Characterizations of almost PP-ring for three important classes of rings

Huda Odetallah*
Hasan Al-Ezeh
Emad Abu Osba
The University of Jordan
School of Science
Mathematics Department
Amman
Jordan
huda.odetallah123442@gmail.com
alezehh@ju.edu.jo
eabuosba@ju.edu.jo

Abstract. A ring is called an almost $pp-$ ring if the annihilator of each element of R is generated by its idempotents. We prove that for a ring R and an Abelian group G, if the group ring RG is an almost $pp-$ ring then so is R, Moreover, if G is a finite Abelian group then $|G|^{-1} \in R$. Then we give a counter example to the converse of this. Also, we prove that RG is an almost $pp-$ring if and only if RH is an almost $pp-$ ring for every subgroup H of G. It is proved that the polynomial ring $R[x]$ is an almost $pp-$ ring if and only if R is an almost $pp-$ ring. Finally, we prove that the power series ring $R[[x]]$is an almost $pp-$ ring if and only if for any two countable subsets S and T of R such that $S \subseteq Ann_R(T)$, there exists an idempotent $e \in Ann_R(T)$ such that $b = be$ for all $b \in S$.
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1. Introduction

All rings considered in this paper are assumed to be commutative with unity $1 \neq 0$, and all groups are Abelian. Recall that a ring R is called a $pf-$ring if every principal ideal is a flat $R-$module. An ideal I of a ring R is called pure if for every $a \in I$, there exists $b \in I$ such that $ab = a$. It is well known that a ring R is a $pf-$ring if and only if $Ann_R(a) = \{x \in R : xa = 0\}$ is a pure ideal for every $a \in R$, see [1]. There are different characterizations of $pf-$rings, see [7] and [3]. A ring R is called a $pp-$ring if for each $a \in R$, $Ann_R(a)$ is generated by an idempotent element in R. These rings were studied extensively in literatures, see [5], [10], and [3]. As a generalization of $pp-$ring, Al-Ezeh in [4] introduced a new class of rings called almost $pp-$rings. A ring R is called an almost $pp-$ring if for each $a \in R$, the annihilator ideal $Ann_R(a)$ is generated by its idempotents.

$*$. Corresponding author

It is known that a ring R is an almost $pp-$ring if and only if for every $a \in R$ and $b \in Ann_R(a)$, there exists an idempotent $e \in Ann_R(a)$ such that $b = eb$, see [4, Theorem 1]. Some properties of almost $pp-$rings were investigated in [4] and [13].

Clearly, every $pp-$ring is an almost $pp-$ring and every almost $pp-$ring is a $pf-$ring. Every $pf-$ring is a reduced ring (a ring has no nonzero nilpotent elements), see [1, Lemma 2]. Al-Ezeh in [4] gave an example of an almost $pp-$ring which is not a $pp-$ring, and another example of a $pf-$ring which is not an almost $pp-$ring.

Our aim in this paper to characterize when group rings RG are almost $pp-$rings. Furthermore, we characterize when polynomial rings and power series rings are almost $pp-$rings.

2. Almost $PP-$Rings

In this section, we establish general results on almost $pp-$rings.

Definition 2.1. *Let R be a ring. Then R is said to be an almost $pp-$ring if for every $a \in R$, the annihilator $Ann_R(a)$ is generated by its idempotents.*

In some research, an almost $pp-$ring is called an "almost weak Baer" ring.

Lemma 2.2. *Let R be a Noetherian ring. Then R is an almost $pp-$ring if and only if R is a $pp-$ring.*

Proof. Clearly, if R is $pp-$ring, then R is an almost $pp-$ring.

