Second type nabla Hukuhara differentiability for fuzzy functions on time scales

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Abstract. In this paper, we introduce a new class of derivative called second type nabla Hukuhara derivative for fuzzy functions on time scales under Hukuhara difference. We prove existence and uniqueness of this derivative and obtain its fundamental properties.
Keywords: fuzzy functions, time scales, Hukuhara difference, nabla Hukuhara derivative.

1. Introduction

In modelling a real world phenomenon, some vagueness or impreciseness occurs due to incomplete information about the parameters which we cannot exactly describes the behaviour of the problem. In order to deal with these impreciseness or vagueness Zadeh [29] introduced the theory called fuzzy sets. The fuzzy set theory is an excellent approach which helps us to deal with fuzzy dynamic models. Fuzzy set theory is the preliminary source to study fuzzy differential equations (Fde's) or interval differential equations. Fde's play a vital role in applications of biology, economics and many other engineering problems where uncertainty arises. Hukuhara [12] initiated the difference between two

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sets called Hukuhara difference and developed the theory of derivatives and integrals for set valued mappings. Later, Puri and Ralescu [21] studied Hukuhara derivative for fuzzy functions using Hukuhara difference and it is the primary approach for studying uncertainty of the dynamical systems. Further, Kaleva [14], studied Fde's under Hukuhara differentiability and also studied existence and uniqueness of the solutions to Fde's using Hukuhara derivative which has a disadvantage that the solutions exist only when the functions have an increasing length of support. To overcome this circumstance, Bede and Gal [6] studied generalizations of the differentiability of fuzzy number valued functions Later, Stefanini and Bede [22] studied generalized Hukuhara differentiability of interval value functions and interval differential equations. Further, Malinowski [19, 20] studied the concept of second type Hukuhara derivative for interval differential equations and interval cauchy problem with second type Hukuhara derivative. Furthermore, Zhang and Sun [28] studied stability of Fde's under second type Hukuhara derivative.

Time scales was initiated by the german mathematician Stefan Hilger [10]. For fundamental theory and applications on time scales calculus and dynamic equations on time scales are found in [1, 7]. For alternative solutions of linear dynamic equations on time scales and boundary value problems for dynamic equations on time scales were studied in [3] and [8]. The important features of time scales are extension, unification and generalization. The theory of time scale calculus is applicable to any field in which dynamic process described by continuous or discrete time models. If we take time scales as real numbers, then the derivative of a function is equal to standard differentiation while, if we take time scales as integers then it turns to backward difference operator or forward difference operator. In some recent studies and applications in economics [5], production, inventory models [4], adaptive control [13], neural networks [17] cellular neural networks [9] suggested nabla derivative is more preferable than delta derivative on time scales.

The multivalued functions on time scales under Hukuhara derivative was introduced in [11]. Hukuhara differentiability of interval-valued functions and interval differential equations on time scales was studied in [18]. Recently Vasavi et. al. [23, 24, 25, 26] introduced Hukuhara delta derivative, second type Hukuhara delta derivative and generalized Hukuhara delta derivatives using Hukuhara difference and studied fuzzy dynamic equations on time scales. With the importance and advantages of nabla derivative, we proposed to develop the theory of fuzzy nabla dynamic equations on time scales. In this context, we introduce second type nabla Hukuhara derivative for fuzzy functions on time scales and study their properties. The rest of this paper is organized as follows. In section 2, we present some definitions, properties, basic results relating to fuzzy sets, calculus of fuzzy functions and time scales calculus. Section 3 introduces second type nabla Hukuhara derivative of fuzzy functions on time scales and establish uniqueness, existence of the derivative and also obtain some properties.

2. Preliminaries

It is important to recall some basic results and definitions related to fuzzy calculus. Let $\Re_k(\Re^n)$ be the family of all convex compact nonempty subsets of $\Re^n$. Denote the set addition and scalar multiplication in $\Re_k(\Re^n)$ as usual. Then $\Re_k(\Re^n)$ satisfies the properties of commutative semigroup [14] under addition with cancellation laws. Further, if $\alpha, \beta \in \Re$ and $S, T \in \Re_k(\Re^n)$, then

$$\alpha(S+T) = \alpha S + \alpha T, \;\; \alpha(\beta S) = (\alpha\beta)S, \;\; 1.S = S,$$

and if $\beta, \alpha \geq 0$ then $(\beta+\alpha)S = \beta S + \alpha S$. Let S and T be two bounded nonempty subsets of $\Re^n$. By using the Pampeiu-Hausdorff metric we defined the distance between S and T as follows

$$d_H(S,T) = \max\{\sup_{s\in S}\inf_{t\in T}\|s-t\|, \sup_{t\in T}\inf_{s\in S}\|s-t\|\}$$

here $||.||$ is the Euclidean norm in $\Re^n$. Then $(\Re_k(\Re^n), d_H)$ becomes a seperable and complete metric space [14].

Define

$$\mathbb{E}_n = \{\mu : \Re^n \to [0,1]/\mu \text{ satisfies(a)-(d) below}\}, \text{ where}$$

(a) If $\exists$ a $t \in \Re^n$ such that $\mu(t) = 1$ then μ is said to be normal,

(b) μ is fuzzy convex,

(c) μ is upper semicontinuous,

(d) the closure of $\{t \in \Re^n/\mu(t) > 0\} = [\mu]^0$ is compact.

For $0 < \lambda \leq 1$, denote $[\mu]^\lambda = \{t \in \Re^n : \mu(t) \geq \lambda\}$, then from the above conditions we have that the λ-level set $[\mu]^\lambda \in \Re_k(\Re^n)$.

According to Zadeh's extension principle we define $g : \mathbb{E}_n \times \mathbb{E}_n \to \mathbb{E}_n$ by

$$g(p,q)(Z) = \sup_{Z=g(p,q)} \min\{p(x), q(y)\}.$$

We know that $[g(p,q)]^\lambda = g([p]^\lambda, [q]^\lambda)$, for all $p, q \in \mathbb{E}_n$ and g is a continuous function. The scalar multiplication $\odot$ and addition $\oplus$ of $p, q \in \mathbb{E}_n$ is defined as

$$[p \oplus q]^\lambda = [p]^\lambda + [q]^\lambda, [k \odot p]^\lambda = k[p]^\lambda, \text{ where } p, q \in \mathbb{E}_n, \; k \in \Re, \; 0 \leq \lambda \leq 1.$$

Theorem 2.1. *[14] If $\mu \in \mathbb{E}_n$, then*

(a) $[\mu]^\lambda \in \Re_k(\Re^n)$ *for all* $0 \leq \lambda \leq 1$,

(b) $[\mu]^{\lambda_2} \subset [\mu]^{\lambda_1}$ *for all* $0 \leq \lambda_1 \leq \lambda_2 \leq 1$,

(c) *If* $\lambda_k \in [0,1]$ *and* $\{\lambda_k\}$ *is a nondecreasing sequence converging to* $\lambda > 0$, *then*

$$[\mu]^\lambda = \bigcap_{k\geq 1} [\mu]^{\lambda_k}.$$

Conversely, if $\{X^\lambda / 0 \leq \lambda \leq 1\}$ *is a subsets of family of* $\Re^n$ *satisfying the above conditions from (a)-(c), then* $\exists a\, x \in \mathbb{E}_n \ni$

$$[\mu]^\lambda = X^\lambda, \quad forall \quad \lambda \in (0,1] \ and$$

$$[\mu]^0 = cl\left\{ \bigcup_{0<\lambda\leq 1} X^\lambda \right\} \subset X^0, \ here\ cl\ is\ the\ closure\ of\ the\ set.$$

Theorem 2.2 ([14])**.** *If sequence* $\{X_n\}$ *converges to* X *in* $\Re_k(\Re^n)$ *and* $d(X_n, X) \to 0$ *as* $n \to \infty$ *then*

$$X = \bigcap_{n\geq 1} cl\left\{ \bigcup_{m\geq n} X_m \right\}.$$

Define $D_H : \mathbb{E}_n \times \mathbb{E}_n \to [0,\infty)$ by

$$D_H(s,t) = \sup_{0\leq\lambda\leq 1} d_H([s]^\lambda, [t]^\lambda),$$

here d_H is the Pampeiu Hausdorff metric defined in $\Re_k(\Re^n)$. Then $(\mathbb{E}_n, D_H)$ is a complete metric space [14].

The following theorem extend the properties of addition and scalar multiplication of fuzzy number valued functions ($\Re_F = \mathbb{E}_1$) to $\mathbb{E}_n$.

Theorem 2.3 ([2])**.** (a) *If* $\tilde{0}$ *is the zero element in* $\Re_F$, *then* $\hat{0} = (\tilde{0}, \tilde{0}, \ldots, \tilde{0})$ *is the zero element in* $\mathbb{E}_n$. *i.e.* $s \oplus \hat{0} = \hat{0} \oplus s = s\ \forall s \in \mathbb{E}_n$;

(b) *For any* $s \in \mathbb{E}_n$ *has no inverse with respect to* '$\oplus$';

(c) *For any* $\beta, \gamma \in \Re$ *with* $\beta, \gamma \geq 0$ *or* $\beta, \gamma \leq 0$ *and* $s \in \mathbb{E}_n$, *then* $(\beta + \gamma) \odot s = (\beta \odot s) \oplus (\gamma \odot s)$;

(d) *For any* $\beta \in \Re$ *and* $s, t \in \mathbb{E}_n$, *we have* $\beta \odot (s \oplus t) = (\beta \odot s) \oplus (\beta \odot t)$;

(e) *For any* $\beta, \gamma \in \Re$ *and* $s \in \mathbb{E}_n$, *we have* $\beta \odot (\gamma \odot s) = (\beta\gamma) \odot s$.

