Reliability bounds of dependent linear consecutive k-out-of-n:G systems

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Abstract. Most of researches in the reliability theory dealt to study the independence between components in a system. But, in many real systems, dependence between the components is one of the intractable realistic assumptions that need to be carefully considered. Then, the main purpose of this paper is to provide sharp upper and lower bounds for the reliability of linear consecutive k-out-of-n:G systems consisting of dependent components with identical or arbitrary distribution functions. Some comparisons are done and many examples are treated to prove the performance of the proposed method.
Keywords: linear consecutive k-out-of-n:G system, upper bound (B_U), lower bound (B_L), reliability, dependent components, Copula.

1. Introduction

Reliability is an important task especially in complex and high technology systems. Problems related to reliability are particularly critical when there are concerns over the consequences of system failures in terms of safety and cost. Studies elaborated for linear consecutive k-out-of-n systems have attracted a great importance on theoretical and practical fields, indifferently. These systems appear primordially in various engineering fields, such as: mechanical, civil, electronic engineering, telecommunication and network domain, etc.

A linear consecutive k-out-of-n:G (F) system (denoted Lin/Con/k/n:G (F)) consists of n linearly arranged components such as the system works (fails) if and only if at least k consecutive components work (fail). Note that there is a duality between the two systems.

Reliability and opened problems which are related to consecutive k-out-of-n:G (F) systems have been widely studied in the literature under various assumptions and have been resolved, either in the binary case (there are only two states: function, failure): [19], [3], [4], [7], [12], or in the multi-state case

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(the system and the components can assume more than two states): [10], [22], [9], [21], [1], [6], [2], ... Where special attention has been paid to i.i.d coherent systems, because computation of reliability characteristics of a system that consists of dependent components is difficult especially when the specific type of dependence is not known.

However, in several real situations, a great number of systems operate with dependence structures. So, it is necessary to study this category. Recently, some authors started to study systems with dependence structures and similarly, evaluating the exact value of reliability for such systems is so difficult. For this reason, conditions oblige us to estimate this reliability via bounds limits approach. Although, it should be noted that obtaining bounds under an assumption of unknown independence for such systems is harder than for independence one. In the literature, there are few papers investigating the system's reliability with dependence assumption. In this domain, [8] obtained the reliability of consecutive k-out-of-n:G system with dependent elements using a matrix formulation. [16] proposed bounds for the reliability and the expectations of lifetimes of coherent systems based on dependent exchangeable components (which means that the joint distribution of the component lifetimes is invariant in law under permutation) using the concept of Samaniego's signature. [15] investigated the properties of coherent systems with dependent components using the concept of hyperminimal and hypermaximal distributions and proved that the lifetime distribution of any coherent system is a generalized mixture of a series (parallel) subsystem lifetime distributions. Then, from well-known properties of mixtures, bounds and moments for the hyperminimal and hypermaximal distributions for coherent systems are obtained. [20] studied the residual lifetime of both linear and circular consecutive k-out-of-n systems with independent but non identical components. They obtained expressions for residual lifetime distributions and mean residual lifetime functions in terms of permanents when $2k \geq n$ for linear systems and $2k + 1 \geq n$ for circular systems. The failure rate functions and their asymptotic behavior of consecutive k-out-n systems, using mixture representations, were investigated by [7]. They studied the two cases: i.i.d component lifetimes, then independent but not identically distributed component lifetimes. They also obtained some results for the case of exchangeable dependent component lifetimes. [5] investigated the bounding systems reliability, especially for k-out-of n:F linear and circular systems. For the k-out-of-n:F system, the upper and lower bounds are illustrated. But for the other structures, they provided upper and lower bounds for only linear consecutive 2-out-of-3:F system and circular 2-out-of-4:F system. [14] derived bounds for the reliability and the expected lifetime of a coherent system with heterogeneous components, (based on $\overline{G_1}$: the average of the reliability functions of the components and on $\overline{G_2}$: the average of the reliability functions of the series systems obtained from the minimal pats sets). They showed by treating some examples that the bounds obtained by $\overline{G_2}$ seem to be better than that from $\overline{G_1}$, but not always (they presented a counter-example). With the particularity that $\overline{G_1}$ does not depend on the system structure, the permutation of the component reliability functions, and the dependence structure, while $\overline{G_2}$ may depend on these three characteristics. In their other paper, [13] extended the bounds obtained in the precedent one to the case where the components are ordered (the usual stochastic order). Then, using these bounds, they studied the optimal allocation of the components at a given system structure in order to improve the system reliability. Moreover a similar procedure was applied to get bounds for mixtures and the generalized proportional hazard rate model when the baseline populations are ordered. Note that, for this last case, the authors didn't know if the proposed bounds remain optimal. Our contribution is devoted to establish reliability bounds for dependent linear consecutive k-out-of-n:G system for any value of n and k satisfying the relationship $2k \geq n$. From well-known properties of coherent systems and using [6] formula, upper and lower bounds are established. The performance of the provided bounds is quite satisfactory and their calculation is very easy.

