Common fuzzy fixed points of α-fuzzy mappings

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Abstract. The aim of this paper is to obtain the common fuzzy fixed points of α-fuzzy mappings satisfying generalized almost Θ-*contraction* in the setting of complete metric space. In this way, we generalize several well known recent and classical results. Finally, we provide an example to show the significance of the investigation of this paper.

Keywords: complete metric space, almost Θ-contractions, fixed point, fuzzy mappings.

1. Introduction and preliminaries

In 1922, Banach [11] presented a revolutionary contraction principle (namely called Banach contraction principle) in which Picard iteration process was used for the evaluation of a fixed point. This principle guarantees the existence and uniqueness of fixed points of certain self-mappings of metric spaces, and provides a constructive method to find those fixed points. The Banach contraction principle was also used to establish the existence of a unique solution for a nonlinear integral equation [22]. For instance, it has been used to show the existence of solutions of nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach spaces and to show the convergence of algorithms in compu-

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tational mathematics. Because of its importance and usefulness for mathematical theory, it has become a very popular tool in solving existence problems in many directions. Several authors have obtained various extensions and generalizations of Banach's theorem by defining a variety of contractive type conditions for self and non-self mappings on metric spaces.

In [12, 13] Berinde studied many kinds of contraction mappings and gave the concept of almost contraction in following way.

Definition 1 ([12])**.** *Let (X, d) be a metric space. A mapping $T : X \to X$ is called an almost contraction if there exists a constant $\lambda \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$.

He also generalized the above almost contraction in this way.

Definition 2 ([13])**.** *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be generalized almost contraction if there exists a constant $\lambda \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Very recently, Jleli and Samet [20] introduced a new type of contraction called Θ-contraction and established some new fixed point theorems for such a contraction in the context of generalized metric spaces.

Definition 3. *Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:*

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if

$$\lim_{n\to\infty} (\alpha_n) = 0;$$

(Θ_3) there exists $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.

A mapping $T : X \to X$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \Longrightarrow \Theta(d(Tx, Ty)) \leq [\Theta(d(x, y))]^k. \tag{1.1}$$

Theorem 4 ([20])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a Θ-contraction, then T has a unique fixed point.*

Later on Hancer et al.[17] modified the above definitions by adding a general condition (Θ_4) which is given in this way:

(Θ_4) $\Theta(\inf A) = \inf \Theta(A)$, for all $A \subset (0, \infty)$ with $\inf A > 0$.

Following Hancer et al. [17], we represent the set of all continuous functions $\Theta : \mathbb{R}^+ \to \mathbb{R}$ satisfying $(\Theta_1) - (\Theta_4)$ conditions by Ω.

For more details on Θ-contractions, we refer the reader to [2, 4, 19, 23, 25, 30].

Following the Banach contraction principle Nadler [24] introduced the concept of multi-valued contractions using the Hausdorff metric and established that a multi-valued contraction possesses a fixed point in a complete metric space. In 1981, Heilpern [18] used the concept of fuzzy set to introduce a class of fuzzy mappings, which is a generalization of the set-valued mappings, and proved a fixed point theorem for fuzzy contraction mappings in metric linear space. It is worth noting that the result announced by Heilpern [18] is a fuzzy extension of the Banach contraction principle. Subsequently, several other authors have studied existence of fixed points of fuzzy mappings, for example, Azam et [8, 9], Bose et al. [14], Chang et al. [15], Cho et al. [16], Qiu et al. [26], Rashwan et al. [27], Shi-sheng [29].

In the following we always suppose that (X, d) is a complete metric space. Moreover, we shall use the following notations which have been recorded from [1, 5, 10, 28, 31]:

Let $CB(X)$ be the family of nonempty, closed and bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

where

$$d(x, A) = \inf_{y\in A} d(x, y).$$

A fuzzy set in X is a function with domain X and values in $[0, 1]$, I^X is the collection of all fuzzy sets in X. If A is a fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of x in A. The α -level set of A is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1],$$
$$[A]_0 = \overline{\{x : A(x) > 0\}}.$$

