Characterization of generalized projective and injective soft modules

Asima Razzaque*
Department of Basic Science
King Faisal University
Saudi Arabia
asima.razzaque@yahoo.com

Inayatur Rehman
Department of Mathematics and Sciences
College of Arts and Applied Sciences
Dhofar University Salalah
Oman
irehman@du.edu.om

M. Iftikhar Faraz
School of Engineering
Grand Valley State University
USA
farazm@gvsu.edu

Kar Ping Shum
Institute of Mathematics
Yunnan University
Kunming, 650091
P. R. China
kpshum@ynu.edu.cn

Abstract. We first introduce the concepts of projective soft LA-modules, free soft LA-modules, split sequence in soft LA-modules and establish various results on projective soft LA-modules. Then, we consider the injective soft LA-modules and give some relevant results by using free soft LA-modules and split sequences in soft LA-modules.
Keywords: soft LA-rings, soft LA-modules, soft LA-ring homomorphism, soft LA-module homomorphism, exact sequence.

1. Introduction

In our daily life, the real world is multifaceted. Thus, there are many problems in different disciplines such as engineering, social sciences, medical sciences etc in the real world and we construct "models" of reality that are simplifications of aspects of the real world. Unluckily, these mathematical models are quite intricate and we are unable to find the precise solutions. Since there are many

*. Corresponding author

uncertainties mixed up with the data. The traditional tools to deal with these uncertainties are applicable only under certain environment. These may be due to the uncertainties of natural environmental phenomena of human awareness about the real world or to the confines of the means used to measure objects. For example, elusiveness or uncertainty in the boundary between states or between urban and rural areas or the exact growth rate of population in a country's rural area or making decision in a machine based environment using database information. Thus, the classical set theory, which is based on crisp and exact case, may not be fully suitable for conducting such problems of uncertainty.

In order to deal with uncertainties, there are many theories, for example, theory of fuzzy sets [28], theory of intuitionistic fuzzy sets [4], theory of vague sets, the theory of interval mathematics [5], [8] and theory of rough sets [11] have been developed, yet difficulties are seem to be still there. It is noted that the theory of soft sets was first proposed by D. Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainties. In order to model vagueness and uncertainties, D. Molodtsov first introduced the concept of soft sets and it has received much attention since its inception. In his well known paper [13], D. Molodtsov presented some fundamental results of the new theory and successfully applied them into several directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, theory of probability etc. A soft set is a collection of approximate description of an object. He also showed how soft set theory is free from parametrization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory and game theory. In fact, the soft systems provides a very general framework with the involvement of parameters. Nowadays, the research work on soft set theory and its applications in various fields are progressing rapidly.

We notice that P. K. Maji [11], [12] in 2002 and 2003 first presented the application of soft sets in decision making based on the reduction of parameters to keep the optimal choice objects. In addition, D. Chen [6] presented a new definition of soft set parametrization reduction and a comparison of it with attributes reduction in rough set theory. A. Sezgin et al. [17], introduced the union soft subnear-rings and union soft ideals of a near-ring. The application of soft sets in algebraic structures was introduced by H. Aktaş and N. Çağman [1]. They discussed the notion of soft groups and derived some basic properties. They also showed that soft groups extends the concept of fuzzy groups. Recently, X. Liu et al.[9], established some useful fuzzy isomorphism theorems of soft rings. They also discussed the fuzzy ideals of soft rings. In [10], X. Liu et al., have considered the isomorphism theorems for soft rings. In [25], Q. M. Sun et al., have discussed the concept of soft modules and investigated some of their basic properties.

The Left Almost Ring (LA-ring) is actually an off shoot of LA-semigroup and LA-group. In fact, an LA-rings is a non-commutative and non-associative algebraic structure and gradually due to its peculiar characteristics it has been emerging as a useful non-associative class which intuitively would be a quite convinent tool to enhance non-associative ring theory. By an LA-ring, we mean a non-empty set R with at least two elements such that $(R,+)$ is an LA-group, $(R,\cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R,+,\cdot)$, we can always obtain an LA-ring $(R,\oplus,\cdot)$ by defining for all $a,b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring.

Furthermore, T. Shah and I. Rehman [22], have discussed left almost ring (LA-ring) of finitely nonzero functions which is in fact a generalization of a commutative semigroup ring. Recently T. Shah and I. Rehman [23], discussed some properties of LA-rings through their ideals and intuitively ideal theory would be a gate way for investigating the application of fuzzy sets, intuitionistics fuzzy sets and soft sets in LA-rings. For example, T. Shah et al., [20], have applied the concept of intuitionistic fuzzy sets and established some useful results. In [16], some computational work through Mace4, has been done and some interesting characteristics of LA-rings have been explored. Recently, in [18], T. Shah et al., have adopted a new approach to apply the Molodtsov's soft set theory to a class of non-associative rings. And in [19], T. Shah and Asima Razzaque have discussed some basic properties regarding soft M-system, soft P-system and soft I-System in a non-associative left almost rings. T. Shah et al. [21], have promoted the concept of LA-modules and establish some results of isomorphisms theorems and direct sum of LA-modules. Also T. Shah and I. Rehman in [22] utilized the both LA-semigroup and LA-ring and generalizes the notion of a commutative semigroup ring. Furthermore, they defined the notion of LA-modules over an LA-ring, which is a non abelian non-associative structure but closer to abelian group. Hence the study of this algebraic structure is completely parallel to modules which are basically the abelian groups. In this aspect, A. Alghamdi and F. Sahraoui [2], have defined and constructed a tensor product of LA-modules, they extended some simple results from the ordinary tensor to the new setting. Also Asima Razzaque et al., [3] have given the concept of exact sequence in LA-modules. For some further study of LA-rings, the readers are referred to ([15], [21], [24]).