Let $S, T \in \mathbb{E}_n$. If $\exists$ a $R \in \mathbb{E}_n$ such that $S = T \oplus R$ then we say that R is the H-difference (Hukuhara difference) of S and T and is denoted by $S \ominus_h T$. For any $S, T, R, U \in \mathbb{E}_n$ and $\alpha \in \Re$, the following holds

(a) $D_H(S,T) = 0 \Leftrightarrow S = T$;

(b) $D_H(\alpha S, \alpha T) = |\alpha| D_H(S,T)$;

(c) $D_H(S \oplus R, T \oplus R) = D_H(S,T)$;

(d) $D_H(S \ominus_h R, T \ominus_h R) = D_H(S,T)$;

(e) $D_H(S \oplus T, R \oplus U) \leq D_H(S,R) + D_H(T,U)$;

(f) $D_H(S \ominus_h T, R \ominus_h U) \leq D_H(S,R) + D_H(T,U)$.

provided the H-difference exists.

Now, we present some fundamental definitions and properties of Hukuhara derivative of fuzzy functions on the compact interval $I = [a,b]$, $a, b \in \Re$.

Definition 2.1 ([6]). *Let $I = [a,b] \subset \Re$ be a compact interval. A mapping $G : I \to \mathbb{E}_n$ is said to be Hukuhara form-I differentiable at $\theta \in I$ if $\exists$ a $G'(\theta) \in \mathbb{E}_n, \ni G(\theta + \hbar) \ominus_h G(\theta)$, $G(\theta) \ominus_h G(\theta - \hbar)$ exists for all $\hbar > 0$ sufficiently small and the limit*

$$\lim_{\hbar \to 0^+} \frac{G(\theta + \hbar) \ominus_h G(\theta)}{\hbar}, \lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta - \hbar)}{\hbar}.$$

exists in the topology of $\mathbb{E}_n$ and equal to $G'(\theta)$. The element $G'(\theta)$ is called the Hukuhara derivative of G at θ in the metric space $(\mathbb{E}_n, D_H)$. Consider only the one-sided derivatives at the end points of I.

Definition 2.2 ([6]). *A mapping $G : I \to \mathbb{E}_n$ is said to be Hukuhara form-II differentialble at $\theta \in I$, if $\exists$ a $G'(\theta) \in \mathbb{E}_n \ni G(\theta) \ominus_h G(\theta + \hbar), G(\theta - \hbar) \ominus_h G(\theta)$ exists, $\forall\, \hbar > 0$ sufficiently small $\ni$ the limit exists*

$$\lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta + \hbar)}{-\hbar}, \lim_{\hbar \to 0^+} \frac{G(\theta - \hbar) \ominus_h G(\theta)}{-\hbar}.$$

and are equal to $G'(\theta)$. Here $G'(\theta)$ is called the Hukuhara form-II derivative at θ.

Remark 2.1 ([14]). A function $G : I \to \mathbb{E}_n$ is said to be differentiable if the multivalued mapping $G_\lambda : I \to \Re_k(\Re^n)$ is Hukuhara differentiable for all $\lambda \in [0,1]$ and

$$[G_\lambda(\theta)]' = [G'(\theta)]^\lambda,$$

where $[G_\lambda]'$ is the H-derivative of G_λ.

Definition 2.3 ([14]). *A mapping $G : I \to \mathbb{E}_n$ is said to be strongly measurable if for each $\lambda \in [0,1]$, the fuzzy function $G_\lambda : I \to \Re_k(\Re^n)$ defined by $G_\lambda(\theta) = [G(\theta)]^\lambda$ is measurable.*

Remark 2.2 ([14]). If $\{\lambda_k\}$ is a nonincreasing sequence converges to 0 for all $x \in \mathbb{E}_n$, then

$$\lim_{k \to \infty} d_H([x]^0, [x]^{\lambda_k}) = 0.$$

Now, we present some fundamental definitions and results of time scales.

Definition 2.4 ([7]). *(a) Any nonempty closed subset of $\Re$ is defined as a time scale which is denoted by $\mathbb{T}$.*

(b) $\rho : \mathbb{T} \to \mathbb{T}$ is the backward jump operator and $\nu : \mathbb{T} \to \mathbb{R}^+$, the graininess operator are defined by

$$\rho(\theta) = \sup\{\theta_0 \in \mathbb{T} : \theta_0 < \theta\},\ \nu(\theta) = \theta - \rho(\theta)\ for\ \theta \in \mathbb{T}.$$

(c) The operator ρ is called left dense if $\rho(\theta) = \theta$, otherwise left scattered.

(d) $\mathbb{T}_k = \mathbb{T} - \{m\}$, if $\mathbb{T}$ has a right scattered minimum m. Otherwise $\mathbb{T}_k = \mathbb{T}$.

(e) A mapping $g^\rho : \mathbb{T} \to \Re$ defined by

$$g^\rho(\theta) = g(\rho(\theta))\ for\ each\ \theta \in \mathbb{T},$$

where $g : \mathbb{T} \to \Re$ is a function.

(f) The interval in time scale $\mathbb{T}$ is defined by

$$\mathbb{T}^{[a,b]} = \{\theta \in \mathbb{T} : a \le \theta \le b\} = [a,b] \cap \mathbb{T}$$

and

$$\mathbb{T}_k^{[a,b]} = \begin{cases} \mathbb{T}^{[a,b]}, & if\ a\ is\ right\ dense; \\ \mathbb{T}^{[\sigma(a),b]}, & if\ a\ is\ right\ scattered. \end{cases}$$

Definition 2.5 ([7]). *Let $g : \mathbb{T} \to \Re$ be a function and $\theta \in \mathbb{T}_k$. Then $g^\nabla(\theta)$ exists as a number provided for any given $\epsilon > 0$, $\exists$ a neighbourhood N_δ of θ (i.e., $N_\delta = (\theta - \delta, \theta + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[g(\rho(\theta)) - g(\theta_0)] - g^\nabla(\theta)[\rho(\theta) - \theta_0]| \le \epsilon|\rho(\theta) - \theta_0|,\ \ for\ all\ \ \theta_0 \in N_\delta,$$

Here, $g^\nabla(\theta)$ is called the nabla derivative of g at θ. Moreover, g is said to be nabla (or Hilger) differentiable on $\mathbb{T}_k$, if $g^\nabla(\theta)$ exists $\forall\ \theta \in \mathbb{T}_k$. The function $g^\nabla : \mathbb{T}_k \to \Re$ is then called the nabla derivative of g on $\mathbb{T}_k$.

Definition 2.6 ([7]). *A mapping $g : \mathbb{T} \to \Re$ is said to be regulated if its left sided limits exists and are finite at all ld-point (left dense points) in $\mathbb{T}$ and its right sided limits exists and are finite at all rd-points (right dense points) in $\mathbb{T}$*

Definition 2.7 ([7]). *Let $g : \mathbb{T} \to \Re$ be a function. g is said to be ld-continuous, if it is continuous at each ld-point in $\mathbb{T}$ and $\lim\limits_{\theta_0 \to \theta^+} g(\theta)$ exists as a finite number for all rd-points in $\mathbb{T}$.*

Lemma 2.1 ([7]). *Let $G : \mathbb{T} \to \Re$.*

(a) If g is ∇-differentiable at $\mathbb{T}$, then g is continuous at θ.

(b) If g is continuous at θ and θ is left scattered, then g is ∇-differentiable and $g^\nabla(\theta) = \frac{g(\theta) - g(\rho(\theta))}{\nu(\theta)}$.

(c) If g is ∇-differentiable at θ, then $g(\rho(\theta)) = g(\theta) + (-1)\nu(\theta)g^\nabla(\theta)$.

3. Nabla Hukuhara differentiability

In this section, first we introduce second type nabla Hukuhara derivative of fuzzy functions on time scales. Later, we establish uniqueness and existence of this derivative and obtain some properties on second type nabla Hukuhara derivative. For further discussion, we use the following notation: for some $\delta > 0$, we define the neighbourhood of $\theta \in \mathbb{T}^{[a,b]}$ by $N_{\mathbb{T}^{[a,b]}}(\theta, \delta) = (\theta - \delta, \theta + \delta) \cap \mathbb{T}^{[a,b]} = N_{\mathbb{T}^{[a,b]}}$

Definition 3.1. *For any given $\epsilon > 0$ $\exists$ a $\delta > 0$, such that the fuzzy function $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ has a unique $\mathbb{T}$-limit $P \in \mathbb{E}_n$ at $\theta \in \mathbb{T}^{[a,b]}$ if $D_H(G(\theta) \ominus_h P, \hat{0}) \leq \epsilon$, for all $\theta \in N_{\mathbb{T}^{[a,b]}}(\theta, \delta)$ and it is denoted by $\mathbb{T} - \lim_{\theta \to \theta_0} G(\theta)$.*

Here $\mathbb{T}$-limit denotes the limit on time scale in the metric space $(\mathbb{E}_n, D_H)$.

