Criteria and geometric properties for bounded univalent functions in the unit disk

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Abstract. In this article, we establish new univalence criteria for normalized analytic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z)/z \neq 0$ in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. Indeed, we prove for any $n \geq 2$ that the condition $|(f(z)/z)^{(n)}| \leq (n!/(n+1))(1 - \sum_{k=2}^{n} k|a_k|)$ is sufficient and sharp for f to be univalent in $\mathbb{U}$. The equality attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$, where $\sum_{k=2}^{n} k|a_k| = 1$. We investigate interesting geometric properties for such classes of functions. Namely, subordinations, inclusions, distortion and growth theorems, area estimate, starlikeness and convexity.
Keywords: univalent functions, univalence criteria, subordination, starlike functions, convex functions, area theorem, distortion theorem.

1. Introduction and preliminaries

It is well known that the condition $\mathrm{Re} f'(z) > 0$ is sufficient for analytic function f to be univalent (one-to-one) in any convex domain. In 1962, MacGregor [1] investigated such functions in the unit disk $\mathbb{U} = \{z : \ |z| < 1\}$, whenever f is normalized by $f(0) = f'(0) - 1 = 0$. In fact, the class of normalized analytic functions, denoted by $\mathcal{A}$, is analytically characterized by functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Afterwards, in [2, 3], the authors studied the subclass

$$F = \{f \in \mathcal{A} : \ |f'(z) - 1| < 1, \ z \in \mathbb{U}\}$$

of the class $\mathcal{R} = \{f \in \mathcal{A} : \mathrm{Re}\, f'(z) > 0, \ z \in \mathbb{U}\}$, for various geometric properties. A function $f \in \mathcal{A}$ is said to be starlike in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

Also, $f \in \mathcal{A}$ is called convex in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

The classes of starlike functions and convex functions are denoted, respectively, by $\mathcal{S}^*$ and $\mathcal{K}$. A function f is said to be subordinate to g, written $f \prec g$, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$, for $z \in \mathbb{U}$. The problems of finding criteria and investigating geometric properties for univalent functions has been extensively studied by many authors, see for example [4-11].

In this article, for every $n \geq 2$, we introduce the classes

$$F_n = \left\{ f \in \mathcal{A} : \left| \left(\frac{f(z)}{z} \right)^{(n)} \right| \leq \frac{n!}{n+1} \left(1 - \sum_{k=2}^{n} k|a_k| \right), \quad \frac{f(z)}{z} \neq 0, \;\; z \in \mathbb{U} \right\}$$

of univalent functions in $\mathbb{U}$. Indeed, we prove that F_n is included in the class F, for every $n \geq 2$. As a motivation of our univalence criteria is that, for $n = 2$ and $f(z)/z \neq 0$ in $\mathbb{U}$, the condition

$$\sum_{k=3}^{\infty} (k-1)(k-2)|a_k| \leq \frac{2}{3}(1 - 2|a_2|) \tag{1.1}$$

is sufficient for $f \in \mathcal{A}$ to be in F_2 and hence it is univalent in $\mathbb{U}$. Also, the condition

$$\left| \left(\frac{f(z)}{z} \right)'' \right| \leq \frac{1}{5} \left(-7|a_2| + \sqrt{10 - |a_2|^2} \right), \quad (z \in \mathbb{U})$$

is sufficient for $f \in \mathcal{A}$ to be starlike in $\mathbb{U}$, where equality attained for $f(z) = z + a_2 z^2$, $(|a_2| = 1/\sqrt{5})$. Moreover, the condition

$$\left| \left(\frac{f(z)}{z} \right)'' \right| \leq \frac{2}{11}(1 - 4|a_2|), \quad (z \in \mathbb{U})$$

is sufficient for $f \in \mathcal{A}$ to be convex in $\mathbb{U}$, where equality attained for $f(z) = z + a_2 z^2$, $(|a_2| = 1/4)$. Further, we investigate several geometric properties for the classes F_n. Mainly, inclusions, subordinations, distortion and growth theorems, area estimate, starlikeness and convexity. As consequences from the subordination results, we obtain univalence criteria in terms of bounded derivatives of f.

The following lemmas are needed in the sequel.

