Vertex (n, k)-choosability of graphs

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Abstract. Let $G = (V, E)$, connected, simple graph of order n and size m and let $V(G) = \{1, 2, ..., n\}$. A graph $G = (V, E)$ is said to be vertex (n, k)-choosable, if there exists a collection of subsets of the vertex set, $\{S_k(v) : v \in V\}$ of cardinality k, such that $S_k(u) \cap S_k(v) = \emptyset$ for all $uv \in E(G)$. This paper initiates a study on vertex (n, k)-choosable graphs and finds the different integer values of k, for which the given graph is vertex (n, k)-choosable.
Keywords: choosability, vertex (n, k)-choosability.

1. Introduction

Throughout this article, unless otherwise mentioned, by a graph we mean a connected, simple graph and any terms which are not mentioned here, the reader may refer to [8]. Let $G = (V, E)$, be a graph of order n and size m, where $V(G) = \{1, 2, ..., n\}$. Given a graph G, a *list assignment* L (or a *list coloring)* of G is a mapping that assigns to every vertex v of G, a finite list $L(v)$ of colors[12]. Also, G is said to be $\mathcal{L}$-*list colorable* if the vertices of G can be properly colored so that each vertex v is colored with a color from $\mathcal{L}(v)$.

Invoking the concept of list-assignments of graphs, the concept of $(a : b)$-choosability was defined and studied in [4].

Definition 1.1. A graph $G = (V, E)$ is $(a : b)$-choosable, if for every family of sets $\{S(v) : v \in V\}$ of cardinality a, there exist subsets $C(v) \subset S(v)$, where $|C(v)| = b$ for every $v \in V$, and $C(u) \cap C(v) = \emptyset$, whenever $u, v \in V$ are adjacent.

The k^{th} choice number of G, denoted by $ch_k(G)$, is the minimum integer n so that G is $(n : k)$-choosable. A graph $G = (V, E)$ is k-choosable if it is $(k : 1)$-choosable. The choice number of G, denoted by $ch(G)$, is equal to $ch_1(G)$. Following this, some interesting studies on choosability of graphs have been done (see [1, 5, 6]).

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Motivated by the studies on $(a:b)$-choosability of graphs, we initiate a study on the vertex (n,k)-choosable graphs, where n is the cardinality of the vertex set of G, and discuss the various parameter for the integer values of k.

2. Vertex (n,k)-choosability of graphs

Definition 2.1. A graph $G=(V,E)$ is said to be *vertex (n,k)-choosable*, if there exists a collection of subsets $\{S_k(v) : v \in V\}$ of $V(G)$ of cardinality k, such that $S_k(u) \cap S_k(v) = \emptyset$ for all $uv \in E(G)$.

Definition 2.2. The maximum value of k for which the given graph G is vertex (n,k)-choosable is called *vertex choice number of* G, and is denoted by $\mathcal{V}_{ch}(G)$.

Not all graphs admit vertex (n,k)-choosability for all values of k. A trivial bound for k is, $k \leq n-1$. One may verify that when $k=n-1$, the only vertex (n,k)-choosable graph is the trivial graph K_2. And, for $k=n-2$, the vertex (n,k)-choosable graph is isomorphic to P_3. However, every graph G of order n is vertex $(n,1)$-choosable. That is, the minimum value of k for which the given graph G is vertex (n,k)-choosable is $k=1$. Hence, finding the positive integer values of k, and also the maximum value of k, where $1 \leq k \leq n$, for which the graph G is vertex (n,k)-choosable is an interesting problem.

First, let us look at the vertex choice number of certain classes of graphs. The following observations are immediate.