Assume that R is an almost $pp-$ring and $a \in R$. Then since R is a Noetherian ring, $Ann_R(a)$ is finitely generated ideal. Also, $Ann_R(a)$ is generated by its idempotents. Hence, $Ann_R(a) = \sum_{i=1}^{n} e_i R = eR$ where $1 - e = \prod_{i=1}^{n}(1 - e_i)$. Thus, R is a $pp-$ring. □

Lemma 2.3. *Let $(R_i)_{i \in I}$ be a family of commutative rings. Then $R = \prod_{i \in I} R_i$ is an almost $pp-$ring if and only if R_i is an almost $pp-$ring for all $i \in I$.*

Proof. Assume that R_i is an almost $pp-$ring for each $i \in I$ and $R = \prod_{i \in I} R_i$. Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be two elements of R such that $y \in Ann_R(x)$. Then, $xy = (x_i)_{i \in I}(y_i)_{i \in I} = (x_i y_i)_{i \in I} = 0$. Since R_i is an almost $pp-$ring for every $i \in I$, there exists an idempotent $e_i \in Ann_{R_i}(x_i)$ such that $y_i = e_i y_i$.

Hence, $y = (y_i)_{i \in I} = (e_i)_{i \in I}(y_i)_{i \in I}$, $(e_i)_{i \in I} \in Ann_R(x)$ and $\left((e_i)_{i \in I}\right)^2 = (e_i)_{i \in I}$.

Therefore, R is an almost $pp-$ring.

Conversely, assume that $R = \prod_{i \in I} R_i$ is an almost $pp-$ring. Let $i \in I$ and let x_i, y_i be two elements of R_i such that $y_i \in Ann_{R_i}(x_i)$.

Consider $x = (\alpha_j)_{j \in I}$, with $\alpha_j = \begin{cases} x_i, & j = i \\ 0, & j \neq i \end{cases}$ and $y = (\beta_j)_{j \in I}$, with $\beta_j = \begin{cases} y_i, & j = i \\ 0, & j \neq i. \end{cases}$. So, $y \in Ann_R(x)$. Since R is an almost $pp-$ring, then there exists an idempotent $e = (e_j)_{j\in I} \in Ann_R(x)$ such that $y = ey$. That is for all $j \in I$, $\beta_j = e_j\beta_j$, $e_j \in Ann_{R_j}(\alpha_j)$ and $e_j^2 = e_j$.

Hence, $y_i = e_i y_i$, $e_i \in Ann_{R_i}(x_i)$ and $e_i^2 = e_i$.

Thus, R_i is an almost $pp-$ring for all $i \in I$. □

The following Lemma is well known, see for example [12, Proposition 3.2.7 and its corollary]

Lemma 2.4. *Let R be a subring of a ring S both with the same identity. Suppose that S is a free $R-$module with a basis G such that G is multiplicatively closed and $1 \in G$. Let $\varepsilon : S \longrightarrow R$ be a map defined by*

$$\varepsilon\left(\sum_{i=0}^{n} a_i g_i\right) = \sum_{i=0}^{n} a_i.$$

Then ε is a ring epimorphism.

Let R and S be two rings such that $R \subseteq S$. Let e be an idempotent in S. Then, $e^2 = e$, and so, $(\varepsilon(e))^2 = \varepsilon(e^2) = \varepsilon(e)$. Hence, $\varepsilon(e)$ is an idempotent in R.

Suppose $a \in R$ and $e \in Ann_S(a)$. Then, $ea = 0$ in S. So, $\varepsilon(e)a = 0$ in R since $a \in R$, $\varepsilon(a) = a$.Therefore $\varepsilon(e) \in Ann_R(a)$ and $(\varepsilon(e))^2 = \varepsilon(e)$.

Theorem 2.5. *Let R be a subring of a ring S both with the same identity. Suppose that S is a free $R-$module with a basis G such that G is multiplicatively closed and $1 \in G$. If S is almost $pp-$ring, then so is R.*

Proof. Let $a, b \in R$ such that $b \in Ann_R(a) \subseteq Ann_S(a)$. Then, since S is an almost $pp-$ring, there exists an idempotent $e \in Ann_S(a)$ such that $b = be$. Taking ε to both sides, we get

$$b = \varepsilon(b) = \varepsilon(b)\varepsilon(e) = b\varepsilon(e),$$

$\varepsilon(e) \in Ann_R(a)$ and $(\varepsilon(e))^2 = \varepsilon(e)$.