In this paper, we initiate the concepts of projective and injective soft LA-modules. Also we discuss the free soft LA-modules, split sequence in soft LA-modules and prove some of their related results.

2. Preliminaries

In this section, we recall some basic definitions and results which are relevant to soft sets and LA-modules.

Definition 1 ([13])**.** *Let U be an initial universe and E be a set of parameters. Then we use $P(U)$ to denote the power set of U and A be a non-empty subset of E. A pair (F,A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.*

In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$approximate elements of the soft set (F, A). Clearly, a soft set is not a set.

For the soft sets, we give the following definitions.

Definition 2 ([14]). *For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if it satisfies the following conditions:*
(i) $A \subseteq B$ and
(ii) for all $e \in A$, $F(e) \subseteq G(e)$.

We write $(F, A)\widetilde{\subset}(G, B)$. Also we call (F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote this soft superset by $(F, A)\widetilde{\supset}(G, B)$.

Definition 3 ([12]). *Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).*

Definition 4 ([12]). *A soft set (F, A) over U is said to be a $NULL$ soft set denoted by Φ if for all $\varepsilon \in A$, $F(\varepsilon) = \emptyset$ (null set).*

Definition 5 ([7]). *Let (F, A) be a soft set .Then, the set supp $(F, A) = \{x \in A \mid F(x) \neq \phi\}$ is called the support of the soft set (F, A). A soft set is said to be non null if its support is not equal to the empty set.*

Definition 6 ([22]). *Let $(R, +, .)$ be an LA-ring with left identity e. An LA-group $(M, +)$ is said to be LA-module over R if $R \times M \to M$ defined as $(a, m) \mapsto am \in M$, where $a \in R$, $m \in M$ satisfies the following conditions:*
(i) $(a + b)\, m = am + bm$,
(ii) $a(m + n) = am + an$,
(iii) $a(bm) = b(am)$,
(iv) $1.m = m$,
for all $a, b \in R$, $m, n \in M$.

Left R LA-module is denoted by $_RM$ or simply M. Right R LA-module can be defined in a similar manner and is denoted by M_R.

In the following, we give a non-trivial example of LA-module over R constructed by T. Shah and I. Rehman in [22]. We observe that the LA-module constructed in this example is not a module.

Example 1 ([22]). Let $(R, +, .)$ be an LA-ring with a left identity and S is a commutative semigroup. Then $R[S] = \{\sum_{j=1}^{n} a_j s_j : a_j \in R, s_j \in S\}$ and the map $R \times R[S] \mapsto R[S]$ defined by $(a, \sum_{j=1}^{n} a_j s_j) \mapsto \sum_{j=1}^{n}(aa_j)s_j$ is an LA-module over R.

Definition 7 ([21]). *Let M be a left R LA-module. Then, we call an LA-subgroup N of M over an LA-ring R is called left R LA-submodule of M, if $RN \subseteq N$, i.e., $rn \in N$ for all $r \in R$ and $n \in N$. This is denoted by $N \leq M$.*

By the above definition, we immediately have the following theorem.

Theorem 1 ([21]). *If A and B are two LA-submodules of an LA-module M over an LA-ring R, then $A \cap B$ is also an LA-submodule of M.*

Corollary 1 ([21]). *The intersection of any number of LA-submodules of an LA-module is a LA-submodule.*

Following is the very useful definition in the study of LA-modules.

Definition 8 ([21]). *Let M, N be LA-modules over an LA-ring R. A map $\varphi : M \longrightarrow N$ is called an LA-module homomorphism(or simply R-homomorphism) if, for all r in R and m, n in M*

(i) $\varphi(m+n) = \varphi(m) + \varphi(n)$

(ii) $\varphi(rm) = r\varphi(m)$

We now describe the LA-modules

Theorem 2 ([21]). *Let $\varphi : M \longrightarrow N$ be an LA-module homomorphism from an LA-module M to an LA-module N, then*

(1) *If A is an LA-submodule of M, then $\varphi(A)$ is an LA-submodule of N.*

(2) *If B is an LA-submodule of N, then $\varphi^{-1}(B)$ is an LA-submodule of M.*

Definition 9 ([21]). *Let M be an LA-module and $A \subset M$ is an LA-submodule. We define quotient module or factor module M/A by $M/A = \{A+m : m \in M\}$. That is, M/A is the set of equivalence classes of elements of M. An equivalence class is denoted by $A+m$ or by $[m]$. Each element in the class $A+m$ is called a representative of the class.*