Remark 3.1. From the above definition, we have

$$\mathbb{T} - \lim_{\theta \to \theta_0} G(\theta) = P \in \mathbb{E}_n \Longleftrightarrow \mathbb{T} - \lim_{\theta \to \theta_0} (G(\theta) \ominus_h P) = \hat{0},$$

where $\hat{0}$ is the zero element in $\mathbb{E}_n$.

Definition 3.2. *A fuzzy mapping $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is continuous at $\theta_0 \in \mathbb{T}^{[a,b]}$, if $\mathbb{T} - \lim_{\theta \to \theta_0} G(\theta) \in \mathbb{E}_n$ exists and $\mathbb{T} - \lim_{\theta \to \theta_0} G(\theta) = G(\theta_0)$, i.e.*

$$\mathbb{T} - \lim_{\theta \to \theta_0} (G(\theta) \ominus_h G(\theta_0)) = \hat{0}.$$

Remark 3.2. If $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is continuous at $\theta \in \mathbb{T}^{[a,b]}$, then for every $\epsilon > 0$, $\exists$ a $\delta > 0$, such that

$$D_H(G(\theta) \ominus_h G(\theta_0), \hat{0}) \leq \epsilon, \text{ for all } \theta \in N_{\mathbb{T}^{[a,b]}}.$$

Remark 3.3. Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ and $\theta_0 \in \mathbb{T}^{[a,b]}$.

(a) If $\mathbb{T} - \lim_{\theta \to \theta_0^+} G(\theta) = G(\theta_0)$, then G is said to be right continuous at θ_0.

(b) If $\mathbb{T} - \lim_{\theta \to \theta_0^-} G(\theta) = G(\theta_0)$, then G is said to be left continuous at θ_0.

(c) If $\mathbb{T} - \lim_{\theta \to \theta_0^+} G(\theta) = G(\theta_0) = \mathbb{T} - \lim_{\theta \to \theta_0^-} G(\theta)$, then G is continuous at θ_0.

Definition 3.3 ([16])**.** *Suppose $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be a fuzzy function and $\theta \in \mathbb{T}_k^{[a,b]}$. Let $G^{\nabla_h}(\theta)$ be an element of $\mathbb{E}_n$ exists provided for any given $\epsilon > 0$, $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ of θ and for some $\delta > 0$ such that*

$$\begin{aligned} &D_H[(G(\theta + \hbar) \ominus_h G(\rho(\theta)), (\hbar + \nu(\theta)) \odot G^{\nabla_h}(\theta)] \leq \epsilon|\hbar + \nu(\theta)|, \\ &D_H[(G(\rho(\theta)) \ominus_h G(\theta - \hbar), (\hbar - \nu(\theta)) \odot G^{\nabla_h}(\theta)] \leq \epsilon|\hbar - \nu(\theta)|, \end{aligned} \tag{1}$$

for all $(\theta-\hbar,\theta+\hbar)\in N_{\mathbb{T}^{[a,b]}}$ *with* $0<h<\delta$ *where* $\nu(\theta)=\theta-\rho(\theta)$. *Then* G *is called nabla Hukuhara form-I (nabla-h) differentiable at* θ *and is denoted by* $G^{\nabla_h}(\theta)$.

or

A fuzzy function $G:\mathbb{T}^{[a,b]}\to\mathbb{E}_n$ *is said to be nabla-h differentiable at* $\theta\in\mathbb{T}_k^{[a,b]}$ *if* $\exists$ *a* $G^{\nabla_h}(\theta)\in\mathbb{E}_n$ *such that the limits*

$$\mathbb{T}-\lim_{\hbar\to0^+}\frac{G(\theta+\hbar)\ominus_h G(\rho(\theta))}{\hbar+\nu(\theta)}\ \&\ \mathbb{T}-\lim_{\hbar\to0^+}\frac{G(\rho(\theta))\ominus_h G(\theta-\hbar)}{\hbar-\nu(\theta)}$$

exists and are equal to $G^{\nabla_h}(\theta)$.

Moreover, if nabla-h derivative exists for each $\theta\in\mathbb{T}_k^{[a,b]}$, *then* G *is nabla-h differentiable on* $\mathbb{T}_k^{[a,b]}$. *We consider only right limit at left scattered points and one-sided limit at the end points of* $\mathbb{T}_k^{[a,b]}$.

The above definition does not exists if the fuzzy function has decreasing diameter. So, in order to overcome this circumstance we introduce and study the second type nabla Hukuhara derivative for fuzzy functions on time scales where the results exist for the functions which are having decreasing diameter.

Definition 3.4. *Let* $G:\mathbb{T}^{[a,b]}\to\mathbb{E}_n$ *is a fuzzy function and* $\theta\in\mathbb{T}_k^{[a,b]}$. *Let* $G^{\nabla^{sh}}(\theta)$ *be an element of* $\mathbb{E}_n$ *exists provided for any given* $\epsilon>0$, $\exists$ *a neighbourhood* $N_{\mathbb{T}^{[a,b]}}$ *of* θ *and for some* $\delta>0$ *such that*

$$D_H[(G(\rho(\theta))\ominus_h G(\theta+\hbar)),-(\hbar+\nu(\theta))\odot G^{\nabla^{sh}}(\theta)]\le\epsilon|-(\hbar+\nu(\theta))|,\tag{2}$$

$$D_H[(G(\theta-\hbar)\ominus_h G(\rho(\theta)),-(\hbar-\nu(\theta))\odot G^{\nabla^{sh}}(\theta)]\le\epsilon|-(\hbar-\nu(\theta))|,\tag{3}$$

for all $(\theta-\hbar,\theta+\hbar)\in N_{\mathbb{T}^{[a,b]}}$ *with* $0<h<\delta$ *where* $\nu(\theta)=\theta-\rho(\theta)$. *Then* G *is called second type nabla Hukuhara form-II differentiable (*∇^{sh}*-differentiable) at* θ *and is denoted by* $G^{\nabla^{sh}}(\theta)$.

or

A fuzzy function $G:\mathbb{T}^{[a,b]}\to\mathbb{E}_n$ *is* ∇^{sh}*-differentiable at* $\theta\in\mathbb{T}_k^{[a,b]}$ *if* $\exists$ *a* $G^{\nabla^{sh}}(\theta)\in\mathbb{E}_n$ *such that the limits*

$$\mathbb{T}-\lim_{\hbar\to0^+}\frac{G(\rho(\theta))\ominus_h G(\theta+\hbar)}{-(\hbar+\nu(\theta))}\ \&\ \mathbb{T}-\lim_{\hbar\to0^+}\frac{G(\theta-\hbar)\ominus_h G(\rho(\theta))}{-(\hbar-\nu(\theta))}$$

exists and are equal to $G^{\nabla^{sh}}(\theta)$. *Moreover, if* ∇^{sh}*-derivative exists for each* $\theta\in\mathbb{T}_k^{[a,b]}$, *then* G *is* ∇^{sh}*-differentiable on* $\mathbb{T}_k^{[a,b]}$. *We consider only right limit at left scattered points and one-sided limit at the end points of* $\mathbb{T}_k^{[a,b]}$.

Note. If both $\mathbb{T}$-limits exists at left scattered point, then the ∇^{sh}-derivative is in $\Re^n$ (crisp). It will restrict the ∇^{sh}-differentiability of fuzzy functions on time scales having left scattered points. To avoid this, we consider only right limit at left scattered points.

Example 3.1. Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_1$ be a function defined by $G(\theta) = \frac{1}{\theta^2} \odot n$, where $n = (1, 3, 5)$ is a fuzzy number.

If $\mathbb{T} = \Re$, then from Definition 3.3 $G : \Re \to \mathbb{E}_n$ is not ∇_h-differentiable at $\theta \in \Re$. Since the H-difference $G(\theta) \ominus_h G(\theta - \hbar), G(\theta + \hbar) \ominus_h G(\theta)$ does not exists. Since the H-difference $G(\theta) \ominus_h G(\theta + \hbar), G(\theta - \hbar) \ominus_h G(\theta)$ exists. Therefore, from Definition 3.4, G is ∇^{sh}-differentiable and we have

$$
\begin{aligned}
G^{\nabla^{sh}}(\theta) &= \lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta + \hbar))}{-\hbar} \\
&= \lim_{\hbar \to 0} \frac{G(\theta - \hbar) \ominus_h G(\theta)}{-\hbar} \\
&= \frac{-2}{\theta^3} \odot (1, 3, 5) = \frac{1}{\theta^3} \odot (-10, -6, -2).
\end{aligned}
$$

Lemma 3.1. *If G is ∇^{sh}-differentiable at θ, then ∇^{sh}- derivative exists and it is unique.*

Proof. Suppose that $G^{\nabla^{sh}_1}(\theta)$ and $G^{\nabla^{sh}_2}(\theta)$ are ∇^{sh}-derivatives of G at θ. Then

$$
D_H[-(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_1}(\theta), G(\rho(\theta)) \ominus_h G(\theta + \hbar)] \leq \frac{\epsilon}{2} |-(\hbar + \nu(\theta))|,
$$

$$
D_H[-(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_2}(\theta), G(\rho(\theta) \ominus_h G(\theta + \hbar))] \leq \frac{\epsilon}{2} |-(\hbar + \nu(\theta))|.
$$

