] such that $f_q \notin$ (h,P_h) $\Longrightarrow f_q \notin P^g$(f,R) $\Longrightarrow$ [f_q]$\cap$(g,P_g)=$\tilde{0}$.

Since R^λ is symmetric $f_p \in$[f_q], i.e $f_p \notin$(g,P_g), a contradiction. Conversely suppose $\forall$ (g,P_g) $\in S_f$(X,P),$\overline{apr}(\underline{apr}$(g,$P_g$))$\subseteq$(g,$P_g$) $\subseteq \underline{apr}(\overline{apr}$(g,$P_g$)).

Let $f_q \in$[f_p] and (g,P_g)=f_p, then $f_p \in\underline{apr}(\overline{apr}$(g,$P_g$)) $\Longrightarrow f_p \in P_h$(f,R) $\Longrightarrow$ [f_p]$\subseteq$(h,P_h)=$\overline{apr}$(g,P_g) $\Longrightarrow f_q \in\overline{apr}$(g,$P_g$) $\Longrightarrow f_q \in P^g$(f,R) $\Longrightarrow$ [f_q]$\cap$(g,P_g)$\neq\tilde{0} \Longrightarrow$ $f_p \in$[f_q] $\Longrightarrow R^\lambda$ is symmetric.

Lemma 4.6. *Let R be a fuzzy soft relation defined on fuzzy soft set(f,P) and* $R^\lambda,\lambda \in$*[0,1] is symmetric then for each* $f_p \in$*(f,P),* $\overline{apr}$*(* f_p*) =[*f_p*].*

Proof. Let (g,P_g)= f_p and (h,P_h)= $\overline{apr}$(g,P_g) $f_k \in$ (h,P_h) $\Longleftrightarrow f_k \in P^g$(f,R) $\Longleftrightarrow$ [f_k] $\cap$ (g,P_g)$\neq\tilde{0} \Longleftrightarrow f_p \in$[$f_k$] $\Longleftrightarrow f_k \in$[$f_p$]. Hence the proof.

Theorem 4.7. *If R is any arbitrary fuzzy soft relation defined on fuzzy soft set (f,P) and* $R^\lambda,\lambda \in$*[0,1] is reflexive and symmetric then following statements are equivalent:*

1) R^λ *is transitive;*

2) $\forall$ (g,P_g)$\in S_f$(X,P),$\underline{apr}$(g,P_g) $\subseteq \underline{apr}(\underline{apr}$(g,$P_g$))$\subseteq$(g,$P_g$) $\subseteq \overline{apr}(\overline{apr}$(g,$P_g$)) $\subseteq\overline{apr}$(g,P_g).

Proof. 1 $\Longrightarrow$ 2. Let (h,P_h)=$\underline{apr}$(g,P_g),(k,P_k)=$\overline{apr}$(g,P),and (l,P_l)=$\overline{apr}$(k,P_k). First, we prove that if (h,P_h)=$\underline{apr}$(g,P_g) then P_h(f,R)=P_g(f,R).

By Lemma 3.14 (h,P_h)$\subseteq$ (g,P_g) $\Longrightarrow$ P_h(f,R) $\subseteq P_g$(f,R) Now let P_g(f,R)-P_h(f,R)$\neq \phi$ and $f_p \in P_g$(f,R)-P_h(f,R)$f_p \in P_g$(f,R) $\Longrightarrow$ [f_p]$\subseteq$ (g,P_g) and $f_p \notin$ P_h(f,R) $\Longrightarrow$ [f_p]$\not\subseteq$ (h,P_h) $\Longrightarrow$ there exist $f_k \in$ [f_p] and $f_k \notin$ (h,P_h).

Since R^λ is transitive [f_k] $\subseteq$[f_p]. Hence [f_k] $\subseteq$ (g,P_g) $\Longrightarrow f_k \in P_g$(f,R) $\Longrightarrow$ $f_k \in \underline{apr}$(g,P_g) $\Longrightarrow f_k \in$ (h,P_h), a contradiction.

This implies P_g(f,R)$\subseteq P_h$(f,R) and hence P_h(f,R)=P_g(f,R). Let $f_p \in$ (h,P_h) $\Longrightarrow f_p \in P_g$(f,R) $\Longrightarrow f_p \in P_h$(f,R) $\Longrightarrow f_p \in \underline{apr}$(h,$P_h$). $\Longrightarrow \underline{apr}$(g,$P_g$) $\subseteq$ $\underline{apr}(\underline{apr}$(g,$P_g$)).

By Theorem 4.3 we have $\underline{apr}(\underline{apr}$(g,$P_g$))$\subseteq$(g,$P_g$) $\subseteq \overline{apr}(\overline{apr}$(g,$P_g$)) To prove (l,$P_l$)$\subseteq$ (k,P_k). Suppose(l,P_l)- (k,P_k)$\neq\phi$ and $f_p \in$(l,P_l)- (k,P_k) $f_p \notin$ (k,P_k) $\Longrightarrow f_p \notin$ P^g(f,R) $f_p \in$(l,P_l) $\Longrightarrow f_p \in P^k$(f,R) $\Longrightarrow$ [f_p] $\cap$ (k,P_k)$\neq\tilde{0}$. Let $f_k \in$[f_p] and $f_k \in$ (k,P_k) $f_k \in$ (k,P_k) $\Longrightarrow$ [f_p] $\cap$ (g,P_g)$\neq\tilde{0}$. Since R^λ is transitive, $f_k \in$[f_p] $\Longrightarrow$ [f_k] $\subseteq$[f_p] $\Longrightarrow$ [f_p] $\cap$ (g,P_g)$\neq\tilde{0} \Longrightarrow f_p \in P^g$(f,R), a contradiction.