The paper is organized as follows: Section 2 is devoted to some notations, assumption and defintions which will be used in the whole paper. In Section 3, we provide sharp upper and lower bounds for the reliability of linear consecutive k-out-of-n:G system with component lifetimes of the systems are dependent and both of an arbitrary joint distribution and identically distributed, by using formulas provided by [6],[20] and [18]. In section 4, we present briefly the context of copula applied in reliability computation. We compare reliability bounds for Lin/con/2/3:G system obtained by our proposed method and the reliability evaluated by using the copula context elaborated by [11]. We also compare our obtained results with those provided by [14] and [13].

In the last section, we study the influence of the number of components on the reliability bounds. In each section, we treat some examples to illustrate the proposed results.

2. Notations and Definitions

Notations 1. n: number of components in the system.

k : the minimum number of consecutive components required for the system to be good.

T_j : lifetime of component j ,$j = 1, \dots, n$.

$F_j(t) = P(T_j \leq t)$: distribution function of T_j.

$R(t) = P(T_{k/n:G} > t)$: reliability of the system.

$T_{k/n:G}$: lifetime of the system.

Assumptions 1. We assume that $T = (T_1, ..., T_n)$ an n-dimensional random vector is positively lower and upper orthant dependent. (In reliability, the component lifetimes are usually positively dependent).

Definition 1. *Let $T = (T_1, ..., T_n)$ an n-dimensional random vector, in [18], the positively lower orthant dependent and positively upper orthant dependent are defined as:*

1. T is positively lower orthant dependent (PLOD) if for all $(t_1, t_2, \ldots, t_n)$ in R^n

$$P(T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n) \geq \prod_{j=1}^{n} P(T_j \leq t_j).$$

2. T is positively upper orthant dependent (PUOD) if for all $(t_1, t_2, \ldots, t_n)$ in R^n

$$P(T_1 > t_1, T_2 > t_2, \cdots, T_n > t_n) \geq \prod_{j=1}^{n} P(T_j > t_j).$$

Definition 2. *A consecutive k-out-of-n:G system consists of n linearly arranged components, this system works if and only if at least k consecutive components work.*

In the following section, we propose bounds of reliability for the dependent linear consecutive k-out-of-n:G system (Lin/con/k/n:G system).

3. Bounds of linear consecutive k-out-of-n:G system

Proposition 1. *Let the Lin/con/k/n:G system with arbitrarily distributed dependent components. For $2k \geq n$ the system's reliability is bounded as follows*

$$B_L \leq R(t) \leq B_U,$$

where

$$B_U = 1 + \sum_{m=1}^{m=n-k} F_{k+m}(t) \left[1 - \max_{m \leq j \leq k+m-1} F_j(t)\right] - \max_{n-k+1 \leq j \leq n} F_j(t). \tag{1}$$

and

$$B_L = (k-n) + \sum_{m=1}^{m=n-k} \max_{m \leq j \leq k+m} F_j(t) + \sum_{m=1}^{m=n-k+1} \prod_{j=m}^{j=k+m-1} [1 - F_j(t)]. \tag{2}$$

Corollary 1. *If distribution functions are identical ($F_j(t) = F(t)\ \forall j$), the above formula can be written in this simple form*

$$\begin{aligned} (k-n) + (n-k)F(t) + (n-k+1)(1-F(t))^k &\leq R(t) \\ &\leq 1 + (n-k-1)F(t) - (n-k)F(t)^2. \end{aligned} \tag{3}$$