Therefore, R is an almost $pp-$ring. □

3. Group rings

Given a ring R and a group G, we will denote the group ring of G over R by RG. Elements of the ring RG are just formal finite sums of the form $\sum_{g\in G} a_g g$ with all but a finite number of a_g are 0_R. We write C_n for the cyclic group of order n, $\mathbb{Z}$ for the ring of integers, $\mathbb{Z}_n$ for the ring of integers modulo n, and $\mathbb{C}$ is the field of complex numbers. The imaginary unit is denoted by $\mathbf{i}$.

The following facts are consequences of Theorem 2.5.

Corollary 3.1. *Let R be a ring and G be a group. If RG is an almost pp−ring, then so is R.*

Proof. $S = RG$ is a free $R-$module with a basis G satisfying the assumptions of Theorem 2.5. □

Corollary 3.2. *If RG is an almost pp−ring and H is a subgroup of G, then RH is an almost pp−ring too.*

Proof. RH is a subring of RG and RG is a free $RH-$module on the set $\{g_1, g_2, ...\}$, the coset representatives of H in G. □

Recall that a group G is called locally finite, if every finitely generated subgroup of G is finite.

Theorem 3.3. *Let G be a locally finite group. If RH is an almost pp−ring for all finite subgroup H of G, then RG is an almost pp−ring.*

Proof. Let $u = \sum_{i=1}^{n} a_i g_i \in RG$ and $v = \sum_{i=1}^{n} b_i g_i \in Ann_{RG}(u)$. Let $H = \langle g_1, ..., g_n \rangle$. Then H is finite since G is locally finite. Since $u, v \in RH$, $v \in Ann_{RH}(u) \subseteq Ann_{RG}(u)$. But RH is an almost $pp-$ring by assumption. Thus there exists an idempotent $e \in Ann_{RH}(u) \subseteq Ann_{RG}(u)$ such that $v = ev$. So, RG is an almost $pp-$ring. □

Corollary 3.4. *Let G be a locally finite group. Then RG is an almost pp−ring if and only if for every finite subgroup H of G, RH is an almost pp−ring.*

Proof. If RG is an almost $pp-$ring, by Corollary 3.2, RH is an almost $pp-$ring for all subgroup H of G. So, in particular, for every finite subgroup H of G, RH is an almost $pp-$ring. By Theorem 3.3, the other direction holds. □

Using the same technique used in Theorem 3.3, we get the following:

Corollary 3.5. *The group ring RG is an almost pp−ring if and only if RH is an almost pp−ring for each subgroup H of G.*

Theorem 3.6. *If RG is an almost pp−ring, then R is an almost pp−ring and the order of each finite order element $g \in G$ is a unit in R.*

Proof. By Corollary 3.1, R is almost $pp-$ring. Now let $g \in G$ with $|g| = n < \infty$. Let H be the cyclic subgroup generated by g. Then, by Corollary 3.2, RH is an almost $pp-$ring too. Now $1 + g + g^2 + ... + g^{n-1} \in Ann_{RH}(1 - g)$.

Since RH is an almost $pp-$ring, there exists an idempotent $e = a_0 + a_1 g + ... + a_{n-1} g^{n-1} \in Ann_{RH}(1 - g)$ such that $\left(1 + g + g^2 + ... + g^{n-1}\right) e = 1 + g + g^2 + ... + g^{n-1}$ and $e(1 - g) = 0$.

Thus, $a_0 = a_1 = ... = a_{n-1}$ since $e = eg$.

So, $\left(a_0 + a_0 g + ... + a_0 g^{n-1}\right)\left(1 + g + ... + g^{n-1}\right) = 1 + g + ... + g^{n-1}$. Hence, $a_0 n = 1$.

Therefore, n is a unit in R. □

Corollary 3.7. *If G is a finite group and RG is an almost pp−ring, then $|G|^{-1} \in R$.*

Proof. Let G be a finite group and $|G| = n = \prod_{i=1}^{k} p_i^{\alpha_i}$ where p_i are distinct primes and $\alpha_i \geq 1$ are positive integers for all $i = 1, ..., k$. Then by Cauchy Theorem, there exists $g_i \in G$ such that $|g_i| = p_i$, for all $i = 1, ..., k$.

Thus, since RG is an almost pp−ring and by Theorem 3.6, $p_i^{-1} \in R$ for all $i = 1, ..., k$. But the product of units is a unit.