Consider

$$
\begin{aligned}
&D_H[G^{\nabla^{sh}_1}(\theta), G^{\nabla^{sh}_2}(\theta)] \\
&= \frac{1}{|-(\hbar + \nu(\theta))|} \Big(D_H[-(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_1}(\theta), -(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_2}(\theta)] \Big) \\
&\leq \frac{1}{|-(\hbar + \nu(\theta))|} \Big(D_H[-(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_1}(\theta), G(\rho(\theta) \ominus_h G(\theta + \hbar)] \\
&+ D_H[G(\rho(\theta)) \ominus_h G(\theta + \hbar), -(\hbar + \nu(\theta)) \odot G^{\nabla^{sh}_2}(\theta)] \Big) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \ \forall \, |-(\hbar + \nu(\theta))| \neq 0.
\end{aligned}
$$

Since $\epsilon > 0$, then $D_H[G^{\nabla^{sh}_1}(\theta), G^{\nabla^{sh}_2}(\theta)] = 0$. Therefore, $G^{\nabla^{sh}_1}(\theta) = G^{\nabla^{sh}_2}(\theta)$. Hence ∇^{sh}-derivative exists and is unique. □

Theorem 3.1. *If $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at θ, then G is continuous when θ is left dense and right continuous when θ is left scattered.*

Proof. Assume that $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$. Let $\epsilon^1 \in (0,1)$. Define

$$\epsilon^1 = \epsilon[1 + \|G^{\nabla^{sh}}(\theta)\|]^{-1}.$$

Since G is ∇^{sh}-differentiable $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ such that

$$\begin{aligned} &D_H[(G(\rho(\theta)) \ominus_h G(\theta+\hbar), -(\hbar+\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon| - (\hbar+\nu(\theta))|, \\ &D_H[(G(\theta-\hbar) \ominus_h G(\rho(\theta)), -(\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon| - (\hbar-\nu(\theta))|, \end{aligned}$$

for all $\hbar \ge 0$ with $(\theta-\hbar, \theta+\hbar) \in \mathbb{N}_T^{[a,b]}$. Therefore, for all $(\theta-\hbar, \theta+\hbar) \in N_T^{[a,b]} \cap (\theta-\epsilon, \theta+\epsilon)$ with $0 \le \hbar < \epsilon$.

$$\begin{aligned} D_H[G(\theta-\hbar), G(\theta)] &= D_H[G(\theta-\hbar) \ominus_h G(\theta), \hat{0}] \\ &= D_H[G(\theta-\hbar) \ominus_h G(\rho(\theta)) + G(\rho(\theta)) \ominus_h G(\theta), \\ &\quad - (\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta) - \nu(\theta) \odot G^{\nabla^{sh}} + \hbar G^{\nabla^{sh}}(\theta)] \\ &\le D_H[G(\theta-\hbar) \ominus_h G(\rho(\theta)), -(\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \\ &\quad + D_H[G(\rho(\theta)) \ominus_h G(\theta), (-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \\ &\quad + D_H[\hbar G^{\nabla^{sh}}(\theta), \hat{0}] \\ &\le \epsilon^1| - (\hbar-\nu(\theta)| + \epsilon^1|(-\nu(\theta)))| + \hbar\|G^{\nabla^{sh}}(\theta)\| \\ &< \epsilon^1(1 + \|G^{\nabla^{sh}}(\theta)\|) \\ &= \epsilon. \end{aligned}$$

Therefore, for θ being left dense or left scattered

$$\mathbb{T} - \lim_{\hbar \to 0^+} G(\theta-\hbar) = G(\theta).$$

For left dense point θ, it is easy to prove that

$$\mathbb{T} - \lim_{\hbar \to 0^+} G(\theta+\hbar) = G(\theta).$$

Hence G is continuous at left dense points and right continuous at left scattered points in $\mathbb{T}_k^{[a,b]}$. □

Theorem 3.2. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be right continuous at θ, θ is left-scattered then G is ∇^{sh}-differentiable at θ and*

$$G^{\nabla^{sh}}(\theta) = \frac{-1}{\nu(\theta)} \odot \left(G(\rho(\theta)) \ominus_h G(\theta)\right).$$

Proof. Let θ be left-scattered and since G is right continuous, then by Theorem 3.1, we have

$$G^{\nabla^{sh}}(\theta) = \mathbb{T} - \lim_{\hbar \to 0^+} \frac{G(\rho(\theta)) \ominus_h G(\theta+\hbar)}{-(\hbar+\nu(\theta))} = \frac{-1}{\nu(\theta)} \odot \left(G(\rho(\theta)) \ominus_h G(\theta)\right).$$

□

Theorem 3.3. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be fuzzy function and let $\theta \in \mathbb{T}_k^{[a,b]}$. If θ is left dense, then G is ∇^{sh}-differentiable at $\mathbb{T}_k^{[a,b]}$ if and only if the limits exists and are equal to $G^{\nabla^{sh}}(\theta)$ i.e.,*

$$\lim_{\hbar\to 0^+} \frac{-1}{h} \odot (G(\theta) \ominus_h G(\theta+\hbar)) = \lim_{\hbar\to 0^+} \frac{-1}{h} \odot (G(\theta-\hbar) \ominus_h G(\theta)) = G^{\nabla^{sh}}(\theta).$$

Proof. Suppose that G is ∇^{sh}-differentiable at θ and θ is ld-point. Since G is ∇^{sh}-differentiable at θ, for any given $\epsilon > 0$, $\exists\ N_{\mathbb{T}^{[a,b]}}$ a neighbourhood of $\theta\ \ni$

$$D_H[(G(\rho(\theta)) \ominus_h G(\theta+\hbar), -(\hbar+\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon|-(\hbar+\nu(\theta))|,$$

$$D_H[G(\theta-\hbar) \ominus_h G(\rho(\theta)), -(\hbar-\nu(\theta)) \odot G^{\nabla^{sh}}(\theta)] \le \epsilon|-(\hbar-\nu(\theta))|,$$

for all, $0 < \hbar < \delta$ with $(\theta-\hbar, \theta+\hbar) \in N_{\mathbb{T}^{[a,b]}}$. Since $\rho(\theta) = \theta$, i.e. $\nu(\theta) = 0$,

$$D_H[(G(\theta) \ominus_h G(\theta+\hbar), -\hbar \odot G^{\nabla^{sh}}(\theta)] \le \epsilon\hbar,$$

$$D_H[(G(\theta-\hbar) \ominus_h G(\theta), -\hbar \odot G^{\nabla^{sh}}(\theta)] \le \epsilon\hbar,$$

for all, $0 < \hbar < \delta$ with $(\theta-\hbar, \theta+\hbar) \in N_{\mathbb{T}^{[a,b]}}$. This implies that

$$D_H\left[\frac{G(\theta) \ominus_h G(\theta+\hbar)}{-\hbar}, G^{\nabla^{sh}}(\theta)\right] \le \epsilon,$$

$$D_H\left[\frac{G(\theta-\hbar) \ominus_h G(\theta)}{-\hbar}, G^{\nabla^{sh}}(\theta)\right] \le \epsilon,$$

for all $0 < \hbar < \delta$ with $(\theta-\hbar, \theta+\hbar) \in N_{\mathbb{T}^{[a,b]}}$. Since ϵ is arbitrary, we have

$$\lim_{\hbar\to 0^+} \frac{-1}{\hbar} \odot (G(\theta) \ominus_h G(\theta+\hbar)) = \lim_{\hbar\to 0^+} \frac{-1}{\hbar} \odot (G(\theta-\hbar) \ominus_h G(\theta)) = G^{\nabla^{sh}}(\theta)$$

Conversely, suppose that for all $0 < \hbar < \delta$ with $(\theta-\hbar, \theta+\hbar) \in N_{\mathbb{T}}$, $\exists$ a neighbourhood $N_{\mathbb{T}^{[a,b]}}$ of θ and θ is left dense such that

$$D_H\left[\frac{G(\theta) \ominus_h G(\theta+\hbar)}{-\hbar}, G^{\nabla^{sh}}(\theta)\right] \le \epsilon,$$
$$D_H\left[\frac{G(\theta-\hbar) \ominus_h G(\theta)}{-\hbar}, G^{\nabla^{sh}}(\theta)\right] \le \epsilon.$$

From the above inequalities, G is ∇^{sh}-differentiable at θ and since θ is ld-point, we have $G^{\nabla^{sh}}(\theta) = G'(\theta)$. □

Theorem 3.4. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ be $\nabla^{sh}-$differentiable and $\theta \in \mathbb{T}_k^{[a,b]}$. Then*

$$G(\rho(\theta)) = G(\theta) \oplus (-1)\nu(\theta)G^{\nabla^{sh}}(\theta).$$

or

$$G(\theta) = G(\rho(\theta)) \ominus_h (-1)(\nu(\theta)G^{\nabla^{sh}}(\theta).$$

Proof. (a) If θ is left dense then $\rho(\theta) = \theta$ and $\nu(\theta) = 0$. Hence

$$G(\rho(\theta)) = G(\theta) \oplus (-1)\nu(\theta)G^{\nabla^{sh}}(\theta).$$

or

$$G(\theta) = G(\rho(\theta)) = G(\rho(\theta)) \ominus_h (-1)\nu(\theta)G^{\nabla^{sh}}(\theta).$$

(b) If θ is left-scattered then $\rho(\theta) < \theta$. From Theorem 3.2 we have

$$G^{\nabla^{sh}}(\theta) = \frac{-1}{\nu(\theta)} \odot [G(\rho(\theta)) \ominus_h G(\theta)] \Rightarrow (-1)\nu(\theta) \odot G^{\nabla^{sh}}(\theta) = G(\rho(\theta)) \ominus_h G(\theta).$$

Thus,

$$G(\rho(\theta)) = G(\theta) \oplus (-1)\nu(\theta)G^{\nabla^{sh}}(\theta)$$

or

$$G(\theta) = G(\rho(\theta)) \ominus_h (-1)(\nu(\theta)G^{\nabla^{sh}}(\theta).$$

□

Example 3.2. Let us consider $\mathbb{T} = \Re$ or $\mathbb{T} = t\mathbb{Z} = \{tk : k \in \mathbb{Z}\}$.