Observation 2.3. The vertex choice number of a path P_n is $\lfloor \frac{n}{2} \rfloor$. That is, P_n is vertex (n,k)-choosable for all k, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Consider two disjoint k-element subsets of $V(P_n)$. Since, the path P_n is a bipartite graph, one k-element set can be assigned to all vertices in the first partition and other k-element set can be assigned to all vertices in the second partition. That is, by atleast two disjoint k-element sets, all vertices of P_n can be covered. Hence, the maximum value of k will be, $\lfloor \frac{n}{2} \rfloor$. Let $k > \lfloor \frac{n}{2} \rfloor$, say, $k = \lfloor \frac{n}{2} \rfloor + 1$. Take any subset V_1 of $V(P_n) = \{u_1, u_2, \ldots, u_n\}$, of cardinality $\lfloor \frac{n}{2} \rfloor + 1$. Let $u_1 \in P_n$ be assigned by this set of cardinality $\lfloor \frac{n}{2} \rfloor + 1$. . Then, for the second vertex u_2, we cannot find a subset V_2 of $V(P_n)$, of order $\lfloor \frac{n}{2} \rfloor + 1$, disjoint from V_1. That is, P_n is not vertex (n,k) choosable for $k = \lfloor \frac{n}{2} \rfloor + 1$. Hence, in general P_n is not vertex (n,k) choosable for any $k > \lfloor \frac{n}{2} \rfloor$.

Observation 2.4. The vertex choice number of the star graph S_n is $\lfloor \frac{n}{2} \rfloor$.

That is, for the star graph S_n, the vertex (n,k)-choosability is possible if there exists two disjoint k-element subsets of $V(S_n)$. Then, one k-element set should necessarily be assigned to the central node and the other k-element set should be assigned to all other nodes that are at a distance one from the central node. Therefore, S_n is vertex (n,k)-choosable for all $k \leq \lfloor \frac{n}{2} \rfloor$.

Proposition 2.5. *The complete graph K_n is vertex (n,k)-choosable if and only if $k=1$.*

Proof. Let $V(G) = \{1, 2, 3, \ldots, n\}$. Clearly there are n number of disjoint one element subsets of $V(G)$, and hence these one element subsets may be assigned to every vertex of K_n in a one-to-one manner. And hence, K_n is vertex $(n, 1)$-choosable. If $k \geq 2$, then the number of k-element subsets are less than n. Hence K_n is not vertex (n, k)-choosable, for $k \geq 2$. Also the vertex choice number of the complete graph K_n is 1, for all n. □

Theorem 2.6. *An even cycle C_n is vertex (n, k)-choosable if and only if $k \leq \frac{n}{2}$.*

Proof. Let $V(C_n) = \{1, 2, \ldots, n\}$, and n be even.

Consider, $f : V(C_n) \to \mathcal{P}(V(C_n)) - \emptyset$ defined by,

$$f(i) = \begin{cases} \{1, 2, \ldots, k\}, \text{if i is odd}, \\ \{k+1, k+2, \ldots, k+k\}, \text{if i is even}. \end{cases}$$

Then, for C_n to be vertex (n, k)-choosable, k should necessarily be such that, $1 \leq k \leq \frac{n}{2}$, if n is even.

Conversely, when n is even and $k > \frac{n}{2} + 1$, we reach a contradiction that whenever $ij \in E(C_n), f(i) \cap f(j) \neq \emptyset$.

□

Remark. An odd cycle C_n is vertex (n, k)-choosable if and only if $\leq \lfloor \frac{n}{2} \rfloor$.

Theorem 2.7. *A complete bipartite graph $K_{m,n}$ is vertex $(m+n, k)$-choosable, for $1 \leq k \leq \frac{m+n}{2}$, if and only if both m and n are simultaneously even or simultaneously odd.*

Proof. Without loss of generality, assume that both m and n are even. Let the vertex set of $K_{m,n}$ be V, where $V = A \cup B$ so that $|A| = m$ and $|B| = n$. Here, $|V| = m + n$. Now we have to find the values of k, for which $K_{m,n}$ is $(m + n, k)$-choosable.

Trivially there exists vertex $(m + n, 1)$ choosability, since there are $m + n$ disjoint one element subsets of V. Hence, $k \geq 1$.

Now, $V = A \cup B$, and every vertices in A is adjacent to all vertices in B. Also, there is no adjacency among the vertices in A and similarly, no two vertices in B are adjacent to each other. Let $A = \{1, 2, \ldots, m\}$ and $B = \{m + 1, m + 2, \ldots, m + n\}$.