So, $\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)^{-1} \in R$ and hence $|G|^{-1} \in R$. □

We will see in Example 3.23 that the converse of this Corollary needs not be true.

Example 3.8. If G is a finite group and $\mathbb{Z}_{p^r}G$ is an almost pp−ring, p is prime integer and $r \geq 1$, then by the previous Corollary $|G|$ is a unit in $\mathbb{Z}_{p^r}$. So, $p \nmid |G|$ and hence $\gcd(p, |G|) = 1$.

Example 3.9. $\mathbb{Z}_{p^r}G$ is not an almost pp−ring for any finite p−group G, where p is prime integer and $r \geq 1$. More generally, if $p \mid n$ and G is a p−group, then $\mathbb{Z}_n G$ is not an almost pp−ring.

The following example shows that if R is an almost pp−ring, it is not necessary that RG is an almost pp−ring.

Example 3.10. $\mathbb{Z}G$ is not almost pp−ring for any nontrivial finite group G.

If R is a ring and G is an Abelian group, then RG is a Noetherian ring if and only if R is a Noetherian ring and G is a finitely generated group, see [9, Theorem 2].

Theorem 3.11. *Let G be a finite Abelian group and n be an integer with $n > 1$. Then the following are equivalent:*

(1) *$\mathbb{Z}_n G$ is an almost pp−ring.*

(2) *$\mathbb{Z}_n G$ is a pp−ring.*

(3) *$\gcd(n, |G|) = 1$ and n is square free.*

Proof. (1) $\Longleftrightarrow$ (2) Since $\mathbb{Z}_n$ is Noetherian ring and G is finite Abelian group, it follows that $\mathbb{Z}_n G$ is Noetherian ring. Then the result follows by Lemma 2.2.

(2) $\Longleftrightarrow$ (3) See [15, Example 1.8]. □

The following Lemma exists in [12, page 134]

Lemma 3.12. $(R_1 \times R_2 \times ... \times R_n)\, G \cong \prod_{i=1}^{n} R_i G$

Theorem 3.13. *If $R = R_1 \times R_2 \times ... \times R_n$, then RG is an almost $pp-$ring if and only if R_iG is an almost $pp-$ring for all $i = 1, ..., n$.*

Proof. The proof follows from Lemma 2.3 and Lemma 3.12. □

Theorem 3.14. *If $R[x]/\left(x^n + a_1x^{n-1} + ... + a_n\right)$ is an almost $pp-$ring, where $a_1, ..., a_n \in R$ and n is a positive integer, then R is an almost $pp-$ring.*

Proof. $S = R[x]/\left(x^n + a_1x^{n-1} + ... + a_n\right)$ is free $R-$module with a basis $\left\{1, x, ..., x^{n-1}\right\}$ satisfying the assumptions of Theorem 2.5. □

The following proposition was introduced by Yi and Zhou [14].

Proposition 3.15. *Let R be a ring. Then*
(1) *If $2^{-1} \in R$, then $RC_2 \cong R \times R$ and $RC_4 \cong R \times R \times \left(R[x]/\left(x^2+1\right)\right)$*
(2) *If $R \subseteq \mathbb{C}$ and $3^{-1} \in R$, then $RC_3 \cong R \times \left(R[x]/\left(x^2+x+1\right)\right)$.*

Theorem 3.16. *RC_2 is almost $pp-$ring if and only if R is almost $pp-$ring and $2^{-1} \in R$.*

Proof. The proof follows from Corollary 3.1, Corollary 3.7 and Lemma 2.3. □

Theorem 3.17. *RC_4 is an almost $pp-$ring if and only if $R[x]/\left(x^2+1\right)$ is an almost $pp-$ring and $2^{-1} \in R$.*

Proof. By Proposition 3.15, if $2^{-1} \in R$ then $RC_4 \cong R \times R \times (R[x]/(x^2+1).)$. So, using Theorem 3.14, we get RC_4 is an almost $pp-$ring if and only if $R[x]/(x^2+1)$. is an almost $pp-$ring. □

Theorem 3.18. *If $R \subseteq \mathbb{C}$, then RC_3 is an almost $pp-$ring if and only if $R[x]/\left(x^2+x+1\right)$ is an almost $pp-$ring and $3^{-1} \in R$.*

Proof. By Proposition 3.15, if $3^{-1} \in R$, then $RC_3 \cong R \times (R[x]/(x^2+x+1).)$. So, using Theorem 3.14, we get RC_3 is an almost $pp-$ring if and only if $R[x]/(x^2+x+1)$. is an almost $pp-$ring. □

Note that, if $G = H \times K$, then $RG = R(H \times K) \cong (RH)K$.