(a) If $\mathbb{T} = \Re$, then from Theorem 3.3 $G : \Re \to \mathbb{E}_n$ is ∇^{sh}-differentiable at $\theta \in \Re$ iff

$$G^{\nabla^{sh}}(\theta) = \lim_{\hbar \to 0^+} \frac{G(\theta - \hbar) \ominus_h G(\theta)}{\hbar} = \lim_{\hbar \to 0} \frac{G(\theta) \ominus_h G(\theta + \hbar)}{\hbar} = G^{'}(\theta).$$

(b) If $\mathbb{T} = t\mathbb{Z}$, then every point $\theta \in \mathbb{T}$ is isolated and

$$\rho(\theta) = \sup\{\theta - nt : n \in \mathbb{N}\} = \theta - t,$$

$$\nu(\theta) = \theta - \rho(\theta) = \theta - (\theta - t) = t.$$

From Theorem 3.2 $G : t\mathbb{Z} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at $\theta \in t\mathbb{Z}$ and

$$G^{\nabla^{sh}}(\theta) = \frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)} = \frac{G(\theta - t) \ominus_h G(\theta)}{-t} = \frac{-1}{t} \odot \Delta G(\theta),$$

where Δ is the forward Hukuhara difference operator.

Theorem 3.5. *Denote* $[G(\theta)]^\lambda = G_\lambda(\theta)$ *for each* $\lambda \in [0, 1]$, *where* $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ *be the fuzzy function and if* G *is* ∇^{sh}*-differentiable, then* G_λ *is also* ∇^{sh}*-differentiable and*

$$[G^{\nabla^{sh}}(\theta)]^\lambda = G_\lambda^{\nabla^{sh}}(\theta).$$

Proof. If θ is left scaterred and G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$, then from Theorem 3.2, we get

$$[G^{\nabla^{sh}}(\theta)]^\lambda = \frac{[G(\rho(\theta))]^\lambda \ominus_h [G(\theta)]^\lambda}{-\nu(\theta)} = \frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)} = G_\lambda^{\nabla^{sh}}(\theta),$$

for each $\lambda \in [0,1]$. If G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ and θ is left dense, then for $\lambda \in [0,1]$, we get

$$[G(\theta - \hbar) \ominus_h G(\theta)]^\lambda = [G_\lambda(\theta - \hbar) \ominus_h G_\lambda(\theta)]$$

and multiplying by $\frac{-1}{\hbar} < 0$ and taking the limit $\hbar \to 0^+$, we have

$$\lim_{\hbar \to 0^+} \frac{-1}{\hbar} \odot [G_\lambda(\theta - \hbar) \ominus_h G_\lambda(\theta)] = G_\lambda^{\nabla^{sh}}(\theta).$$

Similarly, we can prove

$$\lim_{\hbar \to 0^+} \frac{-1}{\hbar} \odot \big[G_\lambda(\theta) \ominus_h G_\lambda(\theta + \hbar)\big] = G_\lambda^{\nabla^{sh}}(\theta).$$

Therefore, from Theorem 3.3, we get $[G^{\nabla^{sh}}(\theta)]^\lambda = G_\lambda^{\nabla^{sh}}(\theta)$. □

Remark 3.4. The above Theorem 3.5, states that if G is ∇^{sh} differentiable then the multivalued mapping G_λ is ∇^{sh}-differentiable for all $\lambda \in [0,1]$, but the converse of the theorem need not be true. That is the existence of H-differences of λ-level sets $[p]^\lambda \ominus_h [q]^\lambda$ does not imply the existence of H-difference of $p \ominus_h q$.

However, for the converse of the theorem we have the following:

Theorem 3.6. *Suppose that $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ satisfy the following conditions:*

(1) *For each $\theta \in \mathbb{T}^{[a,b]}$ and θ is left dense*

  (a) *$\exists$ a $\beta > 0$, $\ni$ the Hukuhara differences $G(\theta - \hbar) \ominus_h G(\theta)$ and $G(\theta) \ominus_h G(\theta + \hbar)$ exists for all $0 < \hbar < \beta$ and for all $\theta - \hbar, \theta + \hbar \in N_{\mathbb{T}^{[a,b]}}$;*

  (b) *the fuzzy mappings G_λ, $\lambda \in [0,1]$, are uniformly ∇^{sh}-differentiable with derivative $G_\lambda^{\nabla^{sh}}$, i.e., to each $\theta \in \mathbb{T}^{[a,b]}$ and $\epsilon > 0$ $\exists$ a $\delta > 0$ such that*

$$D_H \left\{ \frac{G_\lambda(\theta - \hbar) \ominus_h G_\lambda(\rho(\theta))}{-(\hbar - \nu(\theta))}, G_\lambda^{\nabla^{sh}}(\theta) \right\} < \epsilon,$$
$$D_H \left\{ \frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta + \hbar)}{-(\hbar + \nu(\theta))}, G_\lambda^{\nabla^{sh}}(\theta) \right\} < \epsilon,$$

  for all $0 < \hbar < \delta$, $\theta - \hbar, \theta + \hbar \in N_T^{[a,b]}$, $\lambda \in [0,1]$.

(2) *for each $\theta \in \mathbb{T}^{[a,b]}$ and θ is left scattered*

  (a) *the Hukuhara differences $G(\rho(\theta)) \ominus_h G(\theta)$ exists and;*

(b) *the fuzzy mappings* G_λ, $\lambda \in [0,1]$, *are uniformly nabla*^{sh}*-differentiable with derivative* $G_\lambda^{\nabla^{sh}}$, *i.e., to each* $\theta \in \mathbb{T}^{[a,b]}$ *and* $\epsilon > 0 \ \exists$ *a* $\delta > 0$ *such that*

$$D_H\left\{\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}, G_\lambda^{\nabla^{sh}}(\theta)\right\} < \epsilon. \tag{4}$$

Then G *is* ∇^{sh}*-differentiable and its derivative is given by* $G_\lambda^{\nabla^{sh}}(\theta) = [G^{\nabla^{sh}}(\theta)]^\lambda$.

Proof. Case (1): For θ being left dense points in $\mathbb{T}^{[a,b]}$, then the proof is similar to the proof of Theorem 5.1 [14].

Case(2): For θ being left scattered points in $\mathbb{T}^{[a,b]}$, consider $\{G_\lambda^{\nabla^{sh}}(\theta), \lambda \in [0,1]\}$, where $G_\lambda^{\nabla^{sh}}(\theta)$ is convex, compact and nonempty subset of $\Re^n$. If $\lambda_1 \leq \lambda_2$ then by our supposition (a), we have

$$G_{\lambda_1}(\rho(\theta)) \ominus_h G_{\lambda_1}(\theta) \supset G_{\lambda_2}(\rho(\theta)) \ominus_h G_{\lambda_2}(\theta) \tag{5}$$

For $0 < \hbar < \beta$, we have $G_{\lambda_1}^{\nabla^{sh}}(\theta) \supset G_{\lambda_2}^{\nabla^{sh}}(\theta)$. Let $\{\lambda_n\}$ be a nondecreasing sequence coverges to $\lambda > 0$. For $\epsilon > 0$ choose $\hbar > 0 \ni$ the equation (4) holds. Now, let us consider

$$\begin{aligned} D_H(G_\lambda^{\nabla^{sh}}(\theta), G_{\lambda_n}^{\nabla^{sh}}(\theta)) &\leq D_H\left(G_\lambda^{\nabla^{sh}}(\theta), \frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}\right) \\ &\quad + D_H\left(\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}, G_{\lambda_n}^{\nabla^{sh}}(\theta)\right) \\ &< \epsilon + \frac{1}{\nu(\theta)} D_H[G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta), G_{\lambda_n}(\rho(\theta)) \ominus_h G_{\lambda_n}(\theta)] \\ &\quad + \frac{1}{\nu(\theta)} D_H[G_{\lambda_n}(\rho(\theta)) \ominus_h G_{\lambda_n}(\theta), -\nu(\theta) G_{\lambda_n}^{\nabla^{sh}}(\theta)] \\ &< 2\epsilon + \frac{1}{\nu(\theta)} D_H[G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta), G_{\lambda_n}(\rho(\theta)) \ominus_h G_{\lambda_n}(\theta)]. \end{aligned}$$

By our supposition $2(a)$, the rightmost term converges to zero as $n \to \infty$ and hence

$$\lim_{n\to\infty} D_H(G_\lambda^{\nabla^{sh}}(\theta), G_{\lambda_n}^{\nabla^{sh}}(\theta)) = 0.$$