Remark 1. To ensure the reliability value of a system in the tolerable interval $[0,1]$, it's necessary to establish the following relation

$$\begin{aligned} &\max\left\{(k-n)+\sum_{m=1}^{m=n-k}\max_{m\le j\le k+m}F_j(t)+\sum_{m=1}^{m=n-k+1}\prod_{j=m}^{j=k+m-1}[1-F_j(t)],0\right\} \\ (4)\quad &\le\ R(t) \\ &\le \min\left\{1+\sum_{m=1}^{m=n-k}F_{k+m}(t)\left[1-\max_{m\le j\le k+m-1}F_j(t)\right]-\max_{n-k+1\le j\le n}F_j(t),1\right\}. \end{aligned}$$

When components are identically distributed, formula (4) becomes

$$\begin{aligned} &\max\{(k-n)+(n-k)F(t)+(n-k+1)(1-F(t))^k,0\} \\ (5)\qquad &\le\ R(t) \\ &\le \min\{1+(n-k-1)F(t)-(n-k)F(t)^2,1\}. \end{aligned}$$

The bellow property can be used to quantify bounds of the reliability of the studied system

Properties 1.

$$\max_j\{P(T_j\le t)\}\le P(\bigcup_j T_j\le t)\le \min\{\sum_j P(T_j\le t),1\}.$$

Proof of Proposition 1. The reliability of dependent linear consecutive k-out-of-n:G system is defined by

$$R(t)=P(T_{k/n:G}>t).$$

The lifetime of this system can be represented as follows

$$T_{k/n:G}=\max\{T_{[1:k]},T_{[2:k+1]},\cdots,T_{[n-k+1:n]}\},$$

such as $T_{[1:n]}=\min\{T_1,\cdots,T_n\}$.

[6] formula leads to

$$(6)\qquad R(t)=P(T_{k/n:G}>t)=\sum_{i=k}^{n}\left[P(T_{[i-k+1:i]}>t)-P(T_{[i-k+1:i+1]}>t)\right]$$

with $P(T_{[n-k+1:n+1]>t)}=0$.

Certainly, formula (6) gives the expression of reliability for this system, but it is quite difficult to calculate it and it is not straightforward to use it especially for large systems. For this reason, we have resorted to the calculation of the following bounds.

Equation (6) can be written as

$$R(t) = 1 + \sum_{m=1}^{m=n-k} P(\bigcup_{j=m}^{j=k+m} T_j \le t) - \sum_{m=1}^{m=n-k+1} P(\bigcup_{j=m}^{j=k+m-1} T_j \le t) \tag{7}$$

Mathematical substitutions have been done in equation (7) that leads to

$$R(t) = 1 + \sum_{m=1}^{m=n-k} \left[F_{k+m}(t) - P(T_{k+m} \le t \bigcap (\bigcup_{j=m}^{j=k+m-1} T_j \le t)) \right]$$
$$- P\left(\bigcup_{j=n-k+1}^{j=n} T_j \le t \right). \tag{8}$$

We begin by establishing the upper bound.

We have

$$-P(\bigcup_{j=n-k+1}^{j=n} (T_j \le t) \le - \max_{n-k+1 \le j \le n} F_j(t)$$

and

$$- P(T_{k+m} \le t \bigcap (\bigcup_{j=m}^{j=k+m-1} T_j \le t)) \le -P(T_{k+m} \le t) P(\bigcup_{j=m}^{j=k+m-1} T_j \le t)$$
$$\le -F_{k+m}(t) \max_{m \le j \le k+m-1} F_j(t).$$

Thus, the integration of obove inequalities in equation (8), the upper bound of the reliability is given as

$$R(t) \le 1 + \sum_{m=1}^{m=n-k} F_{k+m}(t) \left[1 - \max_{m \le j \le k+m-1} F_j(t) \right] - \max_{n-k+1 \le j \le n} F_j(t). \tag{9}$$

The lower bound expression can be provided as follows:

Using equation (7), we have

$$P(\bigcup_{j=m}^{j=k+m} T_j \le t) \ge \max_{m \le j \le k+m} F_j(t)$$

and

$$-P(\bigcup_{j=m}^{j=k+m-1} T_j \le t) = -1 + P(\bigcap_{j=m}^{j=k+m-1} T_j > t) \ge -1 + \prod_{j=m}^{j=k+m-1} [1 - F_j(t)].$$

Then, we obtain

$$R(t) \ge (k-n) + \sum_{m=1}^{m=n-k} \max_{m \le j \le k+m} F_j(t) + \sum_{m=1}^{m=n-k+1} \prod_{j=m}^{j=k+m-1} [1 - F_j(t)]. \tag{10}$$