Lemma 1.1 ([12]). *If $\omega(z)$ is analytic in $\mathbb{U}$ and $|\omega(z)| \leq 1$ in $\mathbb{U}$, then for each $m \geq 1$, the function $\Phi_m(z)$ defined by*

$$\Phi_m(z) = \int_0^z m u^{m-1} \omega(u) du = z^m \int_0^1 m t^{m-1} \omega(tz) dt = z^m \Psi_m(z)$$

is clearly analytic in $\mathbb{U}$ and moreover, $\Psi_m(z)$ is analytic in $\mathbb{U}$ such that $|\Psi_m(z)| \leq 1$ in $\mathbb{U}$.

Lemma 1.2 ([13])**.** *Let g be a convex function in $\mathbb{U}$, and let $h(z) = g(z) + k\alpha z g'(z)$ for $z \in \mathbb{U}$, where $\alpha > 0$ and k is a positive integer. If $p(z) = g(0) + p_k z^k + p_{k+1} z^{k+1} + \cdots$, $z \in \mathbb{U}$, is holomorphic in $\mathbb{U}$ and $p(z) + \alpha z p'(z) \prec h(z)$, $z \in \mathbb{U}$, then $p(z) \prec g(z)$, $z \in \mathbb{U}$, and this result is sharp.*

Lemma 1.3 ([14, p.7])**.** *For analytic functions $g(z) = \sum_{k=0}^{\infty} b_k z^k$ in $\mathbb{U}$ with $|g(z)| \le 1$, we have*

$$\sum_{k=0}^{\infty} |b_k|^2 \le 1.$$

Lemma 1.4. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then, for every $n \ge 2$, we have*

$$f^{(n)}(z) = z \left(\frac{f(z)}{z}\right)^{(n)} + n \left(\frac{f(z)}{z}\right)^{(n-1)}. \tag{1.2}$$

Proof. From the following expansions

$$f^{(n)}(z) = n!\, a_n + \sum_{k=n+1}^{\infty} \frac{k!}{(k-n)!} a_k\, z^{k-n},$$

$$\left(\frac{f(z)}{z}\right)^{(n)} = \sum_{k=n+1}^{\infty} \frac{(k-1)!}{(k-n-1)!} a_k\, z^{k-n-1},$$

and

$$\left(\frac{f(z)}{z}\right)^{(n-1)} = (n-1)!\, a_n + \sum_{k=n+1}^{\infty} \frac{(k-1)!}{(k-n)!} a_k\, z^{k-n}.$$

It can be easily verified that (1.2) holds. □

2. Univalence and inclusions

Let us start by proving that $F_n \subseteq F$, for every $n \ge 2$.

Theorem 2.1 (Univalence Criteria)**.** *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z)/z \ne 0$ in $\mathbb{U}$ and let*

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \le \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k|a_k|\right), \quad (z \in \mathbb{U}). \tag{2.1}$$

for some $n \ge 2$. Then $f \in F$ and hence it is univalent in $\mathbb{U}$. The inequality (2.1) is sharp, where equality attained for functions of the form

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k, \;\; \textit{where} \; \sum_{k=2}^{n} k|a_k| = 1 \tag{2.2}$$

and for $f(z) = z \pm (1/(n+1)) z^{n+1}$.

Proof. Let

$$\beta_n = \frac{n!}{n+1}\left(1-\sum_{k=2}^{n} k|a_k|\right).$$

Then, for $n=2$, condition (2.1) is equivalent to

$$\left|\left(\frac{f(z)}{z}\right)''\right| \le \beta_2$$

and

$$\left(\frac{f(z)}{z}\right)'' = \beta_2\,\phi_1(z), \tag{2.3}$$

where ϕ_1 is analytic in $\mathbb{U}$ and $|\phi_1(z)| \le 1$ in $\mathbb{U}$. Now, by the virtue of

$$\left(\frac{f(z)}{z}\right)^{(k)}|_{z=0} = k!\,a_{k+1}$$

and using Lemma 1.1, then integrating (2.3) from 0 to z yields

$$\left(\frac{f(z)}{z}\right)' = a_2 + \beta_2 z\int_0^1 \phi_1(tz)\,dt := a_2 + \beta_2 z\phi_2(z). \tag{2.4}$$