From Theorem 2.2 and (5) we have

$$G_\lambda^{\nabla^{sh}}(\theta) = \bigcap_{n\geq 1} cl\left\{\bigcup_{m\geq n} G_{\lambda_m}^{\nabla^{sh}}(\theta)\right\}.$$

If $\lambda = 0$, we can write it as

$$\lim_{n\to\infty} D_H(G_0^{\nabla^{sh}}(\theta), \nabla^{sh} G_{\lambda_n}^{\nabla^{sh}}(\theta)) = 0,$$

where the nondecreasing sequence $\{\lambda_n\}$ tends to zero , and as a result of this

$$G_0^{\nabla^{sh}}(\theta) = cl\left(\bigcup_{n\geq 1} G_{\lambda_n}^{\nabla^{sh}}(\theta)\right).$$

Then from Theorem 2.1, $\exists$ an element $\tilde{u} \in \mathbb{E}_n$ such that

$$[\tilde{u}]^\lambda = G_\lambda^{\nabla^{sh}}(\theta), \quad \lambda \in [0,1].$$

Let $\theta \in \mathbb{T}^{[a,b]}$, $\epsilon > 0$, $\delta > 0$ and $(\theta - \hbar, \theta + \hbar) \in N_T^{[a,b]}$ be as in supposition (b) then, we have

$$D_H\left(\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}, \tilde{u}^\lambda\right) = D_H\left(\frac{G_\lambda(\rho(\theta)) \ominus_h G_\lambda(\theta)}{-\nu(\theta)}, G_\lambda^{\nabla^{sh}}(\theta)\right) < \epsilon$$

Thus, G is ∇^{sh}-differentiable. □

Theorem 3.7. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ defined by $G(\theta) = g(\theta) \odot u$ for all $\theta \in \mathbb{T}^{[a,b]}$, where $u \in \mathbb{E}_n$ and $g : \mathbb{T}^{[a,b]} \to \mathbb{T}_+$ is nabla differentiable at $\theta_0 \in \mathbb{T}^{[a,b]}$. If $g^\nabla(\theta_0) < 0$, then G is ∇^{sh}-differentiable at θ_0 with $G^{\nabla^{sh}}(\theta_0) = g^\nabla(\theta_0) \odot u$.*

Proof. Since g is nabla differentiable at θ_0, then from Lemma 2.1, g is continuous at θ_0. **Case(i):** If θ_0 is left scattered then, we have

$$g^\nabla(\theta_0) = \frac{g(\theta_0) - g(\rho(\theta_0))}{\nu(\theta_0)}.$$

Since $g^\nabla(\theta_0) < 0$, then

$$g(\rho(\theta_0)) - g(\theta_0) = g^\nabla(\theta_0)(-\nu(\theta_0)) > 0$$

It implies that

$$g(\rho(\theta_0)) = g(\theta_0) + g^\nabla(\theta_0)(-\nu(\theta_0)).$$

Now, multiplying the above equation with $u \in \mathbb{E}_n$ on both sides, then we get

$$g(\rho(\theta_0)) \odot u = [g(\theta_0) \odot u] \oplus \left[g^\nabla(\theta_0)(-\nu(\theta_0)) \odot u\right].$$

It implies that $[g(\rho(\theta_0)) \odot u] \ominus_h [g(\theta_0) \odot u] = [g^\nabla(\theta_0)(-\nu(\theta_0))] \odot u$ and then

$$G(\rho(\theta_0)) \ominus_h G(\theta_0) = [g^\nabla(\theta_0)(-\nu(\theta_0))] \odot u.$$

dividing by $(-\nu(\theta_0))$, we have

$$\frac{G(\rho(\theta_0)) \ominus_h G(\theta_0)}{(-\nu(\theta_0))} = [g^\nabla(\theta_0)] \odot u$$

and hence

$$G^{\nabla^{sh}}(\theta_0) = g^\nabla(\theta) \odot u.$$

Case(ii): If θ_0 is left dense, then $g^{\nabla}(\theta_0) = g^{'}(\theta_0) < 0$ and

$$g^{'}(\theta_0) = \lim_{\hbar \to 0^+} \frac{g(\theta_0) - g(\theta_0 - \hbar)}{\hbar}.$$

It follows that for $\hbar > 0$ sufficiently small, we have $g(\theta_0) - g(\theta_0 - \hbar) < o$ and

$$g(\theta_0 - \hbar) = g(\theta_0) + \phi(\theta_0, \hbar)$$

Now, multiplying the above equation with $u \in \mathbb{E}_n$ on both sides, we get

$$g(\theta_0 - \hbar)) \odot u = [g(\theta_0) \odot u] \oplus [\phi(\theta_0, \hbar) \odot u] \, .$$

It implies

$$G(\theta_0 - \hbar) = G(\theta_0) \oplus [(\phi(\theta_0, \hbar) \odot u].$$

Therefore, $G(\theta_0 - \hbar) \ominus_h G(\theta_0)$ exists and hence G is left ∇^{sh}-differentiable at θ_0. Similarly, we can prove G is right ∇^{sh}-differentiable at θ_0.

It follows that, G is ∇^{sh}-differentiable at θ_0 with $G^{\nabla^{sh}}(\theta_0) = g^{\nabla_0}(\theta) \odot u$. □

Example 3.3. Let us define $G(\theta) = \frac{1}{\theta} \odot u \, , \forall \, \theta \in \mathbb{T}^{[1,10]}$, $G : \mathbb{T}^{[1,10]} \to \mathbb{E}_1$ is a fuzzy function and $u = (2, 3, 4)$ is the triangular fuzzy number. Here, $g(\theta) = \frac{1}{\theta}$ and $g^{\nabla}(\theta) = \frac{-1}{\theta(\rho(\theta))} < 0 \, , \forall \, \theta \in \mathbb{T}^{[1,10]}$, from Theorem 3.7, we have $G(\theta)$ is ∇^{sh}-differentiable and $G^{\nabla}(\theta) = \frac{-1}{\theta(\rho(\theta))} \odot u \, , \forall \, \theta \in \mathbb{T}^{[1,10]}$.

Theorem 3.8. *Let $G : \mathbb{T}^{[a,b]} \to \mathbb{E}^1$ defined as $[G(\theta)]^\lambda = [g_\lambda(\theta), h_\lambda(\theta)], \lambda \in [0, 1]$ and $G(\theta)$ is ∇^{sh}-differentiable on $\mathbb{T}^{[a,b]}$. Then g_λ and h_λ are nabla-differentiable on $\mathbb{T}^{[a,b]}$ and*

$$[G^{\nabla^{sh}}(\theta)]^\lambda = [h_\lambda^{\nabla}(\theta), g_\lambda^{\nabla}(\theta)].$$

Proof. If G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ and θ is left scattered, then for any $\lambda \in [0, 1]$,

$$[G(\rho(\theta)) \ominus_h G(\theta)]^\lambda = [g_\lambda(\rho(\theta)) - g_\lambda(\theta), h_\lambda(\rho(\theta)) - h_\lambda(\theta)].$$

and multiplying with $\frac{-1}{\nu(\theta)}$, we get

$$\begin{aligned}
[G^{\nabla^{sh}}(\theta)]^\lambda =& \frac{-1}{\nu(\theta)} \odot [G(\rho(\theta)) \ominus_h G(\theta)]^\lambda \\
&= \left[\frac{1}{-\nu(\theta)} \odot [g_\lambda(\rho(\theta)) - g_\lambda(\theta), h_\lambda(\rho(\theta)) - h_\lambda(\theta)] \right] \\
&= \left[\frac{h_\lambda(\rho(\theta)) - h_\lambda(\theta)}{-\nu(\theta)}, \frac{g_\lambda(\rho(\theta)) - g_\lambda(\theta)}{-\nu(\theta)} \right] \\
&= \left[\frac{h_\lambda(\theta) - h_\lambda(\rho(\theta))}{\nu(\theta)}, \frac{g_\lambda(\theta) - g_\lambda(\rho(\theta))}{\nu(\theta)} \right] \\
&= \left[h_\lambda^{\nabla}(\theta), g_\lambda^{\nabla}(\theta) \right] .
\end{aligned}$$

If G is ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$ and θ is ld-point, then for any $\lambda \in [0,1]$,

$$[G(\theta-\hbar) \ominus_h G(\theta)]^\lambda = [g_\lambda(\theta-\hbar) - g_\lambda(\theta), h_\lambda(\theta-\hbar) - h_\lambda(\theta)]$$

and multiplying with $\frac{-1}{\hbar} < 0$ and taking limits as $\hbar \to 0^+$, we get

$$\begin{aligned}
&\lim_{\hbar\to 0^+} \frac{-1}{\hbar} \odot [G(\theta-\hbar) \ominus_h G(\theta)]^\lambda \\
&\quad = \lim_{\hbar\to 0^+} \frac{-1}{\hbar} \odot [g_\lambda(\theta-\hbar) - g_\lambda(\theta), h_\lambda(\theta-\hbar) - h_\lambda(\theta)] \\
&\quad = \left[\lim_{\hbar\to 0^+} \frac{h_\lambda(\theta-\hbar) - h_\lambda(\theta)}{-\hbar}, \lim_{\hbar\to 0^+} \frac{g_\lambda(\theta-\hbar) - g_\lambda(\theta)}{-\hbar}\right] \\
&\quad = \left[\lim_{\hbar\to 0^+} \frac{h_\lambda(\theta) - h_\lambda(\theta-\hbar)}{\hbar}, \lim_{\hbar\to 0^+} \frac{g_\lambda(\theta) - g_\lambda(\theta-\hbar)}{\hbar}\right] \\
&\quad = [h_\lambda^\nabla(\theta), g_\lambda^\nabla(\theta)].
\end{aligned}$$