Lemma 1 ([21]). *With the canonical operations, by choosing representatives, $(A+m)+(A+n) = A+(m+n)$, the set M/A is an LA-group. A, the equivalence class of $0 \in M$ is the left identity of M/A. The map $\pi : M \longrightarrow M/A$, $\pi(m) = A+m$ is surjective LA-group homomorphism.*

Definition 10 ([21]). *Let M be an LA-module over an LA-ring R. Let A and B be LA-submodules of M. Then M is said to be the internal direct sum of A and B, if every element $m \in M$ can be written in one and only one way as $m = a + b$, where $a \in A$ and $b \in B$. Symbolically, the direct sum is represented by the notation $M = A \oplus B$.*

Definition 11 ([25]). *Let $\{M_i \mid i \in I\}$ be a nonempty family of R-modules, $P = \prod_{i\in I} M_i = \{(x_i) \mid x_i \in M_i\}$ is a direct product set, if the operations on the product are given by $(x_i) + (y_i) = (x_i + y_i)$ and $r(x_i) = (rx_i)$, then P induce a left R module structure called direct product of $\{M_i \mid i \in I\}$, which is denoted by $\prod_{i\in I} M_i$.*

Proposition 1 ([25]). *Let $\{M_i \mid i \in I\}$ be a nonempty family of submodules of M. Then $\cap_{i\in I} M_i$ and $\sum_{i\in I} M_i$ are all submodules of M.*

Definition 12 ([25]). *All the elements (x_i) in the direct product $\prod_{i\in I} M_i$, where x_i is zero for almost all $i \in I$ except finite one, establish a submodule of $\prod_{i\in I} M_i$ which is called direct sum of $\{M_i \mid i \in I\}$, will be denoted by $\coprod_{i\in I} M_i$ or $\oplus_{i\in I} M_i$.*

For the LA-modules over an LA-ring, we state the following theorem.

Theorem 3 ([21]). *Let M be an LA-module over an LA-ring R. If A and B are LA-submodules of M, then M is the internal direct sum of A and B if and only if*

(1) $M = A + B$.

(2) $A \cap B = \{0\}$.

Definition 13 ([25]). *For a sequence of R-homomorphisms and R-modules $\cdots \longrightarrow M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \cdots$ is called an exact sequence if $Imf_{n-1} = kerf_n$ for all $n \in \mathbb{N}$. An exact sequence of the form $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is called a short exact sequence.*

By applying the above definition, we immediately obtain the following Proposition concerning the morphisms of modules.

Proposition 2 ([27]). *Let $f : M \longrightarrow N$ be R-homomorphism for R-modules M and N. Then*

(1) $0 \longrightarrow M \xrightarrow{f} N$ *is exact if and onlt if f is monomorphism.*

(2) $M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is epimorphism.*

(3) $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is an isomorphism.*

Theorem 4 ([26]). *Every left R module is a homomorphic image of free left R module.*

Proposition 3 ([26]). *For a short exact sequence $O \to A \xrightarrow{f} B \xrightarrow{g} C \to O$ of R-modules and homomorphisms, the following statements are equivalent:*

(1) *there exists a homomorphism $\alpha : B \to A$ such that $\alpha f = \perp_A$*

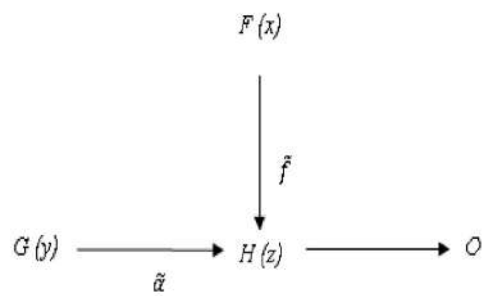
(2) *there exists a homomorphism $\beta : C \to B$ such that $g\beta = \perp_C$*

(3) *Imf is a direct summand of B.*

3. Projective soft LA-modules

We initiate in this section with the definition of projective soft LA-modules.

Definition 14. *Let M be a left LA-module over an LA-ring R. Then (F, A) is called a projective soft LA-module over M, if the given diagram of soft LA-modules and soft LA-homomorphisms with row exact, there exists a soft LA-homomorphism $\widetilde{g} : F(x) \to G(y)$ which makes the completed diagram commutative, that is $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$.*



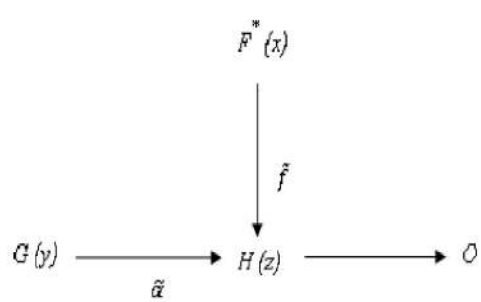
Definition 15. *A soft LA-module* (F^{*}, A) *over* M *is called a free soft LA-module on a basis* $(\bar{X}, A) \neq \phi$*, if there is a map* $\widetilde{\alpha} : \bar{X}(x) \to F^{*}(x)$ *such that given any map* $\widetilde{f} : \bar{X}(x) \to G^{*}(y)$*, where* (G^{*}, B) *is any soft LA-module, there exists a unique soft LA-homomorphism* $\widetilde{g} : F^{*}(x) \to G^{*}(y)$ *such that* $\widetilde{f} = \widetilde{g}\widetilde{\alpha}$*.*

Throughout this paper homomorphism is always considered as a soft LA-homomorphism.

