Fixed point theorem for contraction mappings in probabilistic normed spaces

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Abstract. In this paper, the concept of contractive mappings and $\phi-$ contraction mappings on Menger's probabilistic normed spaces are defined with suitable examples. The unique fixed point theorem for contractive mappings and $\phi-$ contraction mappings are established in Menger's probabilistic normed spaces.

Keywords: Menger's Probabilistic normed spaces, sequential continuity, fixed point, probabilistic bounded.

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1. Introduction and preliminaries

Probabilistic functional analysis has risen as one of the vital mathematical disciplines in view of the needs in dealing with probabilistic models in applied problems. Probabilistic Normed spaces were introduced by Serstnev and its new definition was proposed by C.Alsina, B.Schwerier and A.Sklar ([1], [2]). The general theory of Probabilistic Metric spaces and Probabilistic Normed spaces can be read in ([8], [13], [14]). The theory of probabilistic normed spaces (briefly. PN spaces) is important as a generalization of deterministic results of linear normed spaces and in the study of random operator equations. The PN spaces may also provide us with the suitable tools to contemplate the geometry of nuclear physics and have important applications in quantum particle physics especially in string theory and in ε_∞ theory. Some of the recent developments in different types of probabilistic normed spaces, translation invariant topologies induced by probabilistic norms and linear 2-normed spaces are discussed in ([3], [6], [7], [8]).

A triangle function [4] is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, non-decreasing in each place and has ε_0 as identity, this is, for all F, G and H in Δ^+:

(TF1) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$,

(TF2) $\tau(F, G) = \tau(G, F)$,

(TF3) $F \leq G \Longrightarrow \tau(F, H) \leq \tau(G, H)$,

(TF4) $\tau(F, \varepsilon_0) = \tau(\varepsilon_0, F) = F$.

Moreover, a triangle function is continuous if it is continuous in the metric space (Δ^+, d_S).

Typical continuous triangle functions [4] are

$$\tau_T(F, G)(x) = \sup_{s+t=x}\{T(F(s), G(t))\}$$

and

$$\tau_{T^*}(F, G) = \inf_{s+t=x}\{T^*(F(s), G(t)\}.$$

Here T is a continuous t-norm, i.e. a continuous binary operation on $[0, 1]$ that is commutative, associative, non-decreasing in each variable and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0, 1]$ which is related to the continuous t-norm T through $T^*(x, y) = 1-T(1-x, 1-y)$. Let us recall among the triangular function one has the function defined via $T(x, y) = \min(x, y) = M(x, y)$ and $T^*(x, y) = \max(x, y)$ or $T(x, y) = \Pi(x, y) = xy$ and $T^*(x, y) = \Pi^*(x, y) = x + y - xy$.

Some more examples of t-norms [4] are W and Z, defined respectively by

$$W(x,y) := \max\{x+y-1, 0\}.$$

$$Z(x,y) := \begin{cases} 0, & \text{if } (x,y) \in [0,1] \times [0,1], \\ x, & \text{if } x \in [0,1], y = 1, \\ y, & \text{if } x = 1, y \in [0,1] \end{cases}$$

then we have

$$Z < W < \Pi < M$$

and every t-norm T,

$$Z \leq T \leq M.$$

For every t-norm Π, W, Z and M, it is defined that

$$\Pi_\Pi(F,G)(x) := \Pi(F(x), G(x)),$$

$$\Pi_W(F,G)(x) := W(F(x), G(x)),$$

$$\Pi_Z(F,G)(x) := Z(F(x), G(x)),$$

$$\Pi_M(F,G)(x) := M(F(x), G(x)).$$

A few more interesting examples of t-norms and t-conorms can be found in the recent paper [10].