Here $\overline{B}$ denotes the closure of the set B. Let $\mathcal{F}(X)$ be the collection of all fuzzy sets in a metric space X. For $A, B \in \mathcal{F}(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. We denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless otherwise is stated, where $\chi_{\{x\}}$ is the characteristic function of the crisp set A. If there exists an $\alpha \in [0, 1]$ such that $[A]_\alpha, [B]_\alpha \in CB(X)$, then define

$$p_\alpha(A, B) = \inf_{x\in[A]_\alpha, y\in[B]_\alpha} d(x, y),$$

$$D_\alpha(A,B) = H([A]_\alpha, [B]_\alpha).$$

If $[A]_\alpha, [B]_\alpha \in CB(X)$ for each $\alpha \in [0,1]$, then define

$$p(A,B) = \sup_\alpha p_\alpha(A,B),$$

$$d_\infty(A,B) = \sup_\alpha D_\alpha(A,B).$$

We write $p(x,B)$ instead of $p(\{x\},B)$. A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0,1]$ and $\sup_{x\in V} A(x) = 1$. The collection of all approximate quantities in V is denoted by $W(V)$. Let X be an arbitrary set, Y be a metric space. A mapping T is called fuzzy mapping if T is a mapping from X into $\mathcal{F}(Y)$. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 5. *Let S,T be fuzzy mappings from X into $\mathcal{F}(X)$. A point $u \in X$ is called an α- fuzzy fixed point of T if there exists $\alpha \in [0,1]$ such that $u \in [Tu]_\alpha$. The point $u \in X$ is called a common α- fuzzy fixed point of S and T if there exists $\alpha \in [0,1]$ such that $u \in [Su]_\alpha \cap [Tu]_\alpha$. When $\alpha = 1$, it is called a common fixed point of fuzzy mappings.*

For the sake of convenience, we first state some known results for subsequent use in the next section.

Lemma 6. *Let (X,d) be a metric space and $A,B \in CB(X)$. Then for each $a \in A$,*

$$d(a,B) \leq H(A,B).$$

Lemma 7 ([5])**.** *Let V be a metric linear space, $T : X \to W(V)$ be a fuzzy mapping and $x_0 \in V$. Then there exists $x_1 \in V$ such that $\{x_1\} \subset T(x_0)$.*

2. Main results

Theorem 8. *Let (X,d) be a complete metric space and let S,T be fuzzy mappings from X into $\mathcal{F}(X)$ and for each $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0,1]$ such that $[Sx]_{\alpha_S(x)}$, $[Ty]_{\alpha_T(x)}$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta \in \Omega$, $k \in (0,1)$ and $L \geq 0$ such that*

$$\Theta\left(H\left([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}\right)\right) \leq \Theta(d(x,y),)^k + LM(x,y), \tag{2.1}$$

for all $x,y \in X$ with $H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) > 0$, where

$$M(x,y) = \min\Big\{d\left(x,[Sx]_{\alpha_S(x)}\right), d\left(y,[Ty]_{\alpha_T(y)}\right), d\left(x,[Ty]_{\alpha_T(y)}\right),$$
$$d\left(y,[Sx]_{\alpha_S(x)}\right)\Big\}. \tag{2.2}$$

Then there exists some $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Proof. Let x_0 be an arbitrary point in X, then by hypotheses there exists $\alpha_S(x_0) \in (0,1]$ such that $[Sx_0]_{\alpha_S(x_0)}$ is a nonempty, closed and bounded subset of X. For convenience, we denote $\alpha_S(x_0)$ by α_1. Let $x_1 \in [Sx_0]_{\alpha_S(x_0)}$. For this x_1, there exists $\alpha_T(x_1) \in (0,1]$ such that $[Tx_1]_{\alpha_T(x_1)}$ is a nonempty, closed and bounded subset of X. By Lemma 6, (Θ_1) and (2.1), we have

$$\begin{aligned}\Theta\left(d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right)\right) &\leq \Theta(H\left([Sx_0]_{\alpha_S(x_0)}, [Tx_1]_{\alpha_T(x_1)}\right) \\ &\leq \Theta(d(x_0,x_1))^k + LM(x_0,x_1),\end{aligned}$$

where

$$M(x_0,x_1) = \min\left\{\begin{array}{c} d\left(x_0, [Sx_0]_{\alpha_S(x_0)}\right), d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right), \\ d\left(x_0, [Tx_1]_{\alpha_T(x_1)}\right), d\left(x_1, [Sx_0]_{\alpha_S(x_0)}\right) \end{array}\right\}.$$

From (Θ_4), we know that

$$\Theta\left(d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right)\right) = \inf_{y \in [Tx_1]_{\alpha_T(x_1)}} \Theta(d(x_1,y)).$$