The following result is a crucial result.

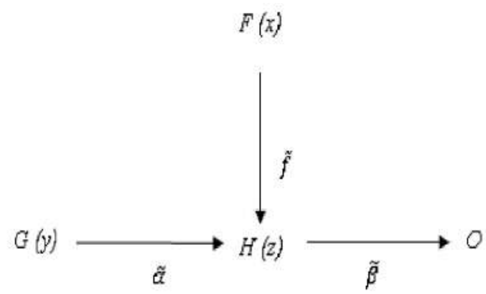
Theorem 5. *Every free soft LA-module is a projective soft LA-module.*

Proof. Let (F^{*}, A) be a free soft LA-module with basis $(\bar{X}, A)$. Let be a diagram of soft LA-modules and soft LA-homomorphism in which row is exact. Let $x \in \bar{X}(x)$ for all $x \in A$. Then $\widetilde{f}(x) \in H(z)$ for all $z \in C, x \in A$ and as $\widetilde{\alpha}$ is onto, so there exists $b \in G(y)$ for all $y \in B$ such that $\widetilde{\alpha}(b) = \widetilde{f}(x)$. Define $\widetilde{g} : \bar{X}(x) \to G(y)$, for all $x \in A$, for all $y \in B$ by $\widetilde{g}(x) = b$ and extend this function $\widetilde{g} : F^{*}(x) \to G(y)$ where $x \in A$, $y \in B$. It can be observed that $\widetilde{\alpha}\widetilde{g}(x) = \widetilde{\alpha}(\widetilde{g}(x)) = \widetilde{\alpha}(b) = \widetilde{f}(x)$. Hence it follows that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$. Therefore it is proved that every free soft LA-module is projective soft LA-module.

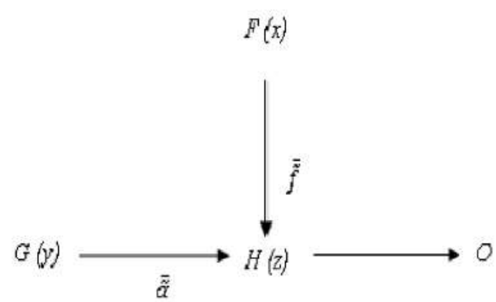


□

Proposition 4. *Let (F, A) be a projective soft LA-module. If the diagram of soft LA-modules and soft LA-homomorphisms the row is exact and $\widetilde{\beta}\widetilde{f} = 0$, then there exist a homomorphism $\widetilde{g} : F(x) \to G(y)$ for all $x \in A$ and $y \in B$ such that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$.*



Proof. Let $\bar{H} = Im\widetilde{\alpha} = ker\widetilde{\beta}$ and $\bar{\widetilde{\alpha}} : G(y) \to \bar{h}(z)$ be the homomorphism induced by $\widetilde{\alpha}$ where $y \in B$ and $z \in C$. So that $\widetilde{\beta}\widetilde{f} = 0$ this implies that $Im\widetilde{f}$ is contained in $ker\widetilde{\beta} = Im\widetilde{\alpha} = \bar{H}$. Therefore $\widetilde{f}$ induces a homomorphism $\bar{\widetilde{f}} : F(x) \to \bar{H}(z)$ where $z \in C$ and $x \in A$, such that if $\widetilde{i} : \bar{H}(z) \to H(z)$ is the inclusion map then $\widetilde{\alpha} = i\bar{\widetilde{\alpha}}$ and $\widetilde{f} = i\bar{\widetilde{f}}$. We have then a diagram in which row is exact. The soft LA-module $F(x)$ for all $x \in A$ is projective soft LA-module, so there exists a homomorphism $\widetilde{g} : F(x) \to G(y)$ such that $\bar{\widetilde{\alpha}}\widetilde{g} = \bar{\widetilde{f}}$ for $x \in A$ and $y \in B$. But $\widetilde{\alpha}\widetilde{g} = i\bar{\widetilde{\alpha}}\widetilde{g} = i\bar{\widetilde{f}} = \widetilde{f}$ .This implies that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$. Hence the theorem is proved.



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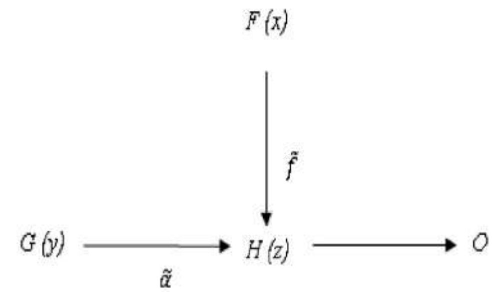
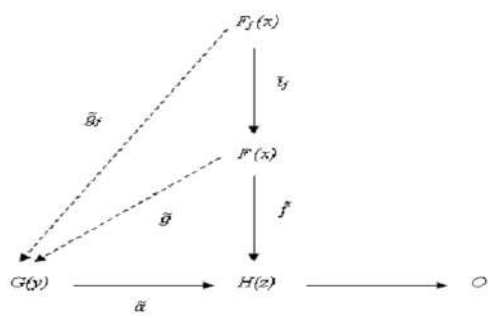
In the following Proposition, we give a characterization for the projective soft LA-modules.

