On (m, n)-fully stable Banach algebra modules

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Abstract. In this paper the concept of fully-(m, n) stable Banach Algebra-module ($F-(m, n)-S-B-A$-module), we study some properties of $F-(m, n)-S-B-A$-module and another characterization have been given.

Keywords: fully stable Banach A-module, fully (m, n)-stable Banach A - module, multiplication $(m, n)-A$-module.

1. Introduction

A non-empty set A is an algebra if, $(A, +, .)$ is a vector space over a field F, $(A, +, \circ)$ is a ring and $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$ for every $\alpha \in F$, for every $a, b \in A$" [1]. In [2]" a ring R is an algebra $\langle R, +, \cdot, -, 0\rangle$ where $+$ and $\cdot$ are two binary operations, $-$ is unary and 0 is nullary element satisfying, $\langle R, +, -, 0\rangle$ is an abelian group, $\langle R, .\rangle$ is a semigroup and $x.(y + z) = (x.y) + (x.z)$ and $(x + y).z = (x.z) + (y + z)$". "Let A be an algebra, recall that a Banach space E is a Banach left A-module ($B - A$ - module) if E is a left A-module, and $\|a.x\| \leq \|a\| \, \|x\| (a \in A, \, x \in E)$" [1]. Following [3] "a map from a left $B - A$-module X into a left Banach A-module Y (A is not necessarily commutative) is said a multiplier (homomorphism) if it satisfies $T(a.x) = a.Tx$ for all $a \in A$, $x \in X$ ". In [4], "a submodule N of an R-module M is said to be stable, if $f(N) \subseteq N$ for each R-homomorphism $f : N \longrightarrow M$. M is called a fully stable module, each submodule of M is stable". "A Banach algebra module M is called $F - S - B - A$-module if for every submodule N of M and for each multiplier $\theta : N \longrightarrow M$ such that $\theta(N) \subseteq N$" [5]. We use the notation $R^{m\times n}$ for the set of all $m \times n$ matrices over R. For $A \in R^{m\times n}$, A^T will denote the transpose of A. In general, for an R-module N, we write $N^{m\times n}$ for the set of

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all formal $m \times n$ matrices whose entries are elements of N. Let M be a right Banach Algebra-module and N be a left R-module. For $x \in M^{l\times m}$, $s \in R^{m\times n}$ and $y \in M^{n\times k}$, under the usual multiplication of matrices, xs (resp. sy) is a well defined element in $M^{l\times m}$ (resp. $N^{n\times k}$). "If $X \subseteq M^{l\times m}$, $S \subseteq R^{m\times n}$ and $Y \subseteq N^{n\times k}$ define

$$\begin{aligned} \ell_{M^{l\times m}}(S) &= \left\{u \in M^{l\times m} \mid us = 0; \forall s \in S\right\}, \\ r_{N^{n\times k}}(S) &= \left\{v \in N^{n\times k} \mid sv = 0; \forall s \in S\right\}, \\ \ell_{R^{m\times n}}(Y) &= \left\{s \in R^{m\times n} \mid sy = 0; \forall y \in Y\right\}, \\ r_{R^{m\times n}}(X) &= \left\{s \in R^{m\times n} \mid xs = 0; \forall x \in X\right\}. \end{aligned}$$

We will write $N^n = N^{1\times n}$, $N_n = N^{n\times 1}$ [6]. In this paper for two fixed positive integers n, m the concept of fully (m, n)-stable Banach algebra modules has been introduced.

2. Fully (m,n)-stable Banach algebra modules

"A left $B - A$-module X is n-generated for $n \in N$ if there exists $x_1, \ldots, x_n \in X$ such that each $x \in X$ can represented as $x = \sum_{k=1}^{n} a_k.x_k$ for some $a_1, \ldots, a_n \in A$. A module which is 1-generated is called a cyclic module" [7] .

Definition 2.1. Let K be $B - A$-module, K is called (m, n)-fully stable B-A-module, if for every n-generated submodule L of K^m and for each multiplier $\theta : L \to K^m$ satisfy $\theta(L) \subseteq L$, for two fixed positive integers n, m.

In [5] "for a nonempty subset M in a left $B - A$ - module X, the annihilater $ann_A(M)$ of M is $ann_A(M) = \{a \in A|a.x = 0\,\forall x \in M\}$".