Theorem 3.19. *If $R \subseteq \mathbb{C}$, then RC_6 is an almost $pp-$ring if and only if $6^{-1} \in R$ and $R[x]/\left(x^2+x+1\right)$ is an almost $pp-$ring.*

Proof. Since $C_6 \cong C_3 \times C_2$, then $RC_6 \cong (RC_3)C_2$.

So, RC_6 is an almost $pp-$ring if and only if $2^{-1} \in RC_3$ and RC_3 is an almost $pp-$ring.

Since $R \subseteq \mathbb{C}$, RC_3 is an almost $pp-$ring if and only if $3^{-1} \in R$ and $R[x]/\left(x^2+x+1\right)$ is an almost $pp-$ring.

Hence, RC_6 is an almost $pp-$ring if and only if $2^{-1} \in RC_3$, $3^{-1} \in R$ and $R[x] \big/ \big(x^2 + x + 1\big)$ is an almost $pp-$ring.

Now, $2^{-1} \in RC_3$ if and only if $2^{-1} \in R$. To see this, assume that $2^{-1} \in RC_3$. Then $2\left(\sum_{i=0}^{2} a_i g^i\right) = 1$ for some $a_i \in R$, $i = 0, 1, 2$. So, $2a_0 = 1$ and $2a_1 = 2a_2 = 0$. Thus, $2^{-1} \in R$. The converse is clear.

Therefore, we are done. □

Theorem 3.20. *Let R be a von Neumann regular ring and G be a locally finite group. Then the following are equivalent:*

(1) *RG is a $pp-$ring.*

(2) *RG is an almost $pp-$ring.*

(3) *The order of every finite subgroup of G is a unit in R.*

Proof. (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (3) Since RG is an almost $pp-$ring, we have RH is an almost $pp-$ring for every subgroup H of G. So, if H is finite subgroup of G, then $|H|^{-1} \in R$.

(3) $\Longrightarrow$ (1) See [15, Proposition 1.9]. □

Lemma 3.21. *Let R_1 and R_2 be two integral domains, and let T be a non-integral domain subring of $R = R_1 \times R_2$ containing the identity element $(1, 1)$. Then T is an almost $pp-$ring if and only if $(0, 1) \in T$.*

Proof. Assume that T is an almost $pp-$ring. Since T is not an integral domain, there are non-zero elements $(a, b), (c, d) \in T$ such that $(a, b) . (c, d) = (0, 0)$. Since $(a, b) \neq (0, 0)$, either $a \neq 0$ or $b \neq 0$, say $a \neq 0$. Thus $c = 0$ and $d \neq 0$. Since $(c, d) \in Ann_T((a, b))$, there exists an idempotent $(x, y) \in Ann_T((a, b))$ such that $(c, d)(x, y) = (c, d)$. So, $dy = d$ and $d \neq 0$ in R_2. Thus $y = 1$. Since $xa = 0$ and $a \neq 0$ in R_1, $x = 0$.So, $(x, y) = (0, 1) \in T$.