The relation (2.4), by integration and multiplying by z, gives

$$f(z) - z = a_2 z^2 + \beta_2 z\int_0^z u\,\phi_2(u)\,du. \tag{2.5}$$

By differentiating both sides of (2.5), we have

$$\begin{aligned} f'(z)-1 &= 2a_2 z + \beta_2\left(z^2\phi_2(z) + \int_0^z u\,\phi_2(u)\,du\right) \\ &= 2a_2 z + \beta_2\left(z^2\phi_2(z) + \frac{1}{2}z^2\int_0^1 2t\,\phi_2(tz)\,dt\right). \end{aligned}$$

Therefore,

$$\left|f'(z)-1\right| < 2|a_2| + \frac{3}{2}\beta_2 = 1,$$

and hence $f \in F$. For $n=3$, we have

$$\left(\frac{f(z)}{z}\right)''' = \beta_3\psi_1(z), \tag{2.6}$$

where ψ_1 is analytic in $\mathbb{U}$ and $|\psi_1(z)| \le 1$ in $\mathbb{U}$. By integration (2.6) from 0 to z, we have

$$\left(\frac{f(z)}{z}\right)'' = 2a_3 + \beta_3 z\int_0^1 \psi_1(tz)\,dt := 2a_3 + \beta_3 z\,\psi_2(z). \tag{2.7}$$

Again by integration (2.7), we get

$$\left(\frac{f(z)}{z}\right)' = a_2 + 2a_3 z + \beta_3 \frac{z^2}{2}\int_0^1 2t\psi_2(tz)\,dt := a_2 + 2a_3 z + \frac{\beta_3}{2} z^2 \psi_3(z). \tag{2.8}$$

Integration (2.8) and then multiplying by z, gives

$$f(z) - z = a_2 z^2 + a_3 z^3 + \frac{\beta_3}{2} z \int_0^z u^2 \psi_3(u)\,du. \tag{2.9}$$

By differentiating both sides of (2.9) and using Lemma 1.1, we have

$$\begin{aligned} f'(z) - 1 &= 2a_2 z + 3a_3 z^2 + \frac{\beta_3}{2}\left(z^3\psi_3(z) + \int_0^z u^2\psi_3(u)\,du\right) \\ &= 2a_2 z + 3a_3 z^2 + \frac{\beta_3}{2}\left(z^3\psi_3(z) + \frac{1}{3} z^3 \int_0^1 3t^2 \psi_3(tz)\,dt\right). \end{aligned}$$

Therefore,

$$\left|f'(z) - 1\right| < 2|a_2| + 3|a_3| + \frac{2}{3}\beta_3 = 1,$$

and hence $f \in F$. In general, if

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \le \beta_n, \quad (z \in \mathbb{U}),$$

then

$$\begin{aligned} f'(z) - 1 &= \sum_{k=2}^{n} k a_k z^{k-1} \\ &\quad + \frac{1}{(n-1)!}\beta_n \left(z^n \varphi_n(z) + \frac{z^n}{n}\int_0^1 n t^{n-1} \varphi_n(tz)\,dt\right), \end{aligned} \tag{2.10}$$

where φ_n is analytic in $\mathbb{U}$ and $|\varphi_n(z)| \le 1$ in $\mathbb{U}$ by Lemma 1.1. Therefore,

$$\left|f'(z) - 1\right| < \sum_{k=2}^{n} k|a_k| + \frac{1}{(n-1)!}\left(1 + \frac{1}{n}\right)\beta_n = 1,$$

and hence $f \in F$. To show that the result is sharp for $n \ge 2$, we consider

$$f_\epsilon(z) = z + \frac{1+\epsilon}{n+1} z^{n+1}, \quad (\epsilon \ge 0). \tag{2.11}$$

Clearly, $a_k = 0$ for $2 \le k \le n$. A computation shows that

$$\left(\frac{f_\epsilon(z)}{z}\right)^{(k)} = \frac{(1+\epsilon)n(n-1)\cdots(n-k+1)}{n+1} z^{n-k}, \ (2 \le k \le n).$$