Similarly, we can prove

$$\lim_{\hbar\to 0^+} \frac{-1}{\hbar}[G(\theta) \ominus_h G(\theta+\hbar)]^\lambda = [h_\lambda^\nabla(\theta), g_\lambda^\nabla(\theta)].$$

Thus, g_λ and h_λ are nabla differentiable on $\mathbb{T}^{[a,b]}$ and $[G^{\nabla^{sh}}(\theta)]^\lambda = [h_\lambda^\nabla(\theta), g_\lambda^\nabla(\theta)]$. □

Example 3.4. Consider the fuzzy function $G(\theta)$ as in Example 3.3. Then $u^\lambda = [2+\lambda,\ 4-\lambda]$ is λ-level set of u and

$$\begin{aligned}
[G(\theta)]^\lambda &= [g_\lambda(\theta), h_\lambda(\theta)] \\
&= \frac{1}{\theta} \odot [2+\lambda,\ 4-\lambda] \\
&= [\frac{1}{\theta}(2+\lambda),\ \frac{1}{\theta}(4-\lambda)].
\end{aligned}$$

From Example 3.3, $G(\theta)$ is ∇^{sh}-differentiable and $G^{\nabla^{sh}}(\theta) = \frac{-1}{\theta(\rho(\theta))} \odot u$. Clearly, $g_\lambda^\nabla(\theta), h_\lambda^\nabla(\theta)$ are nabla differentiable and $g_\lambda^\nabla(\theta) = \frac{-1}{\theta(\rho(\theta))}(2+\lambda)$, $h_\lambda^\nabla(\theta) = \frac{-1}{\theta(\rho(\theta))}(4-\lambda)$. From Example 3.3 and Theorem 3.5, we have

$$\begin{aligned}
[G^{\nabla^{sh}}(\theta)]^\lambda &= \frac{-1}{\theta(\rho(\theta))} \odot u^\lambda \\
&= \frac{-1}{\theta(\rho(\theta))} \odot [2+\lambda,\ 4-\lambda] \\
&= \left[\frac{-1}{\theta(\rho(\theta))}(4-\lambda), \frac{-1}{\theta(\rho(\theta))}(2+\lambda)\right] = [h^\nabla(\theta), g^\nabla(\theta)].
\end{aligned}$$

Hence Theorem 3.8 is verified. And also if $\mathbb{T} = \mathbb{R}$, then $\rho(\theta) = \theta$ and

$$G^{\nabla^{sh}}(\theta) = \frac{-1}{\theta(\rho(\theta))} \odot u^\lambda = \frac{-1}{\theta^2} \odot u^\lambda.$$

If $\mathbb{T} = q^{\mathbb{N}}$, then $\rho(\theta) = \frac{\theta}{q}$ and $G^{\nabla^{sh}(\theta)} = \frac{-1}{\theta(\rho(\theta))} \odot u^\lambda = \frac{-q}{\theta^2} \odot u^\lambda$.

Now, we obtain the ∇^{sh}-derivatives of addition, scalar multiplication and product of second type nabla Hukuhara differentiable for fuzzy functions on time scales.

Theorem 3.9. *Let $G, H : \mathbb{T}^{[a,b]} \to \mathbb{E}^n$ are ∇^{sh}-differentiable at $\theta \in T_k^{[a,b]}$. Then,*

(a) *the sum $G \oplus H : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at θ with*

$$(G \oplus H)^{\nabla^{sh}}(\theta) = G^{\nabla^{sh}}(\theta) \oplus H^{\nabla^{sh}}(\theta);$$

(b) *for any constant λ, $\lambda G : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at θ with*

$$(\lambda \odot G)^{\nabla^{sh}}(\theta) = \lambda \odot G^{\nabla^{sh}}(\theta);$$

(c) *the product $GH : \mathbb{T}^{[a,b]} \to \mathbb{E}_n$ is ∇^{sh}-differentiable at θ with*

$$\begin{aligned}(GH)^{\nabla^{sh}}(\theta) &= G(\rho(\theta))H^{\nabla^{sh}}(\theta) + H(\theta)G^{\nabla^{sh}}(\theta) \\ &= G(\theta)H^{\nabla^{sh}}(\theta) + H(\rho(\theta))G^{\nabla^{sh}}(\theta).\end{aligned}$$

Proof. Since G and H be ∇^{sh}-differentiable at $\theta \in \mathbb{T}_k^{[a,b]}$. Then from Theorem 3.1, G and H are continuous when θ is left dense and right continuous when θ is left scattered. If θ is left scattered, then from Theorem 3.2, we have

$$\frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)} = G^{\nabla^{sh}}(\theta) \tag{6}$$

and

$$\frac{H(\rho(\theta)) \ominus_h H(\theta)}{-\nu(\theta)} = H^{\nabla^{sh}}(\theta). \tag{7}$$

If θ is ld-point, then from Theorem 3.3, we have

$$\lim_{\hbar \to 0^+} \frac{G(\theta) \ominus_h G(\theta + \hbar)}{-\hbar} = \lim_{\hbar \to 0^+} \frac{G(\theta - \hbar) \ominus_h G(\theta)}{-\hbar} = G^{\nabla^{sh}}(\theta) \tag{8}$$

and

$$\lim_{\hbar \to 0^+} \frac{H(\theta) \ominus_h H(\theta + \hbar)}{-\hbar} = \lim_{\hbar \to 0^+} \frac{H(\theta - \hbar) \ominus_h H(\theta)}{-\hbar} = H^{\nabla^{sh}}(\theta). \tag{9}$$

(a) If θ is left scattered and G, H are ∇^{sh}-differentiable at θ, then from Theorem 3.1, $(G \oplus H)$ is right continuous at θ. From Theorem 3.2 and (6), (7), we have

$$\begin{aligned}(G\oplus H)^{\nabla^{sh}}(\theta) &= \frac{(G\oplus H)(\rho(\theta))\ominus_h (G\oplus H)(\theta)}{-\nu(\theta)}\\ &= \frac{[G(\rho(\theta))\oplus H(\rho(\theta))]\ominus_h [G(\theta)\oplus H(\theta)]}{-\nu(\theta)}\\ &= \frac{[G(\rho(\theta))\ominus_h G(\theta)]\oplus [H(\rho(\theta))\ominus_h H(\theta)]}{-\nu(\theta)}\\ &= \frac{G(\rho(\theta))\ominus_h G(\theta)}{-\nu(\theta)}\oplus\frac{H(\rho(\theta))\ominus_h H(\theta)}{-\nu(\theta)}\\ &= G^{\nabla^{sh}}(\theta)\oplus H^{\nabla^{sh}}(\theta).\end{aligned}$$

If θ is left dense and G, H are ∇^{sh}-differentiable at θ, then from (8) & (9), we have

$$\begin{aligned}&\lim_{\hbar\to 0^+}\frac{(G\oplus H)(\theta)\ominus_h (G\oplus H)(\theta+\hbar)}{-\hbar}\\ &= \lim_{\hbar\to 0^+}\frac{(G(\theta)\ominus_h G(\theta+\hbar))\oplus(H(\theta)\ominus_h H(\theta+\hbar))}{-\hbar}\\ &= \lim_{\hbar\to 0^+}\frac{G(\theta)\ominus_h G(\theta+\hbar)}{-\hbar}\oplus\lim_{\hbar\to 0^+}\frac{H(\theta)\ominus_h H(\theta+\hbar)}{-\hbar}\\ &= G^{\nabla^{sh}}(\theta)\oplus H^{\nabla^{sh}}(\theta).\end{aligned}$$

Similarly, we can prove

$$\lim_{\hbar\to 0^+}\frac{(G\oplus H)(\theta-\hbar)\ominus_h (G\oplus H)(\theta)}{-\hbar} = G^{\nabla^{sh}}(\theta)\oplus H^{\nabla^{sh}}(\theta).$$

Therefore, $G\oplus H$ is ∇^{sh}-differentiable at θ and

$$(G\oplus H)^{\nabla^{sh}}(\theta) = G^{\nabla^{sh}}(\theta)\oplus H^{\nabla^{sh}}(\theta).$$

(b) For $\gamma = 0$, the result is obvious. Now, let us assume that $\gamma > 0$.