□

3.1 Numerical examples

We suppose that all components are exponentially distributed with parameter λ ($T_j \sim \xi(\lambda)$)

Case 1: The linear consecutive 2-out-of-3:G system. The reliability bounds using equation (5) are plotted in figure 1

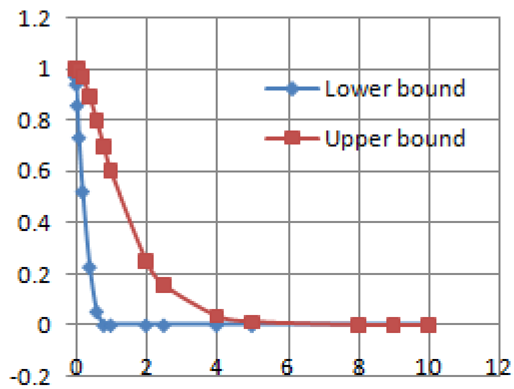

Figure 1: Reliability bounds for linear consecutive 2-out-of-3:G system

Case 2: The linear consecutive 2-out-of-4:G system

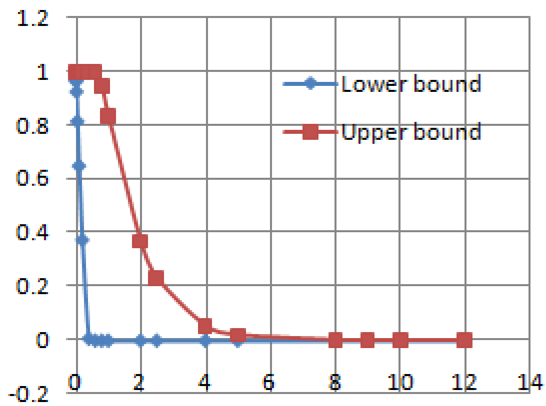

Figure 2: Reliability bounds for linear consecutive 2-out-of-4:G system

Case 3: The linear consecutive 3-out-of-4:G system

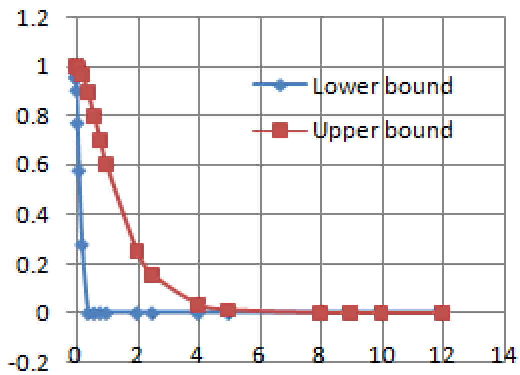

Figure 3: Reliability bounds for linear consecutive 3-out-of-4:G system

Figures 1, 2 and 3 show that for $\lambda t > 5$, the system is less reliable. Then, it's necessary to operate machines until $\lambda t = 5$. We can remark also that when the difference $(n-k)$ decreases, the reliability bounds of the system are getting closer.

4. Comparison with recent results

4.1 Comparison with reliability values using copula

One of the most commonly used methods for modeling dependence between component lifetimes is based on copulas because they contain the information about the dependence structure and can capture the nonlinear dependence. Each copula has its own dependence properties and the detailed review of copulas can be found in [17]. In reliability, the component lifetimes are usually positively dependent, this should be considered while choosing a suitable copula. With the concept of copula, several families of distributions have been constructed such as Gaussian, Clayton, Gumbel, Frank,. . . etc, with the particularity that Gumbel copula and Clayton copula have simple closed form. Also Clayton copula may characterize the joint distribution of the component lifetimes in the context of stress strength interference, for this reason, [11] computed the reliability of dependent consecutive k-out-of-n:G system. The dependency be either linear or non linear. As an example, in his paper, the dependent linear consecutive 2-out-of-3:G was studied using Clayton copula

$$C_l^{cl}(u_1, u_2) = (u_1^{-1} + u_2^{-1} - 1)^{-1}. \tag{11}$$

The component lifetime distribution functions are

$$F_1(t) = 1 - e^{-t}, F_2(t) = 1 - e^{-2t}, F_3(t) = 1 - e^{-3t}.$$

The system's reliability is obtained as

$$\begin{aligned} R(t) &= e^{-2t} - \{(1-e^{-t})^{-1} + (1-e^{-3t})^{-1} - 1\}^{-1} \\ &\quad + \{(1-e^{-t})^{-1} + (1-e^{-2t})^{-1} + (1-e^{-3t})^{-1} - 2\}^{-1}. \end{aligned} \tag{12}$$