Therefore,

$$\left(\frac{f_\epsilon(z)}{z}\right)^{(n)} = \frac{n!}{n+1}(1+\epsilon). \tag{2.12}$$

Letting $\epsilon = 0$ in (2.11) and (2.12) implies that $f_0(z)$ satisfies the equality in (2.1). However, for every $\epsilon > 0$ and $n \geq 2$, we have

$$f'_\epsilon\left(\left(\frac{-1}{1+\epsilon}\right)^{\frac{1}{n}}\right) = 0.$$

Hence f_ϵ is not univalent in $\mathbb{U}$, for $\epsilon > 0$ and the result is sharp. It can be easily check that functions of the form (2.2) are also satisfying the equality in (2.1), where both sides will be zero. This completes the proof of Theorem 2.1. □

Setting $n = 2$ in Theorem 2.1 implies, for $f(z)/z \neq 0$ in $\mathbb{U}$, that

$$\left|\left(\frac{f(z)}{z}\right)''\right| \leq \frac{2}{3}(1 - 2|a_2|), \quad (z \in \mathbb{U})$$

is sufficient condition for f to be in F_2. Now, expanding $(f(z)/z)^{(n)}$ as a Taylor series gives the following corollary.

Corollary 2.2. *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z)/z \neq 0$ in $\mathbb{U}$ and let*

$$\sum_{k=2}^{n} k|a_k| + \frac{n+1}{n!}\sum_{k=n+1}^{\infty} (k-n)_n\,|a_k| \leq 1, \quad (z \in \mathbb{U}), \tag{2.13}$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)...(x+n-1).$$

Then $f \in F_n$.

Proof. Since

$$\left(\frac{f(z)}{z}\right)^{(n)} = \sum_{k=n+1}^{\infty} \frac{(k-1)!}{(k-n-1)!} a_k z^{k-n-1}.$$

Then, we easily conclude that (2.13) is sufficient condition for f to be in F_n. □

As a consequent result from Corollary 2.2, we observe that condition (1.1) is sufficient for f to be in F_2. Note that, $f(z) = z \pm (1/n)z^n \in F_{n-1} \cap F_n$. Indeed, we have the following result.

Theorem 2.3 (Inclusions)**.** *For $n \geq 3$, $F_n \subseteq F_{n-1}$ in the disk $|z| \leq (n+1)/n^2$. Moreover, if $f \in F_n$ with $\sum_{k=2}^{n} k\,|a_k(f)| = 1$, then $f \in F_{n-1}$ in the unit disk $\mathbb{U}$.*

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k \in F_n$. Then

$$\left(\frac{f(z)}{z}\right)^{(n)} = \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right)\mu_1(z),$$

where μ_1 is analytic in $\mathbb{U}$ and $|\mu_1(z)| \le 1$ in $\mathbb{U}$. Therefore, by integration from 0 to z, we observe that

$$\left(\frac{f(z)}{z}\right)^{(n-1)} = (n-1)!\, a_n + \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right) z\,\mu_2(z),$$

where μ_2 is analytic in $\mathbb{U}$ and $|\mu_2(z)| \le 1$ in $\mathbb{U}$. Now, if $|z| \le (n+1)/n^2$, then a simple computation gives

$$\begin{aligned}\left|\left(\frac{f(z)}{z}\right)^{(n-1)}\right| &\le (n-1)!|a_n| + \frac{n!}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right)|z| \\ &\le \frac{(n-1)!}{n}\left(1 - \sum_{k=2}^{n-1} k\,|a_k|\right).\end{aligned}$$

This completes the proof of the first part. To prove the second part, let $f \in F_n$ with $\sum_{k=2}^{n} k\,|a_k| = 1$. Then $(f(z)/z)^{(n)} = 0$ and

$$\left|\left(\frac{f(z)}{z}\right)^{(n-1)}\right| = (n-1)!|a_n| = \frac{(n-1)!}{n}\left(1 - \sum_{k=2}^{n-1} k\,|a_k|\right).$$

Therefore $f \in F_{n-1}$. □

We note that F_n is not included in F_{n-1}, consider the following example

Example 2.4. The function

$$f(z) = z + \frac{1}{4}z^4$$

belongs to F_3. However, it does not belong to F_2.