Hence $\overline{apr}(\overline{apr}(g,P_g))\subseteq\overline{apr}(g,P_g)$.

$2 \Longrightarrow 1$. Let $f_p,f_q,f_r\in R^\lambda$ such that $(f_p,f_q),(f_q,f_r)\in R^\lambda \Longrightarrow f_q\in[f_p]$ and $f_q\in[f_r]$ $\Longrightarrow [f_p]\cap[f_r]\neq\phi$. Let $(h,P_h)=\overline{apr}(f_p)$. By previous Lemma $[f_r]\cap(h,P_h)\neq\tilde{0}$ $\Longrightarrow f_r\in P^h(f,R) \Longrightarrow f_r\in\overline{apr}(h,P_h)$. By our assumption $\overline{apr}((h,P_h))\subseteq(h,P_h) \Longrightarrow$ $f_r\in(h,P_h)=[f_p] \Longrightarrow (f_r,f_p)\in R^\lambda \Longrightarrow R^\lambda$ is transitive.

5. Fuzzy soft topology induced by the fuzzy soft relation

Definition 5.1 ([39])**.** Let $\{P_i:i\in I\}\subset\wp$.

The union of non empty family $\{(f_i,P_i):i\in I\}$ of fuzzy soft sets over the common universe X, denoted by $\bigcup_{i\in I}(f_i,P_i)$ is defined as the fuzzy soft set(h,C) such that $C=\bigcup_{i\in I}P_i$ and for each $c\in C$, $h_c(x)=\bigvee_{i\in I_C}(f_i)_c(x),\forall\ x\in X$, where $I_c=\{i\in I: c\in P_i\}$

Definition 5.2 ([39])**.** Let $\{P_i:i\in I\}\subset\wp$. The intersection of non empty family $\{(f_i,P_i):i\in I\}$ of fuzzy soft sets over the common universe X, denoted by $\bigcap_{i\in I}(f_i,P_i)$ is defined as the fuzzy soft set(h,C) such that $C=\bigcap_{i\in I}P_i$ and for each $c\in C,h_c(x)=\bigwedge_{i\in I_C}(f_i)_c(x),\forall\ x\in X$.

Definition 5.3 ([39])**.** Let (f,P) be a fuzzy soft set defined over the universal set X and the parameter set $P\subset\wp$. τ be the sub family of $S_f(X,P)$. Then τ is called the fuzzy soft topology on (f,P) if the following conditions are satisfied:

1) $\phi,(f,P)\in\tau$;

2) $(f_1,P_1),(f_2,P_2)\in\tau \Longrightarrow (f_1,P_1)\cap(f_2,P_2)\in\tau$;

3) $\{(f_i,P_i):i\in I\}\subset\tau \Longrightarrow \bigcup_{i\in I}(f_i,P_i)\in\tau$.

The pair $(X_f,\tau)$is called the fuzzy soft topological space. Every member of τ is called open fuzzy soft set.

Theorem 5.4. *Let R be any arbitrary surjective fuzzy soft relation defined on fuzzy soft set (f,P) and if for $\lambda\in[0,1],R^\lambda$ is reflexive then $\tau=\{(g,P_g)\in S_f(X,P);$ $\underline{apr}(g,P_g)=(g,P_g)\}$is a fuzzy soft topology defined on (f,P).*

Proof. By Theorem 4.4 we have ϕ, $(f,P)\in\tau$.

Let $(g,P_g),(h,P_h)\in\tau \Longrightarrow \underline{apr}(g,P_g)=(g,P_g)$ and $\underline{apr}(h,P_h)=(h,P_h)$ $(g,P_g)\cap(h,P_h)=\underline{apr}(g,P_g)\cap\underline{apr}(h,P_h)=\underline{apr}((g,P_g)\cap(h,P_h)) \Longrightarrow (g,P_g)\cap(h,P_h)\in\tau$.

Now let $(g_i,P_i)\in\tau$, $\forall i\in I$. By Theorem 4.3, $\underline{apr}(\bigcup(g_i,P_i))\subseteq\bigcup(g_i,P_i)$. Conversely, since $\underline{apr}(g_i,P_i)=(g_i,P_i)$ we have $\bigcup(g_i,P_i)=\bigcup\underline{apr}(g_i,P_i)\subseteq\underline{apr}(\bigcup(g_i,P_i))$.

Hence $\tau=\{(g,P_g)\in S_f(X,P);\ \underline{apr}(g,P_g)=(g,P_g)\}$is a fuzzy soft topology defined on (f,P) and τ is called the fuzzy soft topology induced by the fuzzy soft relation R on (f,P).

6. Conclusion

In this paper we have proposed fuzzy soft relation on fuzzy soft sets. By means of relations on parameter set induced by a fuzzy soft relation, a pair of rough approximate operators are defined. Also we have investigated fuzzy soft topology generated by the rough approximate operators. As a future work with the motivation of ideas presented in this paper one can think of axiomatization of proposed rough approximate operators based on fuzzy soft relation and consider some applications of proposed notions.

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