If θ is left scattered, then from Theorem 3.1, $\gamma\odot G$ is right continuous at θ. From Theorem 3.2 and (6), we have

$$\begin{aligned}(\gamma\odot G)^{\nabla^{sh}}(\theta) &= \frac{\gamma\odot G(\rho(\theta))\ominus_h \gamma\odot G(\theta)}{-\nu(\theta)}\\ &= \gamma\odot\frac{G(\rho(\theta))\ominus_h G(\theta)}{-\nu(\theta)} = \gamma\odot G^{\nabla^{sh}}(\theta).\end{aligned}$$

Since G is ∇^{sh}-differentiable at $\theta\in\mathbb{T}_k^{[a,b]}$ and θ is left dense, then from (8),

$$\lim_{\hbar\to 0^+}\frac{\gamma\odot G(\theta)\ominus_h \gamma\odot G(\theta+\hbar)}{-\hbar} = \gamma\odot\lim_{\hbar\to 0^+}\frac{G(\theta)\ominus_h G(\theta+\hbar)}{-\hbar} = \gamma\odot G^{\nabla^{sh}}(\theta).$$

Similarly, we can prove

$$\lim_{\hbar\to 0^+}\frac{\gamma\odot G(\theta-\hbar)\ominus_h \gamma\odot G(\theta)}{-\hbar} = \gamma\odot G^{\nabla^{sh}}(\theta).$$

(c) If θ is left scattered, then from Theorem 3.1, it is clear that GH is right continuous at θ. From Theorem 3.2, (6) & (7), we have

$$G^{\nabla^{sh}}(\theta) = \frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)} \ \& \ H^{\nabla^{sh}}(\theta) = \frac{H(\rho(\theta)) \ominus_h H(\theta)}{-\nu(\theta)}.$$

$$\begin{aligned}
(GH)^{\nabla^{sh}}(\theta) &= \frac{GH(\rho(\theta)) \ominus_h GH(\theta)}{-\nu(\theta)} \\
&= \frac{G(\theta)[H(\rho(\theta)) \ominus_h H(\theta)] \oplus [G(\rho(\theta)) \ominus_h G(\theta)]H(\rho(\theta))}{-\nu(\theta)} \\
&= G(\theta)\left[\frac{H(\rho(\theta)) \ominus_h H(\theta)}{-\nu(\theta)}\right] \oplus H(\rho(\theta))\left[\frac{G(\rho(\theta)) \ominus_h G(\theta)}{-\nu(\theta)}\right] \\
&= G(\theta)H^{\nabla^{sh}}(\theta) \oplus H(\rho(\theta))G^{\nabla^{sh}}(\theta).
\end{aligned}$$

Since G, H are ∇^{sh}-differentiable and if θ is left dense, then from (8) & (9), we have

$$\begin{aligned}
&\lim_{\hbar\to 0^+} \frac{GH(\theta) \ominus_h GH(\theta+\hbar)}{-\hbar} \\
&\quad = \lim_{\hbar\to 0^+} \frac{G(\theta)[(H(\theta) \ominus_h H(\theta+\hbar)] \oplus [G(\theta) \ominus_h G(\theta+\hbar)]H(\theta)}{-\hbar} \\
&\quad = G(\theta)\lim_{\hbar\to 0^+} \frac{H(\theta) \ominus_h H(\theta+\hbar)}{-\hbar} \oplus \lim_{\hbar\to 0^+} \frac{G(\theta) \ominus_h G(\theta+\hbar)}{-\hbar} \lim_{\hbar\to 0^+} H(\theta+\hbar) \\
&\quad = G(\theta)\lim_{\hbar\to 0^+} \frac{H(\theta) \ominus_h H(\theta+\hbar)}{-\hbar} \oplus \lim_{\hbar\to 0^+} \frac{G(\theta) \ominus_h G(\theta+\hbar)}{-\hbar} H(\theta) \\
&\quad = G(\theta)H^{\nabla^{sh}}(\theta) \oplus G^{\nabla^{sh}}(\theta)H(\theta).
\end{aligned}$$

Similarly, we can prove

$$\lim_{\hbar\to 0^+} \frac{(GH)(\theta-\hbar) \ominus_h (GH)(\theta)}{-\hbar} = G(\theta)H^{\nabla^{sh}}(\theta) \oplus G^{\nabla^{sh}}(\theta)H(\theta).$$

Thus, $(GH)^{\nabla^{sh}}(\theta) = G(\theta)H^{\nabla^{sh}}(\theta) \oplus H(\rho(\theta))G^{\nabla^{sh}}(\theta)$ holds at θ. We get the another product rule in (c) by interchanging G and H and which follows from the last equation. □

4. Conclusions

The fuzzy nabla Hukuhara derivative of form-I (Definition 3.3) does not exists for a fuzzy function of decreasing diameter on time scales. To overcome this shortcoming, in this paper we introduce and study the fundamental properties of second type nabla Hukuhara derivative for fuzzy functions on time scales. In our future work, we propose to study fuzzy nabla integrals on time scales. Further, these concepts can be applied to study the fuzzy nabla dynamic equations on time scales.

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References

[1] R.P. Agarwal, M. Bohner, *Basic calculus on time scales and some of its applicatisons*, Results in Mathematics, 35 (1999), 3-22.

[2] G. A. Anastassiou, S. G. Gal, *On a fuzzy trigonometric approximation theorem of Weierstrass-type*, Journal of Fuzzy Mathematics, 9 (2001), 701-708.

[3] D. R. Anderson, C. C. Tisdell, *Alternative solutions of inhomogeneous second-order linear dynamic equations on time scales*, Journal of Difference Equations and Applications, 17 (2011), 1487-1498.

[4] F. M. Atici, F. Usynal, *A producion-inventory model of HMMS model on time scales*, Applied Mathematics Letters, 21 (2008), 236-243.

[5] F. M. Atici, C. Daniel Biles, Alex Lebedinsky. *An application of time scales to economics, Mathematical and Computer Modelling*, 43 (2006), 718-726.

[6] B. Bede, S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems, 151 (2005), 581-599.

[7] M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Birkhauser, Boston, New York, 2001.

[8] P. W. Eloe, J. Henderson, *Boundary value problems for dynamic equations on time scales*, Nonlinear Interpolation and Boundary Value Problems, 2016, 181-224.

[9] J. Gao, Q. R. Wang, L. W. Zhang, *Existence and stability of almost-periodic solutions forcellular neural networks with time-varying delays in leakage terms on time scales*, Applied Mathematics and Computation, 237 (2015), 639-649.

[10] S. Hilger, *Analysis on measure chains - a unified approach to continuous and discrete calculus*, Results in Mathematics, 18 (1990), 18-56.

[11] S. Hong, *Differentiability of multivalued functions on time scales and applications to multivalued dynamic equations*, Nonlinear Analysis, 71 (2009), 3622-3637.

[12] M. Hukuhara, *Integration des applications measurables dontlavaleurest uncompact convex*, Funkcialaj Ekvacioj, 10 (1967), 205-223.

[13] B. J. Jackson, *Adaptive control in the nabla setting*, Neural, Parallel and Scientific Computations, 16 (2008), 253-272.

[14] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems, 24 (1987), 301-317.

[15] R. Leelavathi, G. Suresh Kumar, M. S. N. Murty, *Nabla inegral for fuzzy functions on time scales*, International Journal of Applied Mathematics, 31 (2018), 669-678.

[16] R. Leelavathi, G. Suresh Kumar, M. S. N. Murty, *Nabla Hukuhara differentiability for fuzzy functions on time scales*, IAENG-International Journal of Applied Mathematics, 49 (2019), 114-121.

[17] B. Liu, Y. Do, H. A. Batarfi, F. E. Alsaadi, *Almost periodic solution for a neutral-type neural networks with distributed leakage delays on time scales*, Neuro Computing, 173 (2016), 921-929.

[18] V. Lupulescu, *Hukuhara differentiability of interval-valued functions and interval differential equations on time scales*, Information Sciences, 248 (2013), 50-67.

[19] M. T. Malinowski, *Interval differential equations with a second type Hukuhara derivative*, Applied Mathematics Letters, 24 (2011), 2118-2123. .

[20] M. T. Malinowski, *Interval Cauchy problem with a second type Hukuhara derivative*, Information Sciences, 213 (2012), 94-105.

[21] M . L. Puri, D. A. Ralescu, *Differentials for fuzzy functions*, Journal of Mathematical Analysis, 91 (1983), 552- 558.

[22] L. Stefanini, B. Bede, *Generalized Hukuhara differentiability of interval-valued functions and interval differential equations*, Nonlinear Analysis, Theory, Methods and Applications, 71 (2009) 1311-1328.

[23] CH. Vasavi, G. Suresh Kumar, M. S. N. Murty, *Fuzzy Hukahara delta differential and applications to fuzzy dynamic equations on time scales*, Journal of uncertain systems, 10 (2016), 163-180.

[24] CH. Vasavi, G. Suresh Kumar, M. S. N. Murty, *Fuzzy dynamic equations on time scales under second type Hukuhara delta derivative*, International Journal of Chemical Sciences, 14 (2016), 49-66.

[25] CH. Vasavi, G. Suresh Kumar, M. S. N. Murty, *Generalized differentiability and integrability for fuzzy set-valued functions on time scales*, Soft Computing, 20 (2016), 1093-1104.

[26] CH. Vasavi, G. Suresh Kumar, M. S. N. Murty, *Fuzzy dynamic equations on time scales under generalized delta derivative via contractive-like mapping principles*, Indian Journal of Science and Technology, 9 (2016), 1-6.

[27] A. Zafer, *On ascillation and nonoscillation of second-order dynamic equations*, Applied Mathematics Letters, 22 (2009), 136-141.

[28] S. Zhang, J. Sun, *Stability of fuzzy differential equations with the second type of Hukuhara derivative*, IEEE transactions on fuzzy systems, 23 (2015), 1323-1328.

[29] L. A. Zadeh,*Fuzzy sets*, Information Control, 8 (1965), 338-353.

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