Notation. Let X be a $B - A$-module

1. $L_{x_1,x_2,\ldots,x_n} = \{\oplus l_{x_i}|n \in N, x_i \in X, i = 1, 2, \ldots, n\}$,
   $K_{y_1,y_2,\ldots,y_n} = \{\oplus k_{y_i}|k \in K, y_i \in X, i = 1, 2, \ldots, n\}$,
2. $\ell_{A^{m\times n}} L_{x_1,x_2,\ldots,x_n} = \{a \in A^{m\times n}, a.(\oplus l_{x_i}) = 0, \forall \oplus l_{x_i} \in L_{x_1,x_2,\ldots,x_n}\}$,
   $\ell_{A^{m\times n}} K_{y_1,y_2,\ldots,y_n} = \{a \in A^{m\times n}, a.(\oplus k_{y_i}) = 0, \forall k_{y_i} \in K_{y_1,y_2,\ldots,y_n}\}$.

Proposition 2.2. *A $B - A$-module M is fully-(m, n) stable, if and only if any two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ of M_n, if $\beta_j \notin \sum_{i=1}^{n} A\alpha_i$, for each $j = 1, \ldots, m$ implies $\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}) \nsubseteq \ell_{A^n}(\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\})$.*

Proof. Assume that K is $F-(m, n)-S-B-A$-module and there exist two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ of M_n such that if $K_{y_j} \notin \sum_{i=1}^{n} A\alpha_i$, for each $j = 1, \ldots, m$ and

$$\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}) \subseteq \ell_{A^n}(\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}).$$

Define $f:\sum_{i=1}^n \alpha_i A \longrightarrow M^m$ by $f(\sum_{i=1}^n a_i L_{x_i}) = \sum_{i=1}^n a_i K_{y_i}$.

Let $L_{x_i} = (k_{1i}, k_{2i}, \ldots, k_{ni})$. If $\sum_{i=1}^n a_i L_{x_i} = 0$, then $\sum_{i=1}^n a_i k_{ij} = 0, j = 1, 2, \ldots, m$, implies that $rL_{x_j} = 0$ where $r = (r_1, \ldots, r_n)$ and hence $r \in \ell_{A^n}\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$. By assumption $rK_{y_j} = 0, j = 1, \ldots, m$ so $\sum_{i=1}^n r_i K_{y_i} = 0$. This shows that f is well defined. It is an easy matter to see that f is multiplier. Fully-(m,n) stability of M implies that there exists $t = (t_1, \ldots, t_n) \in A^n$ such that $f(\sum_{i=1}^n r_i L_{x_i}) = \sum_{k=1}^n t_k(\sum_{i=1}^n r_i L_{x_i}) = \sum_{k=1}^n \sum_{i=1}^n (t_k r_i) L_{x_i}$ for each $\sum_{i=1}^n r_i L_{x_i} \in \sum_{i=1}^n AL_{x_i}$.

Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 in the i-th position and 0 otherwise. $K_{y_i} = f(L_{x_i}) = \sum_{k=1}^n t_k L_{x_i} \in \sum_{i=1}^n AL_{x_i}$, which is contradiction. Conversely assume that there exists n-generated $B-A$-submodule of M^m and multiplier $\mu : \sum_{i=1} AL_{x_i} \to M^m$ such that $\mu(\sum_{i=1}^n AL_{x_i}) \nsubseteq \sum_{i=1}^n AL_{x_i}$. Then there exists an element $\beta(= \sum_{i=1}^n r_i L_{x_i}) \in \sum_{i=1}^n AL_{x_i}$ such that $\mu(K_y) \notin \sum_{i=1}^n AL_{x_i}$. Take $K_{y_i} = K_y$, $j = 1, \ldots, m$, then we have m-element subset $\{\mu(K_y), \ldots, \mu(K_y)\}$, such that $\mu(K_y) \notin \sum_{i=1}^n AL_{x_i}$, $j = 1, \ldots, m$. Let $\eta = (t_1, \ldots, t_n) \in \ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\})$, then $\eta\alpha_j = 0$, i.e $\sum_{i=1}^n t_i a_{ij} = 0$, for each $j = 1, \ldots, m$, $L_{x_j} = (a_{1j}, a_{2j}, \ldots, a_{nj})$ and $\{\mu(K_y), \ldots, \mu(K_y)\}\eta = \sum_{k=1}^n t_k \mu(K_y) = \sum_{k=1}^n t_k \mu(\sum_{i=1}^n r_i L_{x_i}) = \sum_{k=1}^n \mu(\sum_{i=1}^n t_k r_i L_{x_i}) = 0$ hence $\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}) \subseteq \ell_{A^n}(\{\mu(K_y), \ldots, \mu(K_y)\})$, thus $\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}) \subseteq \ell_{A^n}(\{\mu(K_{y_1}), \ldots, \mu(K_{y_1,y_2,\ldots,y_m})\}$ which is a contradiction. Thus M is $F-(m,n)-S-B-A$-module. □