We recall the definition of probabilistic normed space (briefly, PN space) as given in [2], together with the notation that will be needed [13]. We shall consider the space of all distance probability distribution functions (briefly, d.f.'s), namely the set of all left-continuous and non-decreasing functions from $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ into $[0,1]$ such that $F(0) = 0$ and $F(+\infty) = 1$, will be denoted by Δ^+, while the subset $\mathcal{D}^+ \subset \Delta^+$ will denote the set of all proper distance d.f.'s, namely those for which $\ell^- F(+\infty) = 1$. Here $\ell^- f(x)$ denotes the left limit of the function f at the point $x \in \mathbb{R}$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions i.e., $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $\mathbb{R}$. For any $a \geq 0$, ε_a is the d.f. given by

$$\varepsilon_a = \begin{cases} 0, & \text{if } x \leq a, \\ 1, & \text{if } x > a. \end{cases}$$

The space Δ^+ can be metrized in several ways [13], but we shall here adopt the Sibley metric d_S. If F, G are d.f.'s and h is in $]0,1[$, let $(F,G;h)$ denote the condition:

$$G(x) \leq F(x+h) + h \text{ for every} x \in \left]0, \frac{1}{h}\right[.$$

Then the Sibley metric is defined by

$$d_S(F,G) := \inf\{h \in\,]0,1[\, : \ \text{both } (F,G;h) \text{ and } (G,F;h) \text{ hold }\}. \tag{1.1}$$

Since every d.f. F is bounded, $0 \leq F(t) \leq 1$, for all $t \in \overline{\mathbb{R}}$, one has $d_S \leq 1$.

Definition 1.1 ([1], [4]). A Menger's Probabilistic Normed Space (briefly, a Menger's PN space), is a quadruple (X, ν, τ, T), where X is a real vector space, τ is a triangle function, T is a t-norm and the mapping $\nu : X \to \triangle^+$ satisfies the conditions:

1. $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in X);
2. $\nu_{\alpha p}(t) \geq \nu_p\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R}\backslash\{0\}$ and $t \in \mathbb{R}$;
3. $\nu_{p+q}(s+t) \geq \tau_T\left(\nu_p(s), \nu_q(t)\right)$ for all $p, q \in X$ and $s, t \in \mathbb{R}$.

The function ν is called the Menger's probabilistic norm.

Example 1.1 ([1]). Let $(V, \|.\|)$ be a normed space and define $\nu_p := \varepsilon_{\|p\|}$. Let τ be a triangle function such that

$$\tau\left(\varepsilon_a, \varepsilon_b\right) = \varepsilon_{a+b},$$

for all $a, b \geq 0$ and let τ^* be a triangle function with $\tau \leq \tau^*$. For instance, it suffices to take $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where T is a continuous t-norm and T^* is its t-conorm. Then (V, ν, τ, τ^*) is a Menger's PN space.

Definition 1.2 ([8]). Let (X, ν, τ, T) be a Menger's PN space, and (x_n) be a sequence of X then the sequence (x_n) is said to be convergent to x if for all $t > 0$ and $\lambda \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $\nu_{x_n - x}(t) > 1 - \lambda$ for every $n > n_0$.

Definition 1.3 ([8]). Let (X, ν, τ, T) be a Menger's PN space, and (x_n) be a sequence of X then the sequence (x_n) is said to be Cauchy sequence if for all $t > 0$ and $\lambda \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $\nu_{x_n - x_m}(t) > 1 - \lambda$ for all $n, m > n_0$.

Definition 1.4 ([8]). A Menger's PN space (X, ν, τ, T) is said to be complete if every Cauchy sequence in X is convergent to some point in X.

A compltete Menger's PN space is called Menger probabilistic Banach space.

Definition 1.5 ([8]). Let (X, ν, τ, T) be a Menger's PN space, E be a subset of X, then the closure of E is defined as $\overline{E} = \{x \in X : \text{ there exists } (x_n) \subset E \text{ such that } x_n \to x\}$, that is, for $\alpha \in (0,1)$ and $r > 0, x \in \overline{E}$ there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ one has $\nu_{x_n - x}(r) \geq \alpha$.