Now, the same example is traited using our proposed approach and applying formula (4), we obtain

$$\max\{-1 + F_3(t) + [1 - F_1(t)][1 - F_2(t)] + [1 - F_2(t)][1 - F_3(t)], 0\}$$
$$\leq \; R(t) \; \leq \min\{1 - F_3(t)F_2(t), 1\}. \tag{13}$$

The figure 4 shows curves of upper and lower bounds elaborated by the proposed approach and the exact value of reliability using the Clayton copula method.

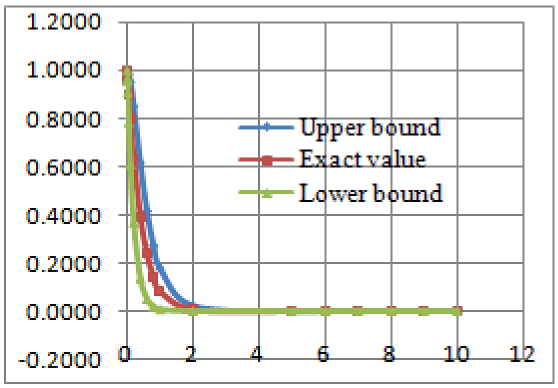

Figure 4: Lower, upper bounds and exact value of Lin/con/2/3:G system

We can remark that there is a well concordance between the results obtained by the two methods. The bounds come very close to exact value of the system's reliability. And when n , k increase, our approach is more flexible, because it didn't need many computations and the time computation is less than the time used in the copula approach.

4.2 Comparaison with Miziula and Navarro bounds

Let us compare the procedure described here and the procedure derived by [14] and [13]. We consider the following examples: linear consecutive 2-out-of-3:G system and linear consecutive 2-out-of-4:G system, with

$$F_1(t) = 1 - e^{-t}, F_2(t) = 1 - e^{-2t}, F_3(t) = 1 - e^{-3t}, F_4(t) = 1 - e^{-4t}$$

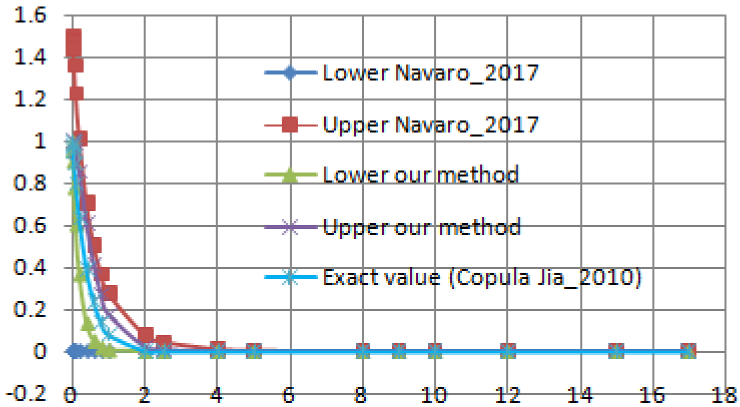

Figure 5: Reliability bounds for linear consecutive 2-out-of-3:G

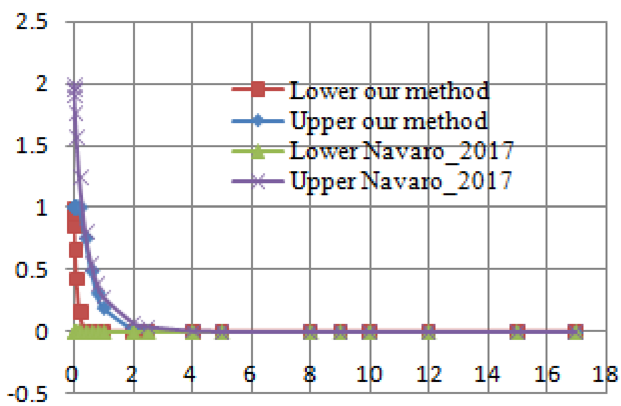

Figure 6: Reliability bounds for linear consecutive 2-out-of-4:G

Figures 5 and 6 show that the bounds obtained in our paper are better than those derived by [13].