Now, assume that $(0, 1) \in T$. Then, $(1, 1) - (0, 1) = (1, 0) \in T$. Consider any $(0, 0) \neq (a, b) \in T$. If $a \neq 0, b \neq 0$, $Ann_T((a, b)) = \{(0, 0)\}$. If $a = 0$, $b \neq 0$, $Ann_T((a, b)) = (1, 0)T$ and if $a \neq 0$, $b = 0$, $Ann_T((a, b)) = (0, 1)T$. Also if $a = b = 0$, then $Ann_T((a, b)) = T$ So, $Ann_T((a, b))$ is generated by its idempotents for all $(a, b) \in T$. Hence T is an almost $pp-$ring. □

Theorem 3.22. *Let R be an integral domain and let $Q(R)$ denotes the quotient field of R. Consider the polynomial $x^2 + a_1x + a_2 \in R[x]$ with α, β are its roots in some field extension, and $\alpha - \beta$ is a unit in R. Then $R[x] \big/ \big(x^2 + a_1x + a_2\big)$ is an almost $pp-$ring if and only if either $\alpha \in R$ or $\alpha \notin Q(R)$.*

Proof. Let $T = R[x] \big/ \big(x^2 + a_1x + a_2\big)$ and $x^2 + a_1x + a_2 = (x - \alpha)(x - \beta)$. By hypothesis, $\alpha \neq \beta$. First suppose $\alpha \notin Q(R)$. Then $x^2 + a_1x + a_2$ is irreducible over $Q(R)$ and hence it is irreducible over R since the polynomial is monic. Thus, T is an integral domain. In particular T is an almost $pp-$ring.

If $\alpha \in Q(R)$, then define $\Phi : R[x] \longrightarrow Q(R) \times Q(R)$ by $\Phi(f(x)) = (f(\alpha), f(\beta)) \in Q(R) \times Q(R)$. Then Φ is a ring homomorphisim with $Ker(\Phi) = \big(x^2 + a_1x + a_2\big)$. Hence, T is a subring of $Q(R) \times Q(R)$.

Assume now that T is an almost $pp-$ring, and so it follows by Lemma 3.21 that $(0,1) \in T$.

Thus there exists $ax+b \in R[x]$ such that $a\alpha + b = 0$ and $a\beta + b = 1$. But since $x^2 + a_1x + a_2 = (x-\alpha)(x-\beta)$, $\alpha + \beta = -a_1$ and $\alpha\beta = a_2$. So,

$$2(b-1)b = 2(-a\beta)(-a\alpha) = 2a^2\alpha\beta = 2a^2a_2.$$

Also,

$$2(b-1)b = (2b-2)b = -(1+a(\alpha+\beta)b = -(1-aa_1)b = (aa_1-1)b.$$

So,

$$2a^2a_2 = (aa_1-1)b.$$

Thus,

$$-b = 2a^2a_2 - aa_1b.$$

Hence

$$\alpha = -\frac{b}{a} = 2aa_2 - a_1b \in R.$$

So, $\alpha \in R$.

Now, assume that $\alpha \in R$,and define $p(x) = \dfrac{x-\alpha}{\beta-\alpha}$. Then $p(x) \in R[x]$, since $\beta - \alpha$ is a unit. But $\Phi(p(x)) = (p(\alpha), p(\beta) = (0,1)$. Thus it follows by Lemma 3.21 that T is an almost $pp-$ring. □

Example 3.23. Let $S = \left\{\frac{n}{3^k} : n,k \in \mathbb{Z},\ k \geq 0\right\}$. Then S is a subring of $\mathbb{Q}$. Set $R = \left\{a + \sqrt{3}b\mathbf{i} : a,b \in S\right\}$. Then R is a subring of $\mathbb{C}$ with $\frac{1}{3} \in R$. Because R is a domain, it is certainly almost $pp-$ring.

Let $f(x) = x^2 + x + 1 \in R[x]$. Then $\alpha = \dfrac{-1+\sqrt{3}\mathbf{i}}{2} \notin R$.

Let $r = 2\sqrt{3}\mathbf{i}$, $s = -\left(3 + \sqrt{3}\mathbf{i}\right)$. Then $r, s \in R$ and $\alpha = \dfrac{s}{r} \in Q(R)$.

Since $(\alpha-\beta)^{-1} = \left(\sqrt{3}\mathbf{i}\right)^{-1} = -\dfrac{\sqrt{3}}{3}\mathbf{i} \in R$, $RC_3 \cong R \times \left(R[x] / \left(x^2+x+1\right)\right)$ is not an almost $pp-$ring.

The above example shows that RC_3 is not an almost $pp-$ring although 3 is a unit in R and R is an almost $pp-$ring.