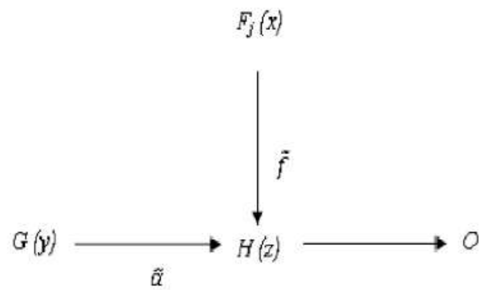
Proposition 5. *The soft LA-module (F, A) is projective soft LA-module if and only if (F_j, A) is a projective soft LA-module for every $j \in J$.*

Proof. Suppose that every (F_j, A) is a projective soft LA-module for every $j \in J$. Consider, a diagram with row exact. We have a soft LA-homomorphism $\widetilde{f}\,\widetilde{i}_j : F_j(x) \to H(z)$ for all $x \in A, z \in C$ and $j \in J$. $F_j(x)$ being projective soft LA-module, so there exists a homomorphism $\widetilde{g}_j : F_j(x) \to G(y)$ such that $\widetilde{\alpha}\widetilde{g}_j = \widetilde{f}\,\widetilde{i}_j$for all $x \in A, y \in B$ and $j \in J$. Now define $\widetilde{g} : F(x) \to G(y)$ for all $x \in A, y \in B$ by $\widetilde{g}(\varepsilon) = \sum_j \widetilde{g}_j \widetilde{\pi}_j(\varepsilon)$, for $\varepsilon \in F(x)$, $x \in A$.





It can be observed that the sum on the right hand side is finite. Then $\widetilde{g}$ is a soft homomorphism. Now to show projective soft LA-module, so for $\varepsilon \in F(x)$ where $x \in A$, $\widetilde{\alpha}\widetilde{g}(\varepsilon) = \widetilde{\alpha}(\sum_j \widetilde{g}_j \widetilde{\pi}_j(\varepsilon)) = \sum_j \widetilde{\alpha}\widetilde{g}_j \widetilde{\pi}_j(\varepsilon) = \sum_j \widetilde{f}\,\widetilde{i}_j \widetilde{\pi}_j(\varepsilon) = \widetilde{f}(\sum_j \widetilde{i}_j \widetilde{\pi}_j(\varepsilon)) = \widetilde{f}((\sum_j \widetilde{i}_j \widetilde{\pi}_j)(\varepsilon)) = \widetilde{f}(\varepsilon)$. Therefore it shows that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}$. Hence, we have proved that (F, A) is projective soft LA-module. Conversely, suppose that $F(x)$ for all $x \in A$ is projective soft LA-module. For any $j \in J$, consider a diagram with row exact. Then for all $x \in A$ and $z \in C$, $\widetilde{f}\widetilde{\pi}_j : F(x) \to H(z)$ is a homomorphism and $F(x)$ being projective soft LA-module, there exists a homomorphism $\widetilde{g} : F(x) \to G(y)$ such that $\widetilde{\alpha}\widetilde{g} = \widetilde{f}\widetilde{\pi}_j$where $x \in A$ and $y \in B$. Now let take $\widetilde{g}_j = \widetilde{g}\widetilde{i}_j$ which is a homomorphism from $F_j(x) \to G(y)$, then $\widetilde{\alpha}\widetilde{g}_j = \widetilde{\alpha}\widetilde{g}\widetilde{i}_j = \widetilde{f}\widetilde{\pi}_j\widetilde{i}_j = \widetilde{f}$. Hence it is proved that $F_j(x)$ is projective soft LA-modules.



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Definition 16. *A short exact sequence of the form* $O \to F(x) \overset{\widetilde{i}}{\to} G(y) \overset{\widetilde{j}}{\to} H(z) \to O$ *of soft LA-modules and soft homomorphism is said to splits or split sequence of soft LA-modules, if any of the following these condition holds*

(*i*) *there exists a homomorphism* $\widetilde{\gamma} : G(y) \to H(z)$ *such that* $\widetilde{\gamma}\widetilde{i} = \perp_{F(x)}$

(*ii*) *there exists a homomorphism* $\widetilde{\theta} : H(z) \to G(y)$ *such that* $\widetilde{j}\widetilde{\theta} = \perp_{H(z)}$

(*iii*) $Im\widetilde{i}$ *is a direct summand of* $G(y)$

where for all $x \in A, y \in B$ *and* $z \in C$.

We now state the following theorem concerning the soft LA-modules.