We say, E is sequentially closed if $E = \overline{E}$,

Definition 1.6 ([8]). The probabilistic radius R_A of a nonempty set A in a Menger's PN space (X, ν, τ, T) is defined by

$$R_A := \begin{cases} \ell^- \varphi_A(t), & t \in [0, +\infty[\\ 1, & t = +\infty \end{cases}$$

where $\varphi_A(t) := \inf\{\nu_p(t) : p \in A\}$.

Definition 1.7 ([8]). A nonempty set A in a Menger's PN space (X, ν, τ, T) is said to be:

(a) certainly bounded, if $R_A(t_0) = 1$ for some $t_0 \in]0, +\infty[$;

(b) perhaps bounded, if one has $R_A(t) < 1$ for every $t \in]0, +\infty[$, but

$$\lim_{t\to+\infty} R_A(t) = 1;$$

(c) perhaps unbounded, if $R_A(t_0) > 0$ for some $t_0 \in]0, +\infty[$ and

$$\lim_{t\to+\infty} R_A(t) \in]0, 1[;$$

(d) certainly unbounded, if $\lim_{t\to+\infty} R_A(t) = 0$, i.e., if $R_A = \varepsilon_\infty$. Moreover, the set A will be said to be distributionally bounded (henceforth $\mathcal{D}-$bounded) if either (a) or (b) holds, i.e.,if $R_A \in \mathcal{D}^+$; otherwise, i.e., if R_A belongs to $\Delta^+ \setminus \mathcal{D}^+$, A will be said to be $\mathcal{D}-$unbounded.

Definition 1.8 ([8]). Let (X, ν, τ, T) and (Y, μ, τ, T) be two Menger's PN spaces then a mapping $T : X \to Y$ is said to be sequentially continuous if a sequence (x_n) in X converges to $x \in X$ implies (Tx_n) converges to Tx in Y.

Lemma 1.9 ([8]). Let (X, ν, τ, T) be a Menger's PN space. If $|\alpha| \le |\beta|$, then $\nu_{\alpha x} \ge \nu_{\beta x}$ for every $x \in X$.

Definition 1.10 ([8]). Let (X, ν, τ, T) be a Menger's PN space and A be a nonempty subset of X then A is said to be probabilistically bounded if for each $r \in (0, 1)$ there exists $t > 0$ such that $\nu_x(t) > 1 - r$ for all $x \in A$.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. In connection with the function φ we consider the following properties [5]:

1. φ is monotonic increasing;
2. $\varphi(t) < t$ for all $t > 0$;
3. $\varphi(0) = 0$;
4. φ is continuous;
5. $\{\varphi^n(t)\}$ converges to 0 for all $t \ge 0$;
6. $\sum_{n=0}^{\infty}\{\varphi^n(t)\}$ converges for all$t > 0$;
7. $t - \varphi(t) \to \infty$ as $t \to \infty$;
8. φ is subadditive.

Lemma 1.11 ([5]). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function then:

1. φ is monotonic increasing and $\varphi(t) < t$ for all $t > 0$ implies $\varphi(0) = 0$.
2. $\varphi(t) < t$ for all $t > 0$ and φ is continuous implies $\varphi(0) = 0$.
3. φ is monotonic increasing and $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$ then $\varphi(t) < t$ for all $t > 0$.

Definition 1.12 ([5], [11])**.** A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if φ is monotonic increasing and $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$.

Lemma 1.13 ([5], [11])**.** 1. Any comparison function φ satisfies $\varphi(0) = 0$;

2. Any subadditive comparison function φ is continuous;
3. If φ is a comparison function then, for any $k \in \mathbb{N}, \varphi^k$ is a comparison function;
4. If φ is a comparison function then, the function $s : \mathbb{R}_+ \to \mathbb{R}_+$ such that $s(t) = \sum_{k=0}^{\infty} \varphi^k(t)$ for $t \in \mathbb{R}_+$ satisfies that s is monotonic increasing and $s(0) = 0$.