3. Subordination and an area theorem

Theorem 3.1 (Subordination)**.** *Let $f \in \mathcal{A}$ be satisfying*

$$f^{(n)}(z) \prec n!\, a_n + \beta(n+1)z, \quad (z \in \mathbb{U}). \tag{3.1}$$

Then,

$$\left(\frac{f(z)}{z}\right)^{(n-1)} \prec (n-1)!\, a_n + \beta z, \quad (z \in \mathbb{U}), \tag{3.2}$$

for every $n \ge 3$ and $\beta \in \mathbb{C}$. The result is sharp.

Proof. In view of Lemma 1.4, the subordination (3.1) can be written as

$$\frac{z}{n}\left(\frac{f(z)}{z}\right)^{(n)}+\left(\frac{f(z)}{z}\right)^{(n-1)}\prec (n-1)!\,a_n+\frac{n+1}{n}\beta z,\quad (z\in\mathbb{U}).$$

Applying Lemma 1.2 for $p(z)=(f(z)/z)^{(n-1)}$, $g(z)=(n-1)!\,a_n+\beta z$, $k=1$ and $\alpha=1/n$ to the above subordination yields the desired. □

Corollary 3.2. *If $f\in\mathcal{A}$ satisfies*

$$|f^{(n)}(z)-n!\,a_n|<\frac{(n+1)!}{n^2}\left(1-\sum_{k=2}^{n}k\,|a_k|\right),\quad (z\in\mathbb{U}), \tag{3.3}$$

for some $n\geq 3$, then $f\in F_{n-1}$.

Proof. Set

$$\beta=\frac{(n-1)!}{n}\left(1-\sum_{k=2}^{n}k\,|a_k|\right).$$

Then, in view of (3.3), $f^{(n)}(z)-n!\,a_n\prec\beta(n+1)z$ and hence (3.1) holds. Therefore, from (3.2), we obtain

$$\left|\left(\frac{f(z)}{z}\right)^{(n-1)}\right|<\frac{(n-1)!}{n}\left(1-\sum_{k=2}^{n-1}k\,|a_k|\right).$$

Hence, $f\in F_{n-1}$. □

For the case $n=2$, we have the following

Theorem 3.3. *If $f\in\mathcal{A}$ satisfies*

$$f''(z)-2\,a_2\prec\frac{3}{2}\,(1-2|a_2|)\,z, \tag{3.4}$$

or equivalently, $|f''(z)-2\,a_2|<\frac{3}{2}\,(1-2|a_2|)$, $(z\in\mathbb{U})$, then $f\in F$. In particular, if $a_2=0$ and $|f''(z)|<3/2$ then $f\in F$ and hence it is univalent in $\mathbb{U}$.

Proof. In view of (3.4), we have $f''(z)=2\,a_2+\frac{3}{2}\,(1-2|a_2|)\,w(z)$, where w is analytic function in $\mathbb{U}$ and $|w(z)|\leq|z|$ in $\mathbb{U}$ by Schwarz lemma. Therefore,

$$\begin{aligned}
|f'(z)-1| &= \left|\int_0^z f''(s)\,ds\right|\\
&\leq |z|\int_0^1 |f''(tz)|\,dt\\
&\leq |z|\int_0^1\left(2|a_2|+\frac{3}{2}(1-2|a_2|)|z|t\right)\,dt\\
&\leq 2|a_2||z|+\frac{3}{4}(1-2|a_2|)|z|^2\\
&< \frac{3}{4}+\frac{1}{2}|a_2|\leq 1.
\end{aligned}$$

Thus $f \in F$. □

Next, we study the area covered by functions in F_n. Applying Lemma 1.3 to $(f(z)/z)^{(n)}$, where $f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k \in F_n$, implies that

$$\sum_{k=n+1}^{\infty} \left(\frac{(k-n)_n\,(n+1)}{n!\left(1-\sum_{j=2}^{n} j\,|a_j|\right)} \right)^2 |a_k|^2 \leq 1.$$

Using this inequality, we may derive the following theorem.