Theorem 6. *Every soft LA-module is a homomorphic image of a free soft LA-module.*

Proof. Let (F, A) be a soft LA-module over M. Let $(\bar{X}, B) \neq \phi$ be soft set of the elements of which are in one to one correspondence with the elements of (F, A). Let the elements of $(\bar{X}, B)$ corresponding to the element $\varepsilon \in (F, A)$ be denoted $\bar{x_\varepsilon}$. Let (F^*, B) be the free soft LA-module over N with basis $(\bar{X}, B)$. For all $x \in A$ and $y \in B$, let $\widetilde{f} \colon \bar{X}(y) \to F(x)$ be the map given by $\widetilde{f}(\bar{x_\varepsilon}) = \varepsilon$. Then by definition 15, for all $x \in A$ and $y \in B$, $\widetilde{g} \colon F^*(y) \to F(x)$ is the unique homomorphism which satisfies $\widetilde{g}(\bar{x_\varepsilon}) = \varepsilon = \widetilde{f}(\bar{x_\varepsilon})$. Thus $\widetilde{g}$ is an epimorphism and hence it is proved that (F, A) is a homomorphic image of (F^*, B). □

Remark 1. *Since every soft LA-module is homomorphic image of a free soft LA-module and every free soft LA-module is a projective soft LA-module, then we have the following lemma.*

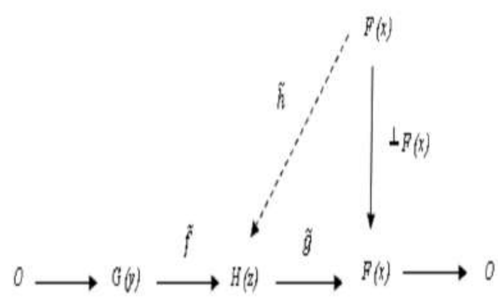
Lemma 2. *Every soft LA-module is a homomorphic image of a projective soft LA-module.*

Proof. The proof follows straightforwardly by theorem 6 and theorem 5. □

We now give below a characterization theorem of projective soft LA-modules.

Theorem 7. *A soft LA-module (F, A) is projective soft LA-module if and only if every exact sequence $O \to G(y) \to H(z) \to F(x) \to O$ splits for all $x \in A, y \in B$ and $z \in C$.*

Proof. Suppose that for all $x \in A$, $F(x)$ is a projective soft LA-module. Consider the following exact sequence, then by definition of projective soft LA-module, there exists a homomorphism $\widetilde{h} : F(x) \to H(z)$ where $x \in A$ and $z \in C$, such that $\widetilde{g}\widetilde{h} = \perp_{F(x)}$which shows that the sequence splits. Now conversely, suppose that the sequence of the form $O \to G(y) \to H(z) \to F(x) \to O$ splits. Since every soft LA-module being homomorphic image of a free soft LA-module. Let (F^*, D) being free soft LA-module and $\widetilde{\alpha} : F^*(t) \to F(x)$ be an epimorphism for all $t \in D$ and $x \in A$. If $G(y)$ denotes the kernel $\widetilde{\alpha}$, we get an exact sequence $O \to G(y) \overset{\widetilde{i}}{\to} F^*(t) \overset{\widetilde{\alpha}}{\to} F(x) \to O$ which by hypothesis splits. Thus $F^*(t) \cong F(x) \oplus G(y)$. The soft LA-module (F^*, D) being free is projective soft LA-module and hence this implies that $F(x)$ and $G(y)$ are projective. Hence, the theorem is proved.



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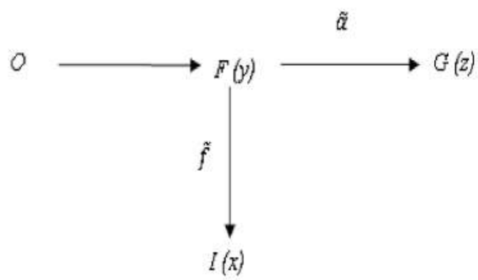
4. Injective soft LA-modules

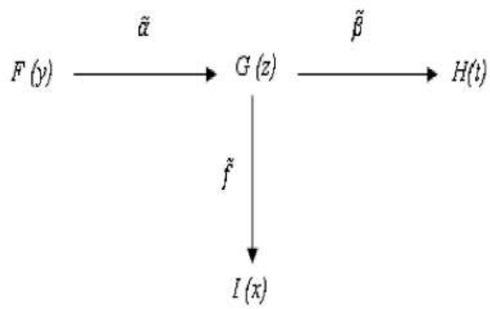
In this section, we define the injective soft LA-modules and establish some relevant results.

Definition 17. *Let (I, A) be a soft LA-module over M, (I, A) is called injective soft LA-module, if given diagram of soft LA-modules and soft LA-homomorphisms with row exact, then there exists a homomorphism $\widetilde{g} : G(z) \to I(x)$ for all $z \in C$ and $x \in A$, which makes the completed diagram commutative that is $\widetilde{g}\widetilde{\alpha} = \widetilde{f}$.*



For the injective soft LA-modules, we have the following Propositions.