Corollary 2.3. *Let M be an $F-(m,n)-S-B-A$-module, then for any two m-element subsets $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}$ and $\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\}$ of M_n, $\ell_{A^n}(\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m}\}) \subseteq \ell_{A^n}(\{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m}\})$ implies that $AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_m} = AK_{y_1} + AK_{y_1,y_2} + AK_{y_1,y_2,\ldots,y_m}$.*

In [9], "$AB-A$ - module X is said to satisfy Baer criterion if each submodule of X satisfies Baer criterion, that is for every submodule N of X and A-multiplier $\theta : N \to X$, there exists an element a in A such that $\theta(n) = an$ for all $n \in N$".

Definition 2.4. A $B-A$ - module X is said to satisfy Baer (m,n)-criterion if each submodule of X satisfies Baer (m,n)-criterion, that is for every n-generated submodule L of X and A- multiplier $\theta : L \to X^m$, there exists an element a in A such that $\theta(l) = al$ for all $l \in L$".

Proposition 2.5. *If X satisfies Baer $(m,1)$-criterion and $\ell_A(L \cap M) = \ell_A(L) + \ell_A(M)$ for each n-generated submodules of X^m, then X satisfies Baer (m,n)-criterion.*

Proof. Let $P = Ax_1 + Ax_2 + \ldots + Ax_n$ be an n-generated submodule of X^m and $f : P \to X^m$ a multiplier. We use induction on n. It is clear that M satisfies Bear (m,n) - criterion, if $n = 1$. Suppose that M satisfies Bear (m,n)-criterion for all k-generated submodule of X^m, for $k \leq n-1$. Write $L = Ax_1$, $M = Ax_2 + \ldots + Ax_n$, then for each $w_1 \in L$ and $w_2 \in M$, $f|_L(w_1) = y_1 w_1$,

$f|_M(w_2) = y_2w_2$ for some $y_1, y_2 \in A$. It is clear $y_1 - y_2 \in \ell_A(L \cap M) = \ell_A(L) + l_A(M)$. Suppose that $y_1 - y_2 = z_1 + z_2$ with $z_1 \in \ell_A(L), z_2 \in \ell_A(M)$ and let $y = y_1 - z_1 = y_2 + z_2$. Then for any $w = w_1 + w_2 \in P$ with $w_1 \in L$ and $w_2 \in M$, $f(w) = f(w_1) + f(w_2) = w_1y_1 + w_2y_2 = w_1(y - 1 - z_1) + w_2(y_2 + z_2) = w_1y + w_2y = (w_1 + w_2)y = wy$. □

Proposition 2.6. *Let X be a $B - A$ - module. Then X satisfies Baer (m, n) criterion if and only if $r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$ for n-element subset $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ of X_n.*

Proof. Suppose that Baer (m, n)-criterion holds for n-generated submodule of X^m let $L_{x_i} = (k_{i1}; k_{i2}, \ldots, k_{im})$, for each $i = 1, \ldots, n$ and $K_y = \{K_{y_1}, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_n}\} \in r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n})$, $K_{y_i} = (a_{1i}, a_{2i}, \ldots, a_{ni})$.

Define $\mu : AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n} \to X_m$ by $\mu(\sum_{i=1}^n a_iL_{x_i}) = \sum_{i=1}^n a_iK_{y_i}$. If $\sum_{i=1}^n a_iL_{x_i}$, then $\sum_{i=1}^n a_ik_{ij} = 0$. $j = 1, \ldots, m$, this implies that $rL_{x_i} = 0$ where $r = (r_1, \ldots, r_n)$ and hence $r \in \ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n})$. By assumption $rL_{x_i} = 0$, $i = 1, \ldots, n$ so $\sum_{i=1}^n a_iK_{y_i} = 0$. This show that f is well defined. It is an easy matter to see that μ is an multiplier. By assumption there exists $t \in A$ such that $\mu(\sum_{i=1}^n a_iL_{x_i}) = t(\sum_{i=1}^n a_iK_{y_i}) = \sum_{i=1}^n (ta_i)K_{y_i}$ for each $\sum_{i=1}^n a_iL_{x_i} \in \sum_{i=1}^n AL_{x_i}$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 in the i-th position and 0 otherwise. $K_{y_i} = \mu(\sum_{i=1}^n L_{x_i}) = \sum_{i=1}^n tL_{x_i} \in \sum_{i=1}^n AL_{x_i}$ which is contradiction. This implies that $r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) \subseteq AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, the other inclusion is trivial.