Theorem 3.4 (An Area Theorem)**.** *The area of the image of* $\mathbb{U}$ *under each function in* F_n *satisfies*

$$A \leq \pi \left\{ 1 + \sum_{k=2}^{n} k|a_k|^2 + \frac{1}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right)^2 \right\}.$$

Equality attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$, *where* $\sum_{k=2}^{n} k|a_k| = 1$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k$ be in the class F_n. It is well known, (see [2]), that the area of the image of $\mathbb{U}$ under f is given by

$$A = \pi \left\{ 1 + \sum_{k=2}^{\infty} k|a_k|^2 \right\}.$$

By making use of Lemma 1.3 and Corollary 2.2, we obtain

$$\begin{aligned} A &= \pi \left\{ 1 + \sum_{k=2}^{n} k|a_k|^2 + \sum_{k=n+1}^{\infty} k|a_k|^2 \right\} \\ &\leq \pi \left\{ 1 + \sum_{k=2}^{n} k|a_k|^2 + \frac{\left(1-\sum_{k=2}^{n} k\,|a_k|\right)^2}{n+1} \sum_{k=n+1}^{\infty} \left(\frac{(k-n)_n\,(n+1)}{n!\left(1-\sum_{j=2}^{n} j\,|a_j|\right)} \right)^2 |a_k|^2 \right\} \\ &\leq \pi \left\{ 1 + \sum_{k=2}^{n} k|a_k|^2 + \frac{1}{n+1}\left(1 - \sum_{k=2}^{n} k\,|a_k|\right)^2 \right\}. \end{aligned}$$

This completes the proof. □

We may observe, for $f \in F_2$, that

$$A \leq \frac{4\pi}{3}\left(1 - |a_2| + \frac{10}{4}|a_2|^2\right)$$

and the maximum value of A is $(3/2)\pi$ which attained at $|a_2| = 1/2$ for the function $f(z) = z + a_2 z^2$, $(z \in \mathbb{U},\ |a_2| = 1/2)$. In general, the maximum value of A for functions in F_n is $\pi(1 + (1/n))$ which attained for the function $f(z) = z + a_n z^n$, $(|a_n| = 1/n)$.

4. Distortion, starlikeness and convexity

The following theorem introduces bounds for functions in F_n and for their derivatives. It will be useful for investigating starlike and convex functions in F_n.

Theorem 4.1 (Distortion and Growth)**.** *Let* $f(z) = z+\sum_{k=2}^{\infty} a_k z^k \in F_n$. *Then, for* $|z| = r < 1$ *and* $g(r) = \sum_{k=2}^{n} |a_k| r^k + \frac{1}{n+1}\left(1 - \sum_{k=2}^{n} k|a_k|\right) r^{n+1}$, *we have*

$$|f(z) - z| \leq g(r); \tag{4.1}$$

$$\left|f'(z) - 1\right| \leq g'(r); \tag{4.2}$$

$$r - g(r) \leq |f(z)| \leq r + g(r); \tag{4.3}$$

$$1 - g'(r) \leq |f'(z)| \leq 1 + g'(r). \tag{4.4}$$

Equalities in (4.1), (4.2), (4.3) *and (4.4) are attained for the functions* $f(z) = z + \sum_{k=2}^{n} a_k z^k$, *where* $\sum_{k=2}^{n} k|a_k| = 1$.

Proof. In view of relation (2.10), we have

$$f'(z) - 1 := \sum_{k=2}^{n} k a_k z^{k-1} + \left(1 - \sum_{k=2}^{n} k|a_k|\right) z^n\, W(z),$$

where

$$W(z) = \frac{n}{n+1}\left(\varphi_n(z) + \frac{1}{n}\int_0^1 n t^{n-1}\varphi_n(tz)\, dt\right)$$

is analytic in $\mathbb{U}$ and $|W(z)| \leq 1$ in $\mathbb{U}$. Therefore,

$$|f'(z) - 1| \leq \sum_{k=2}^{n} k|a_k||z|^{k-1} + \left(1 - \sum_{k=2}^{n} k|a_k|\right)|z|^n$$

and

$$\begin{aligned}
|f(z) - z| &= \left|\int_0^z (f'(s) - 1)\,\mathrm{d}s\right| \\
&\leq |z|\int_0^1 |f'(tz) - 1|\,\mathrm{d}t \\
&\leq |z|\int_0^1 \left\{\sum_{k=2}^{n} k|a_k||z|^{k-1}t^{k-1} + \left(1 - \sum_{k=2}^{n} k|a_k|\right)|z|^n t^n\right\}\mathrm{d}t \\
&= \sum_{k=2}^{n} |a_k||z|^k + \frac{1}{n+1}\left(1 - \sum_{k=2}^{n} k|a_k|\right)|z|^{n+1}.
\end{aligned}$$