Proposition 6. *If* (I, A) *is an injective soft LA-module, then given diagramwith row exact and* $\widetilde{f}\widetilde{\alpha} = 0$, *there exists a homomorphism* $\widetilde{g} : H(t) \to I(x)$ *such that the completed diagram is commutative* $\widetilde{g}\widetilde{\beta} = \widetilde{f}$ *for all* $x \in A$ *and* $t \in D$.

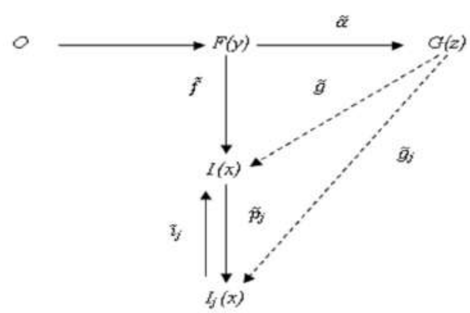


Proof. Let $X(z) \subseteq G(z) = ker\widetilde{\beta} = Im\widetilde{\alpha}$ where $z \in C$. Then $\widetilde{\beta}$ induces a monomorphism $\overline{\widetilde{\beta}} : G(z)/X(z) \to H(t)$ for all $t \in D$ and $z \in C$, given by $\overline{\widetilde{\beta}}(\varepsilon + X(z)) = \widetilde{\beta}(\varepsilon)$, where $\varepsilon \in G(z)$. Let $\overline{\widetilde{\beta}}(\varepsilon_1 + X(z)) = \overline{\beta}(\varepsilon_2 + X(z)) \Rightarrow \widetilde{\beta}(\varepsilon_1) = \widetilde{\beta}(\varepsilon_2) \Rightarrow \widetilde{\beta}(\varepsilon_1) - \widetilde{\beta}(\varepsilon_2) = 0 \Rightarrow \widetilde{\beta}(\varepsilon_1 - \varepsilon_2) = 0 \Rightarrow \varepsilon_1 - \varepsilon_2 \in ker\widetilde{\beta} = X(z) \Rightarrow \varepsilon_1 - \varepsilon_2 \in X(z) \Rightarrow \varepsilon_1 - \varepsilon_2 + X(z) = X(z) \Rightarrow \varepsilon_1 + X(z) = \varepsilon_2 + X(z)$, hence it shows that $\overline{\widetilde{\beta}}$ is a monomorphism. Also $\widetilde{f}\widetilde{\alpha} = 0 \Rightarrow \widetilde{f}(\widetilde{\alpha}(\varepsilon')) = 0(\varepsilon') = 0 \Rightarrow \widetilde{f}(\widetilde{\alpha}(\varepsilon')) = 0 \Rightarrow \widetilde{\alpha}(\varepsilon') \in ker\widetilde{f}$ but $\widetilde{\alpha}(\varepsilon') \in Im\widetilde{\alpha} \Rightarrow Im\widetilde{\alpha} \subseteq ker\widetilde{f} \Rightarrow X = Im\widetilde{\alpha} \subseteq ker\widetilde{f}$ and, therefore $\widetilde{f}$ induces a homomorphism $\overline{\widetilde{f}} : G(z)/X(z) \to I(x)$ for all $x \in A$ and $z \in C$, by $\overline{\widetilde{f}}(\varepsilon + X(z)) = \widetilde{f}(\varepsilon)$, where $\varepsilon \in G(z)$. Let $\widetilde{\pi} : G(z) \to G(z)/X(z)$ denotes the natural projection. Then $\overline{\widetilde{f}}\widetilde{\pi}(\varepsilon) = \overline{\widetilde{f}}(\widetilde{\pi}(\varepsilon)) = \overline{\widetilde{f}}(\varepsilon + X(z)) = \widetilde{f}(\varepsilon)$

for all $\varepsilon \in G(z)$, hence $\overset{\sim}{\bar{f}}\widetilde{\pi} = \widetilde{f}$ and $\overset{\sim}{\bar{\beta}}\widetilde{\pi}(\varepsilon) = \overset{\sim}{\bar{\beta}}(\widetilde{\pi}(\varepsilon)) = \overset{\sim}{\bar{\beta}}(\varepsilon + X(z)) = \widetilde{\beta}(\varepsilon)$ for all $\varepsilon \in G(z)$, therefore $\overset{\sim}{\bar{\beta}}\widetilde{\pi} = \widetilde{\beta}$. Since $I(x)$ is an injective soft LA-module, so there exists a homomorphism $\widetilde{g} : H(t) \to I(x)$ for all $t \in D$ and $x \in A$ such that $\overset{\sim}{\bar{f}} = \widetilde{g}\overset{\sim}{\bar{\beta}}$ then $\widetilde{g}\widetilde{\beta} = \widetilde{g}\overset{\sim}{\bar{\beta}}\widetilde{\pi} = \overset{\sim}{\bar{f}}\widetilde{\pi} = \widetilde{f}$. Hence proved. □