Choose a k-element subset of V for a vertex in A. For example, let $f(1) = \{1, 2, \ldots, k\}$. Since i and j are not adjacent for all $i, j \in \{1, 2, \ldots, m\}$, it is possible to choose the same set for each vertex in A. That is, $f(i) = \{1, 2, \ldots, k\}$ for all i such that $1 \leq i \leq m$. Since, every vertices in A is adjacent to all other vertices in B, we cannot give the same set to any element in B. Hence, we need other k-element set. For this, let $f(m + i) = \{k + 1, k + 2, \ldots, k + k\}$ for all i such that $1 \leq i \leq n$. This is possible since, no two vertices in B are adjacent to each other. Hence, if there are two disjoint k element sets then vertex (n, k)-choosability is possible for $K_{m,n}$. Which gives $k \leq \frac{m+n}{2}$.

Now, suppose that $K_{m,n}$ is $(m+n, k)$-choosable for $1 \leq k \leq \frac{m+n}{2}$, and let m is odd and n is even. That is add $fracm + n2$ is not an integer. Hence, the complete bipartite graph $K_{m,n}$ is vertex $(m+n, k)$-choosable for $1 \leq k \leq \frac{m+n}{2}$, if and only if, both m and n are simultaneously even or simultaneously odd. □

Theorem 2.8. *A tree of order n is vertex (n, k)-choosable if and only if $k \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Let T be the given tree of order n and let $V(T) = \{v_1, v_2, \ldots, v_n\}$. Apply BFS algorithm to the given tree T, by choosing a vertex v_i with maximum degree as root. If there are more than one vertices $v_j \in V(T)$ of maximum degree, choose one such vertex arbitrarily. Without loss of generality, denote the chosen root vertex as v_1. Then, by the choice of v_1, there will be $|deg(v_1)|$ number of vertices in the first level. Define a function $f : V(T) \to \mathcal{P}(V(T)) - \emptyset$ by $f(v_1) = \{1, 2, \ldots, i\}$, $1 \leq i \leq \frac{n}{2}$. Let v_k^j denote any vertex in j^{th} level which is adjacent to vertices in the $(j-1)^{th}$ level. Hence, the vertex v_1 can be denoted by v_1^0.

Define

$$f(v_k^j) = \begin{cases} \{1, 2, \ldots, k\}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, & \text{if j is even,} \\ \{k+1, k+2, \ldots, k+k\}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, & \text{if j is odd.} \end{cases}$$

With this labeling the tree T admits vertex (n, k)-choosability $\forall k$ such that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Conversely, it is sufficient to prove that if $k > \lfloor \frac{n}{2} \rfloor$, then the tree T is not vertex (n, k)-choosable.

If possible, let $k = \lfloor \frac{n}{2} \rfloor + 1$ Let $f(v_1) = \left\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor + 1\right\}$.

Let v_m^1 be any vertex in the first level adjacent to the root vertex v_1. Then, we should necessarily have,

$$f(v_m^1) = \left\{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \ldots, 2(\lfloor \frac{n}{2} \rfloor + 1)\right\}.$$

Clearly, $|f(v_m^1)| < \lfloor \frac{n}{2} \rfloor + 1$, a contradiction. Hence, $k \leq \lfloor \frac{n}{2} \rfloor$. □

Theorem 2.9. *The complete r-partite graph $K(m_1, m_2, \ldots, m_r)$ is vertex $(m_1 + m_2 + \cdots + m_r, k)$-choosable for, $1 \leq k \leq \lfloor \frac{m_1+m_2+\cdots+m_r}{r} \rfloor$.*

Proof. Denote the given complete r-partite graph $K_{m_1, m_2, \ldots, m_r}$ by G. Here, $|S(V)| = m_1 + m_2 + \cdots + m_r$. Now, let $V(G) = A_1 \cup A_2 \cup \cdots \cup A_r$. Since, every vertex in A_i is adjacent to all other vertices in A_j, for all $i \neq j$. Hence, atleast r k-element sets are needed. First choose a k-element set for the first set A_1. Since there is no adjacency between any pair of vertices in A_1, the same set can be choosen for all vertices in A_i. similarlly for each A_i, this method can be followed. That is, only r k-element sets are needed to cover all the vertices in G. Hence, $k \leq \lfloor \frac{m_1+m_2+\cdots+m_r}{r} \rfloor$. □

Theorem 2.10. *Any unicyclic graph G of order n with the unique cycle C_p is vertex (n,k)-choosable if and only if $k \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Let G be a unicyclic graph of order n with the unique cycle C_p. Suppose that p is even. By theorem 2.4, an even cycle C_n is vertex (n,k)-choosable if and only if, $k \leq \frac{n}{2}$. Hence, the cycle C_p alone is vertex (n,k)-choosable in G, for $k \leq \lfloor \frac{n}{2} \rfloor$. We note that $G - C_p$ is a forest. Consider the components of $G - C_p$.