4. Polynomial rings

Let R be a reduced ring and $h(x) = h_0 + h_1x + ... + h_nx^n \in R[x]$. Then $Ann_{R[x]}(h(x)) = N[x]$, where N is the annihilator of the ideal generated by $h_0, h_1, ..., h_n$ (i.e. $N = Ann_R(h_0, h_1, ..., h_n) = \bigcap_{i=0}^{n} Ann_R(h_i)$). Moreover, if $f(x) = a_0 + a_1x + ... + a_mx^m \in Ann_{R[x]}(h(x))$, then $a_ih_j = 0$, $\forall i = 0,1,...,m$, $j = 0,1,...,n$, see [6, Theorem 10].

Theorem 4.1. *If $R[x]$ or $R\left[x, x^{-1}\right]$ are almost pp− rings, then so is R.*

Proof. $R[x]$ and $R[x, x^{-1}]$ are free $R-$modules with bases $\{x^i : i = 0, 1, ...\}$ and $\{x^i : i = 0, \pm 1, ...\}$, respectively, satisfying the assumptions of Theorem 2.5. □

Theorem 4.2. *The polynomial ring $R[x]$ is an almost pp−ring if and only if the ring R is an almost pp−ring.*

Proof. By Theorem 4.1, if $R[x]$ is an almost $pp-$ring then R is an almost $pp-$ring. So, let R be an almost $pp-$ring and $f(x) = a_0 + a_1 x + ... + a_n x^n \in R[x]$. Then, since R is a reduced ring, $Ann_{R[x]}(f(x)) = N[x]$ where $N = \bigcap_{i=0}^{n} Ann_R(a_i)$.

Now, let $g(x) = b_0 + b_1 x + ... + b_m x^m \in Ann_{R[x]}(f(x))$. Then, $b_j \in Ann_R(a_i)$ for all $j = 0, ..., m$, $i = 0, ..., n$. But since R is an almost $pp-$ring, there is an idempotent $e_{ji} \in Ann_R(a_i)$ such that $b_j e_{ji} = b_j$ for all $j = 0, ..., m$, $i = 0, ..., n$. Taking $e_j = \prod_{i=0}^{n} e_{ji}$, we have $e_j \in N$ and $b_j e_j = b_j$ for all $j = 0, ..., m$. Taking $1 - e = \prod_{j=0}^{m}(1 - e_j)$, we have $e \in N$ is an idempotent and $b_j e = b_j$ for all $j = 0, ..., m$.

Thus $eg(x) = g(x)$ and $e \in N[x] = Ann_{R[x]}(f(x))$ and hence $R[x]$ is an almost $pp-$ring. □

The ring of Laurent polynomials $R\left[x, x^{-1}\right]$ is the localization of the polynomial ring at the multiplicative set consisting of the non negative powers of x. The ring R is an almost $pp-$ring if and only if for every $R \subset S \subset Q(R)$, S is an almost $pp-$ring where $Q(R)$ is the total quotient ring of R, see [4, Theorem 3].

So, by Theorem 4.2 and Theorem 4.1, $R\left[x, x^{-1}\right]$ is an almost $pp-$ring if and only if R is an almost $pp-$ring.

In fact, the Laurent polynomial ring $R\left[x, x^{-1}\right]$ is isomorphic to the group ring of the group $\mathbb{Z}$ of integers over R. Thus, $R\mathbb{Z}$ is an almost $pp-$ring if and only if R is an almost $pp-$ring.

5. Power series rings

Let $R[[x]]$ be the power series ring over the ring R. For any reduced ring R, it was proved in Brewer [6] that $Ann_{R[[x]]}(a_0 + a_1 x \ldots) = N[[x]]$ where N is the annihilator of the ideal generated by the coefficients $a_0, a_1, ...$ Moreover if $b_0 + b_1 x + ... \in Ann_{R[[x]]}(a_0 + a_1 x \ldots)$, then $b_i a_j = 0$ for all $i = 0, 1, ...;\ j = 0, 1, \ldots$

Theorem 5.1. *If $R[[x]]$ is an almost pp−ring, then so is R.*

Proof. $R[[x]]$ is free $R-$module with basis $\left\{x^i : i = 0, 1, ...\right\}$ satisfying the assumption of Theorem 2.5. □

Theorem 5.2. *The power series $R[[x]]$ is an almost pp−ring if and only if for any two countable sets $S = \{b_0, b_1, ...\}$ and $T = \{a_0, a_1, ...\}$ such that $S \subseteq$*

$Ann_R(T)$, there exists an idempotent $e \in Ann_R(T)$ such that $b_i = b_i e$ for all $i = 0, 1, ...$

Proof. Assume that $R[[x]]$ is an almost $pp-$ring.