Example 1.14 ([5][11])**.** Some examples for the function φ satisfying the properties mentioned in definition (1.11):

1. Define $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi(t) = kt$ for $k \in [0, 1]$ then φ satifies all the conditions in definition (1.11).
2. Define $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi(t) = \frac{t}{t+1}$ then φ is monotonic increasing,$\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$ and $t - \varphi(t) \to \infty$ as $t \to \infty$.

2. Contraction and $\varphi-$contraction mappings in Menger's PN spaces

Definition 2.1 ([12])**.** Let (X, ν, τ, T) be a Menger's PN space, a subset L of X of the form $\{x + ty; t \in \mathbb{R}_+\}$ where $x, y \in X$ and $y \neq 0$ is called a line.

Definition 2.2. Let (X, ν, τ, T) be a Menger's PN space,$f : X \to X$ is said to be a contraction on X if and only if there is a $k \in (0, 1)$ such that $\nu_{fx-fy}(kt) \geq \nu_{x-y}(t)$ for every $x, y \in X$ and $t > 0$.

Example 2.3. Let (X, ν, τ, T) be a Menger's PN space, and S be a subset of $L = \{x + \alpha y; \alpha \in \mathbb{R}_+\}$. Define $f : S \to X$ by $f(x + \alpha y) = (\frac{\alpha}{1+\alpha})y$ for $x, y \in X$, then f is a contraction on X.

Lemma 2.4. Let (X, ν, τ, T) be a Menger's PN space then, every contraction $f : X \to X$ is sequentially continuous.

Proof. Since f is a contraction on X, we have for $k \in (0, 1)$ $\nu_{fx-fy}(kt) \geq \nu_{x-y}(t)$, for all $x, y \in X$ and $t > 0$.

Let (x_n) be a sequence in X such that $x_n \to x$ i.e., for all $t > 0$ and $\lambda \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ we have $\nu_{x_n-x}(t) > 1 - \lambda$.

As $x_n \in X$ for all n, we have $\nu_{fx_n - fx}(kt) \geq \nu_{x_n - x}(t) > 1 - \lambda$ for every $n > n_0$.

Hence $fx_n \to fx$. So f is sequentially continuous. □

Now we prove a unique fixed point theorem for contractive mappings in Menger's PN space.

Theorem 2.5. Let (X, ν, τ, T) be a Menger's probabilistic Banach space and E be a nonempty closed and probabilistic bounded subset of X. Let $f : E \to E$ be a contraction then f has a unique fixed point on X.

Proof. Since f is a contraction on X, for $k \in (0, 1), \nu_{fx-fy}(kt) \geq \nu_{x-y}(t)$ for all $x, y \in X$ and $t > 0$.

Let $x_0 \in X$. Construct a sequence $\{x_n\}$ depending on x_0. Let $x_1 = fx_0, x_2 = fx_1, x_3 = fx_2, ..., x_n = fx_{n-1}$ then $fx_0 = x_1, f^2x_0 = f(fx_0) = fx_1 = x_2$, ..., $f^n x_0 = x_n$.

We have, $\nu_{f^2(x)-f^2(y)}(kt) = \nu_{f(fx)-f(fy)}(kt) \geq \nu_{fx-fy}(t) \geq \nu_{x-y}\left(\frac{t}{k}\right)$.

Similarly, $\nu_{f^3x-f^3y}(kt) \geq \nu_{x-y}\left(\frac{t}{k^2}\right)$.

Continuing like this we get

$$\nu_{f^n x - f^n y}(kt) \geq \nu_{x-y}\left(\frac{t}{k^{n-1}}\right).$$

We have E is probabilistically bounded then for each $r \in (0, 1)$ there exists $t > 0$ such that $\nu_x(t) > 1 - r$ for all $x \in E$.

We prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $m, n > 0$ with $m > n$. Take $m = n + p$ then

$$\begin{aligned}
&\nu_{x_n - x_{n+p}}(kt) = \nu_{(x_n - x_{n+1})+(x_{n+1}-x_{n+p})}(kt)\\
&\geq \tau_T\{\nu_{x_n - x_{n+1}}\left(\frac{kt}{2}\right), \nu_{x_{n+1}-x_{n+p}}\left(\frac{kt}{2}\right)\}\\
&= \tau_T\{\nu_{f^n x_0 - f^n x_1}\left(\frac{kt}{2}\right), \nu_{(x_{n+1}-x_{n+2})+(x_{n+2}-x_{n+p})}\left(\frac{kt}{2}\right)\}\\
&\geq \tau_T\{\nu_{x_0 - x_1}\left(\frac{t}{2k^{n-1}}\right), \tau_T\{\nu_{f^n x_1 - f^n x_2}\left(\frac{kt}{4}\right), \nu_{(x_{n+2}-x_{n+p})}\left(\frac{kt}{4}\right)\}\}\\
&\geq \tau_T\{\nu_{x_0 - x_1}\left(\frac{t}{2k^{n-1}}\right), \tau_T\{\nu_{x_1 - x_2}\left(\frac{t}{4k^{n-1}}\right), \nu_{(x_{n+2}-x_{n+p})}\left(\frac{kt}{4}\right)\}\}\\
&\geq \tau_T\{\nu_{x_0 - x_1}\left(\frac{t}{2k^{n-1}}\right), \nu_{x_1 - x_2}\left(\frac{t}{4k^{n-1}}\right), \tau_T\{\nu_{(x_{n+2}-x_{n+3})+(x_{n+3}-x_{n+p})}\left(\frac{kt}{4}\right)\}\}\\
&> \tau_T\{1 - r, 1 - r, ..., 1 - r\} = 1 - r.
\end{aligned}$$

So, $\{x_n\}$ is a Cauchy sequence in X. Therefore, $\{x_n\}$ converges to some point $x \in X$.

Since f is sequentially continuous, we have $\lim_{n\to\infty} fx_n = fx$ and $fx = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = x$. Hence f has a fixed point in X.

It remains to prove that such a fixed point is unique.

Let $y \in X$ with $y \neq x$ such that $fy = y$ then, $\nu_{x-y}(kt) = \nu_{fx-fy}(kt) \geq \nu_{x-y}(t)$ implies $\nu_{\frac{1}{k}(x-y)}(t) \geq \nu_{x-y}(t)$ implies $|\frac{1}{k}| \leq 1$ implies $k \geq 1$, a contradiction to $k \in (0,1)$. Hence the fixed point of f is unique. □

Definition 2.6. Let (X, ν, τ, T) be a Menger's PN space then a mapping $f : X \to X$ is said to be a $\varphi-$contraction if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\nu_{fx-fy}(\varphi(t)) \geq \nu_{x-y}(\varphi(t/c))$ for $0 < c < 1$, for all $x, y \in X$ and $t > 0$.

Example 2.7. Let (X, ν, τ, T) be a Menger's PN space. Define $f : X \to X$ by $f(t) = t$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi(t) = \frac{t}{c}$, for $0 < c < 1$ then φ is a comparison function and hence f is a $\varphi-$contraction on X.

Lemma 2.8. Let (X, ν, τ, T) be a Menger's PN space then, every $\varphi-$contraction $f : X \to X$ is sequentially continuous.

Proof. Let (x_n) be a sequence in X such that $x_n \to x$ in X i.e., for all $t > 0$ and $\lambda \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$, we have $\nu_{x_n-x} > 1-\lambda$.

Since f is a $\varphi-$contraction, we have, for $0 < c < 1$, for all $x, y \in X$ and $t > 0$

$$\nu_{fx_n-fx}(\varphi(t)) \geq \nu_{x_n-x}(\varphi(t/c)) > 1-\lambda, \quad \text{for every} \quad n > n_0.$$

Hence f is sequentially continuous. □

Theorem 2.9. Let (X, ν, τ, T) be a Menger's probabilistic Banach space and E be a nonempty closed and $\mathcal{D}-$ bounded subset of X. Let $f : E \to E$ be a $\varphi-$contraction then f has a unique fixed point on X.