The estimates (4.3) and (4.4) are immediate consequences from (4.1) and (4.2), respectively. □

For functions $f \in F_n$, one can write

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \leq \frac{n!}{n+1}\left(1-\sum_{k=2}^{n} k\,|a_k|\right)\alpha,$$

for some $0 \leq \alpha \leq 1$. In the next results, we find values for α such that $f \in F_n$ is starlike or convex in the unit disk $\mathbb{U}$.

Theorem 4.2 (Starlikeness)**.** *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z)/z \neq 0$ in $\mathbb{U}$ be satisfying*

$$\begin{aligned}\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| &\leq \frac{n!}{2(n^2+2n+2)}\left\{-(2n+3)\sum_{k=2}^{n} k\,|a_k|\right.\\ &+ \left.\sqrt{4(n^2+2n+2)-(n-1)^2\left(\sum_{k=2}^{n} k\,|a_k|\right)^2}\right\}\end{aligned}$$

for some $n \geq 2$ and for every $z \in \mathbb{U}$. Then $f \in \mathcal{S}^*$. Equality attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$ such that $\sum_{k=2}^{n} k\,|a_k| = \sqrt{4/5}$.

Proof. Let

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \leq \frac{n!}{n+1}\left(1-\sum_{k=2}^{n} k\,|a_k|\right)\alpha, \tag{4.5}$$

for some $\alpha \geq 0$. In view of relation (2.10) and from (4.2) and (4.5), we observe that

$$\begin{aligned}|f'(z)-1| &\leq \sum_{k=2}^{n} k|a_k||z|^{k-1} + \alpha\left(1-\sum_{k=2}^{n} k|a_k|\right)|z|^n\\ &< \sum_{k=2}^{n} k|a_k| + \alpha\left(1-\sum_{k=2}^{n} k|a_k|\right).\end{aligned}$$

Also, from (2.10), (4.1) and (4.5), we obtain

$$\begin{aligned}\left|\frac{f(z)}{z}-1\right| &\leq \sum_{k=2}^{n} |a_k||z|^{k-1} + \frac{\alpha}{n+1}\left(1-\sum_{k=2}^{n} k|a_k|\right)|z|^n\\ &< \frac{1}{2}\sum_{k=2}^{n} k|a_k| + \frac{\alpha}{n+1}\left(1-\sum_{k=2}^{n} k|a_k|\right).\end{aligned}$$

Now,

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\le \left|\arg f'(z)\right| + \left|\arg\frac{f(z)}{z}\right| \\ &< \arcsin\left\{\sum_{k=2}^{n} k|a_k| + \alpha\left(1-\sum_{k=2}^{n} k|a_k|\right)\right\} \\ &+ \arcsin\left\{\frac{1}{2}\sum_{k=2}^{n} k|a_k| + \frac{\alpha}{n+1}\left(1-\sum_{k=2}^{n} k|a_k|\right)\right\}.\end{aligned}$$

The last bound is equal to $\pi/2$ if

$$\left\{\sum_{k=2}^{n} k|a_k|+\alpha\left(1-\sum_{k=2}^{n} k|a_k|\right)\right\}^2+\left\{\frac{1}{2}\sum_{k=2}^{n} k|a_k|+\frac{\alpha}{n+1}\left(1-\sum_{k=2}^{n} k|a_k|\right)\right\}^2=1.$$