For the vertex (n,k)-choosabiliy of trees, we need two distinct k-element subsets. We can choose the same sets that are assigned for the vertices in the cycle, for the vertices in the tree also. For this, let $\{1, 2, \ldots, p\}$ be the vertex set of C_p and $\left\{ j_1, j_2, \ldots, j_{p_j} \right\}$ be the vertex set of the tree with the root vertex j in the cycle. Also, we have $p + p_1 + p_2 + ... + p_p = n$. Then, if there are two distinct k-element sets, then the vertex (n,k)-choosabiliy of the cycle C_p is given by,

$$f(i) = \begin{cases} \{1, 2, \ldots, k\}, \text{if i is odd,} \\ \{k+1, k+2, \ldots, k+k\}, \text{if i is even,} \end{cases}$$

where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Then, in the tree if j is odd, then for j_1, we can choose the set assigned for even vertices in the cycle C_p. By applying BFS algorithm, we can seen that, by two distinct k element sets, we can cover all the vertices in the tree. Hence, the unicyclic graph G of order n is vertex (n,k)-choosable for $k \leq \lfloor \frac{n}{2} \rfloor$, if the unique cycle C_p is even. Now, let p be odd. That is C_p is an odd cycle.

We have an odd C_n is vertex (n,k)-choosable if and only if $k \leq \lfloor \frac{n}{2} \rfloor$. First label the vertices of C_p by $\lfloor \frac{n}{2} \rfloor$ element subsets of $V(G)$. Next, consider the remaining vertices in the tree. If the root vertex of the tree is an even(odd) vertex in the cycle, then for the next vertex in the tree, we can choose the set assigned for the neighbouring odd (even) vertices in the cycle. Using these two sets all vertices in the tree can be labelled. In a similar manner all the trees attached with the vertices of C_p can be labelled.

Hence, the unicyclic graph G of order n with the unique cycle C_p is vertex (n,k)-choosable if and only if $k \leq \lfloor \frac{n}{2} \rfloor$. This completes the proof. □

Theorem 2.11. *A graph G is vertex (n,k)-choosable, if it does not contain a complete subgraph K_m of order $m \geq \lfloor \frac{n}{k} \rfloor + 1$.*

Proof. Let $G = (V, E)$ be a vertex (n,k)-choosable graph. Suppose that G contains a complete subgraph of order $m = \lfloor \frac{n}{k} \rfloor + 1$. Since, G is vertex (n,k)-choosable, every vertex of G can choose a set of k elements. Let $V(K_m) = \{1, 2, \ldots, m\}$. Now, define the function $f : V(G) \to \mathcal{P}(V(G)) - \emptyset$. Consider the vertex 1 in the complete graph, and let $f(1) = \{1, 2, \ldots, k\}$. Since, 1 is adjacent to all the remaining vertices i, where $i = 2, 3, \ldots, m$ in K_m, they cannot chose the same set $f(1)$. That is, for all vertices in the complete graph K_m we need disjoint k element sets. Hence, atleast m disjoint k-element sets are needed to cover all the vertices in K_m. Since G is a connected graph and K_m is a complete subgraph of G, atleast one vertex in K_m will be adjacent to a vertex not in K_m. Hence we have, $mk < n$. This implies $m < \frac{n}{k}$. That is, $m = \lfloor \frac{n}{k} \rfloor + 1 < \frac{n}{k}$, which is a contradiction. Hence, a graph G is vertex (n, k)-choosable, if it does not contain a complete subgraph of order $m \geq \lfloor \frac{n}{k} \rfloor + 1$. □

3. Conclusion

In this paper, we introduced a new concept namely, vertex (n, k)-choosability of graph. We also discussed the vertex (n, k)-choosability of certain fundamental graph classes. There is a wide scope for further investigation on the vertex (n, k)-choosability of many other graph classes, graph operations and graph products. The edge (m, k)-choosability is another interesting area for further investigation.

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