Proof. Since f is a $\varphi-$contraction on X, then there exists a comparison function φ for $0 < c < 1$, for all $x, y \in X$ and $t > 0$

$$\nu_{fx-fy}(\varphi(t)) \geq \nu_{x-y}(\varphi(t/c)).$$

We have

$$\begin{aligned}\nu_{f^2(x)-f^2(y)}(\varphi(t)) &= \nu_{f(fx)-f(fy)}(\varphi(t)) \\ &\geq \nu_{fx-fy}(\varphi(t/c)) \\ &\geq \nu_{x-y}\left(\varphi(t/c^2)\right).\end{aligned}$$

Similarly, $\nu_{f^3x-f^3y}(\varphi(t)) \geq \nu_{x-y}\left(\varphi(t/c^3)\right)$. Continuing like this we get,

$$\nu_{f^nx-f^ny}(\varphi(t)) \geq \nu_{x-y}(\varphi(t/c^n)).$$

Let $x_0 \in X$. Construct a sequence $\{x_n\}$ depending on x_0. Let $x_1 = fx_0, x_2 = fx_1, x_3 = fx_2, ..., x_n = fx_{n-1}$ then $fx_0 = x_1, f^2x_0 = f(fx_0) = fx_1 = x_2$, $..., T^nx_0 = x_n$.

We prove that $\{x_n\}$ is a Cauchy sequence in X.

Let $m, n > 0$ with $m > n$. Take $m = n + p$ then

$$\begin{aligned}
\nu_{x_n - x_{n+p}}(\varphi(t)) &= \nu_{(x_n - x_{n+1}) + (x_{n+1} - x_{n+p})}(\varphi(t)) \\
&\geq \tau_T\{\nu_{x_n - x_{n+1}}(\varphi(t)), \nu_{x_{n+1} - x_{n+p}}(\varphi(t))\} \\
&= \tau_T\{\nu_{f^n x_0 - f^n x_1}(\varphi(t)), \nu_{(x_{n+1} - x_{n+2}) + (x_{n+2} - x_{n+p})}(\varphi(t))\} \\
&\geq \tau_T\{\nu_{x_n - x_1}(\varphi(t)), \tau_T\{\nu_{(x_{n+1} - x_{n+2})}(\varphi(t)), \nu_{(x_{n+2} - x_{n+p})}(\varphi(t))\}\} \\
&> \tau_T\{1 - r, 1 - r\} = 1 - r.
\end{aligned}$$

So, $\{x_n\}$ is a Cauchy sequence in X. Therefore, $\{x_n\}$ converges to some point $x \in X$.

Since f is sequentially continuous, we have $\lim_{n\to\infty} fx_n = fx$ and $fx = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = x$. Hence f has a fixed point in X.

Now, we have to prove that such a fixed point is unique.

Let $y \in X$ with $y \neq x$ such that $fy = y$ then, for $t > 0$,

$$\begin{aligned}
\nu_{x-y}(\varphi(t)) &= \nu_{fx-fy}(\varphi(t)) \\
&\geq \nu_{x-y}(\varphi(t/c)) \\
&\geq \nu_{fx-fy}\left(\varphi(t/c^2)\right) \\
&\dots \\
&\geq \nu_{x-y}(\varphi(t/c^n)).
\end{aligned}$$

We have $\lim_{n\to\infty} \varphi(t/c^n) = \infty$ and as a consequence $\nu_{x-y}(\infty) = 1$ if and only if ν_x is in D^+. i.e., $\lim_{t\to\infty} \nu_x(t) = 1$. Then, $\nu_{x-y}(t) = 1$ for all $t > 0$ if and only if $\nu_{x-y} = \varepsilon_0$ if and only if $x = y$. This completes the proof. □

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