A computation shows that the root α of the above equation is given by

$$\begin{aligned}\alpha &= \frac{n+1}{2(n^2+2n+2)\left(1-\sum_{k=2}^{n} k\,|a_k|\right)}\left\{-(2n+3)\sum_{k=2}^{n} k\,|a_k|\right. \\ &+ \left.\sqrt{4(n^2+2n+2)-(n-1)^2\left(\sum_{k=2}^{n} k\,|a_k|\right)^2}\right\}\end{aligned}$$

and $\alpha \ge 0$ if

$$\sum_{k=2}^{n} k\,|a_k| \le \sqrt{\frac{4}{5}}.$$

Therefore, for the defined root α, we have

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}$$

and hence $f \in \mathcal{S}^*$. Finally, (4.5) is equivalent to the assumption condition for the defined root α and this completes the proof of Theorem 4.2. □

For $n = 2$, the above result is reduced to

Corollary 4.3. *Any function* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *with* $f(z)/z \neq 0$ *in* $\mathbb{U}$, *satisfies*

$$\left|\left(\frac{f(z)}{z}\right)''\right| \le \frac{1}{5}\left(-7|a_2| + \sqrt{10-|a_2|^2}\right), \quad (z \in \mathbb{U})$$

is starlike in $\mathbb{U}$. *Equality attained for the function* $f(z) = z + a_2\, z^2$, $(|a_2| = 1/\sqrt{5})$.

Theorem 4.4 (Convexity). *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with $f(z)/z \neq 0$ in $\mathbb{U}$, be satisfying*

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \leq \frac{n!}{2n^2+n+1}\left(1-\sum_{k=2}^{n} k^2\,|a_k|\right), \quad (z \in \mathbb{U}). \tag{4.6}$$

Then $f \in \mathcal{K}$. Equality attained for the functions $f(z) = z + \sum_{k=2}^{n} a_k z^k$ such that $\sum_{k=2}^{n} k^2\,|a_k| = 1$.

Proof. Let

$$\left|\left(\frac{f(z)}{z}\right)^{(n)}\right| \leq \alpha\beta_n,$$

for some $\alpha \geq 0$ and

$$\beta_n = \frac{n!}{n+1}\left(1-\sum_{k=2}^{n} k\,|a_k|\right).$$

In view of relation (2.10), we have

$$f'(z) = 1 + \sum_{k=2}^{n} k a_k z^{k-1} + \frac{\alpha\beta_n}{(n-1)!}\left(z^n\varphi_n(z) + \int_0^z u^{n-1}\varphi_n(u)\,\mathrm{d}u\right).$$

Differentiating the previous relation gives

$$f''(z) = \sum_{k=2}^{n} k(k-1)a_k z^{k-2} + \frac{z^{n-1}}{(n-1)!}\alpha\beta_n\left[(n-1)\varphi_{n-1}(z) + (n+1)\varphi_n(z)\right]$$

where $z^n\varphi_n'(z) = (n-1)z^{n-1}(\varphi_{n-1}(z) - \varphi_n(z))$ by Lemma 1.1. Hence, from the above relations and estimate (4.4), we obtain

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{\sum_{k=2}^{n} k(k-1)|a_k| + \frac{2n}{(n-1)!}\alpha\beta_n}{1-\sum_{k=2}^{n} k a_k - \frac{n+1}{n!}\alpha\beta_n}.$$

A mild computation shows that the last bound is less than or equal to 1 if

$$\alpha \leq \frac{(n+1)\left(1-\sum_{k=2}^{n} k^2|a_k|\right)}{(2n^2+n+1)\left(1-\sum_{k=2}^{n} k|a_k|\right)}.$$

This yields

$$\alpha\beta_n \leq \frac{n!}{2n^2+n+1}\left(1-\sum_{k=2}^{n} k^2\,|a_k|\right).$$

Thus, condition (4.6) implies that f is convex in $\mathbb{U}$. □

Theorem 4.4, when $n = 2$, is reduced to

Corollary 4.5. *Any function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with $f(z)/z \neq 0$ in $\mathbb{U}$, satisfies*

$$\left|\left(\frac{f(z)}{z}\right)''\right| \leq \frac{2}{11}(1-4|a_2|), \quad (z \in \mathbb{U})$$

is convex in $\mathbb{U}$. Equality attained for $f(z) = z + a_2\,z^2$, ($|a_2| = 1/4$).

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