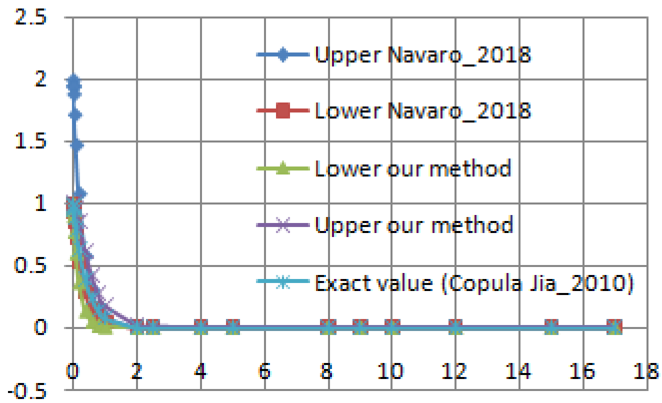

Figure 7: Reliability bounds for linear consecutive 2-out-of-3:G

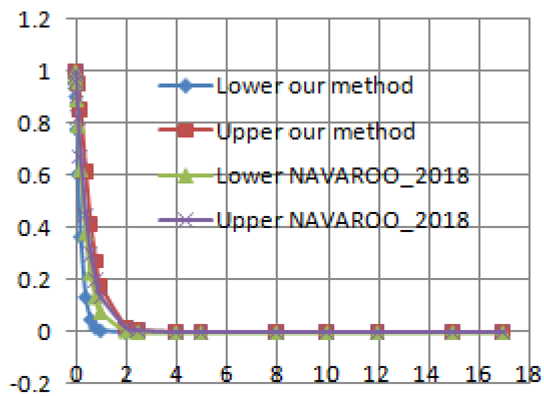

Figure 8: Reliability bounds for linear consecutive 2-out-of-3:G (using (Equ11) in Navarro (2018))

While figure 7 (plotted using $\overline{G_2}$) and figure 8 (plotted using (Equ11) in Navarro 2018) show that the bounds provided by [14] are little better than our bounds, but they stay very close.

5. Influence of system components on reliability bounds

In this section, we will evaluate the difference between the reliability bounds for the case $F_j(t) = F(t)\ \forall j$. In order to determine the influence of variations of n and k on the difference of bounds.

$$(14)\quad I = B_U - B_L = (n-k+1) - F(t) - (n-k)F(t)^2 - (n-k+1)(1-F(t))^k.$$

Where

$$\begin{aligned} B_L &= \max\{(k-n)+(n-k)F(t)+(n-k+1)(1-F(t))^k, 0\} \\ &= (k-n)+(n-k)F(t)+(n-k+1)(1-F(t))^k, \\ B_U &= \min\{1+(n-k-1)F(t)-(n-k)F(t)^2, 1\} \\ &= 1+(n-k-1)F(t)-(n-k)F(t)^2. \end{aligned}$$

We assume that the component lifetimes are exponentially distributed with parameter $\lambda = 1$, and for some different values of n and k we obtain the following figure.

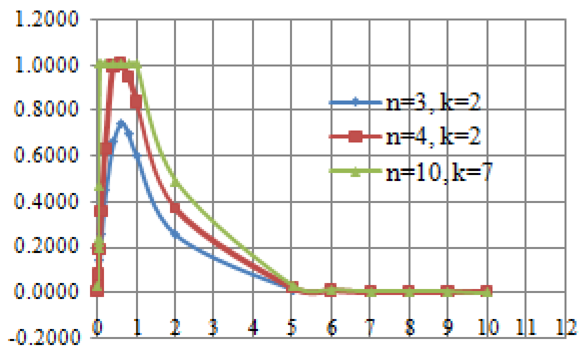

Figure 9: Difference interval for various systems

Figure 9 evaluates the maximum difference between B_U and B_L for different values of n and k.

6. Conclusion

The following results inspired from this work can be drawn as follows

- The development of the reliability bounds of a linear consecutive k-out-of-n:G system in which k consecutive components are positively dependent and arbitrarily distributed.
- The previous results are always valid for any values of k and n satisfying the relationship $2k \geq n$.
- Reliability system using copula approach belongs to the interval of reliability bounds using the developed approach.
- Comparisons between our obtained results and those of [14] and [13] were done. Our approach was applied corresponding to their hypotheses, the corresponding results show a pertinence and a robustness.

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