Let $\{b_0, b_1, ...\} \subseteq Ann_R(a_0, a_1, ...)$. Let $g(x) = b_0 + b_1x + ...$ and $f(x) = a_0 + a_1x \dots$ Then $g(x) \in Ann_{R[[x]]}(f(x))$. Therefore, there exists an idempotent $e \in Ann_{R[[x]]}(f(x)) \cap R$ such that $eg(x) = g(x)$. Thus $e \in Ann_R(a_0, a_1, ...)$ and $eb_i = b_i$, for all $i = 0, 1, ...$

Conversely, the ring R is an almost $pp-$ring because for all $b \in Ann_R(a)$ there exists an idempotent $e \in Ann_R(a)$ such that $be = b$, and so, R is a reduced ring.

Let $g(x) = b_0 + b_1x + ...$, $f(x) = a_0 + a_1x \dots \in R[[x]]$ such that $g(x) \in Ann_{R[[x]]}(f(x))$. Then $g(x) f(x) = 0$.

Thus $b_i a_j = 0$ for all $i = 0, 1, ...; j = 0, 1, ...$

So, $\{b_0, b_1, ...\} \subseteq Ann_R(a_0, a_1, ...)$. By assumption, there exists an idempotent $e \in Ann_R(a_0, a_1, ...)$ such that $eb_i = b_i$ for all $i = 0, 1, ...$

Hence $eg(x) = g(x)$ and $e \in Ann_{R[[x]]}(f(x))$.

Thus $R[[x]]$ is an almost $pp-$ring. □

Example 5.3. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots$ be the direct sum. Then R is an almost $pp-$ring since every element in R is an idempotent. But $R[[x]]$ is not an almost $pp-$ring, because $S = \{(0, 1, 0, ..), (0, 0, 1, 0, ...), (0, 0, 0, 1, 0, ...), ...\} \subseteq Ann_R((1, 0, 0, ...))$ and there are no idempotent elements in R that can fix the set S.

Corollary 5.4. *If R has a finite number of idempotents, then $R[[x]]$ is an almost pp−ring if and only if R is an almost pp−ring.*

Proof. If $R[[x]]$ is an almost $pp-$ring, then by Theorem 5.1 R is an almost $pp-$ring.

Conversely, assume that R is an almost $pp-$ring and let $g(x) = b_0 + b_1x + ...$, $f(x) = a_0 + a_1x \dots \in R[[x]]$ such that $g(x) \in Ann_{R[[x]]}(f(x))$. Since R is reduced ring, then $bi \in Ann_R(a_j)$ for all $i = 0, 1, ...; j = 0, 1, \dots$

So, there exists an idempotent $e_{ji} \in Ann_R(a_j)$ such that $e_{ji}b_i = b_i$ for all $i = 0, 1, ...; j = 0, 1, \dots$ But R has a finite number of idempotent and so, we can find an idempotent $e \in Ann_R(a_j)$ such that $eb_i = b_i$ for all $i = 0, 1, ...;$ $j = 0, 1, \dots$ Hence, $eg(x) = g(x)$ and $e \in Ann_{R[[x]]}(f(x))$.

Thus, $R[[x]]$ is an almost $pp-$ring. □

Kim in [11] proved that if R is a Noetherian ring, then $R[[x]]$ is a $pp-$ring if and only if R is a $pp-$ring, see [11, Theorem 4]. We now give an analogue result for almost $pp-$rings.

Corollary 5.5. *If R is a Noetherian ring, then $R[[x]]$ is an almost pp−ring if and only if R is an almost pp−ring.*

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