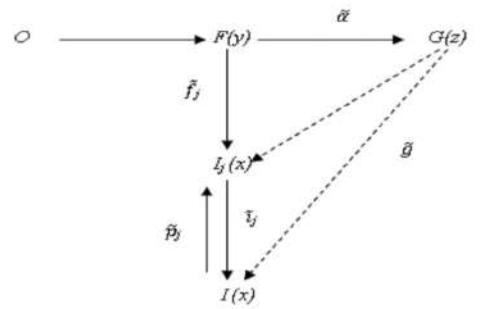
Proposition 7. *Let $(I_j, A)_{j\in J}$ be a family of soft LA-modules and $I(x) = \prod_{j\in J} I_j(x)$ (=direct product of I_j) for all $x \in A$. Then $I(x)$ is an injective soft LA-module if and only if $I_j(x)$ is an injective soft LA-module.*

Proof. Since for all $x \in A$, $I(x) = \prod_{j\in J} I_j(x)$, there exists a homomorphism $\widetilde{i}_j : I_j(x) \to I(x)$ and $\widetilde{p}_j : I(x) \to I_j(x)$ such that $\widetilde{p}_j\widetilde{i}_j = \perp_{I_j}$and $\widetilde{p}_k\widetilde{i}_j = 0$, the zero map if $j \neq k$. Let $\widetilde{\alpha} : F(y) \to G(z)$ for all $y \in B$ and $z \in C$ be a monomorphism of soft LA-modules. Suppose that every $I_j(x)$ for all $x \in A$, is an injective soft LA-module, by considering the following diagram.



Let $\widetilde{f} : F(y) \to I(x)$ be a homomorphism for $y \in B$ and $x \in A$, then $\widetilde{p}_j\widetilde{f} : F(y) \to I_j(x)$ is a homomorphism and as $I_j(x)$ being injective soft LA-module, then there exists a homomorphism $\widetilde{g}_j : G(z) \to I_j(x)$ such that $\widetilde{g}_j\widetilde{\alpha} = \widetilde{p}_j\widetilde{f}$, where $z \in C$ and $x \in A$. Now define $\widetilde{g} : G(z) \to I(x)$ by $\widetilde{g}(\varepsilon) = (\widetilde{g}_j(\varepsilon)), \varepsilon \in G(z)$. Then $\widetilde{g}$ is a homomorphism and for $\varepsilon' \in F(y)$ where $y \in B$, $\widetilde{g}(\widetilde{\alpha}(\varepsilon')) = (\widetilde{g}_j\widetilde{\alpha}(\varepsilon')) = \widetilde{p}_j\widetilde{f}(\varepsilon') = \widetilde{f}(\varepsilon')$, which shows that $\widetilde{g}\widetilde{\alpha} = \widetilde{f}$. Hence proved that $I(x)$ is an injective soft LA-module. Conversely, suppose that $I(x)$ is an injective soft LA-module. For any $j \in J$, let $\widetilde{f}_j : F(y) \to I_j(x)$ be a homomorphism where $y \in B$ and $x \in A$.

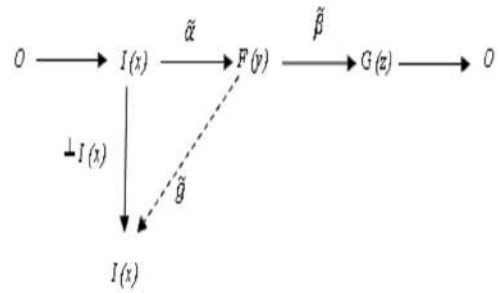
The soft LA-module $I(x)$ being injective soft LA-module, there exists a homomorphism $\widetilde{g} : G(z) \to I(x)$ where $z \in C$ and $x \in A$, such that $\widetilde{g}\widetilde{\alpha} = \widetilde{i}_j\widetilde{f}_j$. Then $\widetilde{p}_j\widetilde{g} : G(z) \to I_j(x)$ is a homomorphism such that $\widetilde{p}_j\widetilde{g}\widetilde{\alpha} = \widetilde{p}_j\widetilde{i}_j\widetilde{f}_j = \widetilde{f}_j$. Hence we have proved that $I_j(x)$ is an injective soft LA-module. □



Proposition 8. *If* (I, A) *is an injective soft LA-module, then every exact sequence of the form* $O \to I(x) \overset{\widetilde{\alpha}}{\to} F(y) \overset{\widetilde{\beta}}{\to} G(z) \to O$ *splits for every* $x \in A, y \in B$ *and* $z \in C$.

Proof. Consider the following diagram,

since $I(x)$ is an injective soft LA-module, so there exists a homomorphism $\widetilde{g} : F(y) \to I(x)$ such that $\widetilde{g}\widetilde{\alpha} = \perp_{I(x)}$. Hence we have proved that the exact sequence of the form $O \to I(x) \overset{\widetilde{\alpha}}{\to} F(y) \overset{\widetilde{\beta}}{\to} G(z) \to O$ splits.



□

5. Conclusion

Our paper can be regarded as a systematic study of soft LA-modules. We study soft LA-modules by giving the concepts of split sequence in soft LA-modules, free soft LA-modules, projective and injective soft LA-modules and their related properties. One can further develop the theory of homological algebra of soft LA-modules by defining functors, pullback and pushouts etc

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