Conversely, assume that $r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, for each $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ in X_n.

Then for each multiplier $f : AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n} \longrightarrow X_m$ and $s = (s_1, \ldots, s_n) \in \ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n})$, $\sum_{k=1}^n s_k(\sum_{i=1}^n t_iL_{x_i}) = 0$, for each $\sum_{i=1}^n t_iL_{x_i} \in \sum_{i=1}^n AL_{x_i}$, hence

$$\sum_{k=1}^n s_kf(\sum_{i=1}^n t_iL_{x_i}) = \sum_{k=1}^n f(\sum_{i=1}^n s_kt_iL_{x_i}) = 0,$$

thus $f(\sum_{i=1}^n t_iL_{x_i}) \in r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, for some $t \in A$. Then X satisfies Baer (m, n)-criterion. □

Corollary 2.7. *Let X be a $B - A$ - module. Then X is $F - (m, n) - S - B - A$ - module if and only if $r_{X_n}\ell_{A^n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$ for n-element subset $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ of X_n*

Following [8], let A be a unital Banach algebra and let $\alpha > 1$. A-module X is called quasi α-injective if, $\varphi : N \to X$ is A-module homomorphisms such that

$\|\varphi\| \leq 1$, there exists A-module homomorphism $\theta : X \to X$, such that $\theta \circ i = \varphi$ and $\|\theta\| \leq \alpha$ where i is an isometry from submodule N of X. We shall say that X is quasi injective if it is quasi α - injective for some α".

The concepts quasi $(m,n) - \alpha$ - injective for some α and multiplication $(m,n) - B - A$ - module has been introduced.

Definition 2.8. Let A be a unital Banach algebra and let $\alpha > 1$. A-module X is called quasi $(m,n) - \alpha$ - injective if,$\varphi : N \longrightarrow X$ is A - module homomorphisms such that $\|\varphi\| \leq 1$, there exists A - module homomorphism $\theta : X \longrightarrow X$, such that $\theta \circ i = \varphi$ and $\|\theta\| \leq \alpha$ where i is an isometry from n - generated submodule N of X. We shall say that X is quasi (m,n) - injective if it is quasi $(m,n) - \alpha$ - injective for some α.

Definition 2.9. $B - A$-module X is called multiplication $(m,n) - A$ - module if each n - generated submodule of X is of the form KX_n for some ideal K of $A^{m\times n}$.

Proposition 2.10. *Let X be multiplication $(m,n) - B - A$ - module. If X is quasi $(m,n) - B - A$ -module then X is $F - (m,n) - S - B - A$-module.*

Proof. Let N be n-generated submodule of X, let $\alpha > 1$ and f be any A-module homomorphism from N to X^m such that $\|f\| \leq 1$. Since X is multiplication $(m,n) - B - A$-module, then $N = KX_n$, and since X is quasi $(m,n) - B - A$ - module, then there exist A-module homomorphism $g : X^m \longrightarrow X^m$ such that $f(N) = g(N) = g(KX_n) = Kg(X_n) \subseteq KX_n = N$. □

References

[1] S. Petrakis, *Introduction to Banach algebras and the Gelfand-Naimark theorems*, special subject II and III Aristotle, University of Thessaloniki Department of Mathemayics, 2008.

[2] G. Ramesh, *Banach algebras*, Department of Mathematics, I. I. T. Hyderabad, ODF Estate, Yeddumailaram, A. P, India 502205, 2013.

[3] J. Bracic, *Simple multipliers on Banach modules*, University of Ljubljana, Slovenia, Glasgow Mathematical Journal Trust, 2003.

[4] M.S. Abbas, *On fully stable modules,* Ph. D. Thesis, University of Baghdad, Iraq, 1990.

[5] Samira Naji Kadhim, *On fully stable Banach algebra modules and fully pseudo stable Banach algebra modules*, Baghdad Science Journal, 15 (2018).

[6] M.S. Abbas, Ali M. Mohammed, *A note on fully (m,n)-stable modules,* International Electronic Journal of Algebra, 6 (2009), 65-73,

[7] J. Bracic, *Local operators on Banach modules,* University of Ljubljana, Slovenia, Mathematical Proceedings of the Royal Irish Academy, 2004.

[8] Z.M. Zhu, J.L. Chen, X.X. Zhang, *On* (m, n)*-quasi-injective modules,* Acta Math. Univ. Comenianae Vol. LXXIV, 1 (2005), 25-36.

[9] Ali M.J. Mohammed, M. Ali, *Fully stable Banach algebra module*, Mathematical Theory and Modeling, 6 (2016), 136-139.

Accepted: 15.04.2019