Thus

$$\begin{aligned}\inf_{y \in [Tx_1]_{\alpha_T(x_1)}} \Theta(d(x_1,y)) \leq [\Theta(d(x_0,x_1)]^k \\ + L\min\left\{\begin{array}{c} d\left(x_0, [Sx_0]_{\alpha_S(x_0)}\right), d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right), \\ d\left(x_0, [Tx_1]_{\alpha_T(x_1)}\right), d\left(x_1, [Sx_0]_{\alpha_S(x_0)}\right) \end{array}\right\}.\end{aligned}$$

Then, from above there exists $x_2 \in [Tx_1]_{\alpha_T(x_1)}$ such that

$$\begin{aligned}\Theta(d(x_1,x_2)) &\leq [\Theta(d(x_0,x_1)]^k \\ &\quad + L\min\left\{d(x_0,x_1), d(x_1,x_2), d(x_0,x_2), d(x_1,x_1)\right\} \\ &= [\Theta(d(x_0,x_1)]^k. \end{aligned} \tag{2.3}$$

For this x_2 there exists $\alpha_S(x_2) \in (0,1]$ such that $[Sx_2]_{\alpha_S(x_2)}$ is a nonempty, closed and bounded subset of X. By Lemma 6, (Θ_1) and (2.1), we have

$$\begin{aligned}\Theta\left(d\left(x_2, [Sx_2]_{\alpha_S(x_2)}\right)\right) &\leq \Theta\left(H\left([Tx_1]_{\alpha_T(x_1)}, [Sx_2]_{\alpha_S(x_2)}\right)\right) \\ &= \Theta\left(H\left([Sx_2]_{\alpha_S(x_2)}, [Tx_1]_{\alpha_T(x_1)}\right)\right) \\ &\leq \Theta(d(x_2,x_1))^k + +LM(x_2,x_1)\end{aligned}$$

where

$$M(x_2,x_1) = \min\left\{\begin{array}{c} d(x_2,x_1), d\left(x_2, [Sx_2]_{\alpha_S(x_2)}\right), d\left(x_1, [Tx_1]_{\alpha_T(x_1)}\right), \\ d\left(x_2, [Tx_1]_{\alpha_T(x_1)}\right), d\left(x_1, [Sx_2]_{\alpha_S(x_2)}\right) \end{array}\right\}.$$

From (Θ_4), we know that

$$\Theta\left[d\left(x_2,[Sx_2]_{\alpha_S(x_2)}\right)\right]=\inf_{y_1\in[Sx_2]_{\alpha_S(x_2)}}\Theta(d(x_2,y_1)).$$

Thus

$$\begin{aligned}&\inf_{y_1\in[Sx_2]_{\alpha_S(x_2)}}\Theta(d(x_2,y_1))\leq\Theta\left[d(x_1,x_2)\right]^k\\&+L\min\left\{\begin{array}{c}d(x_2,x_1),d\left(x_2,[Sx_2]_{\alpha_S(x_2)}\right),d\left(x_1,[Tx_1]_{\alpha_T(x_1)}\right),\\d\left(x_2,[Tx_1]_{\alpha_T(x_1)}\right),d\left(x_1,[Sx_2]_{\alpha_S(x_2)}\right)\end{array}\right\}.\end{aligned}$$

Then, from above there exists $x_3\in[Sx_2]_{\alpha_S(x_2)}$ such that

$$\begin{aligned}\Theta(d(x_2,x_3))&\leq[\Theta(d(x_1,x_2)]^k\\&+L\min\left\{d(x_2,x_1),d\left(x_2,x_3\right),d\left(x_1,x_2\right),d\left(x_2,x_2\right),d\left(x_1,x_3\right)\right\}\\&=[\Theta(d(x_1,x_2)]^k.\end{aligned}\tag{2.4}$$

So, continuing recursively, we obtain a sequence $\{x_n\}$ in X such that

$$x_{2n+1}\in[Sx_{2n}]_{\alpha_S(x_{2n})}\quad\text{and}\quad x_{2n+2}\in[Tx_{2n+1}]_{\alpha_T(x_{2n+1})}\tag{2.5}$$

with

$$\Theta(d(x_{2n+1},x_{2n+2}))\leq[\Theta(d(x_{2n},x_{2n+1})]^k\tag{2.6}$$

and

$$\Theta(d(x_{2n+2},x_{2n+3}))\leq[\Theta(d(x_{2n+1},x_{2n+2})]^k,\tag{2.7}$$

for all $n\in\mathbb{N}$. From (2.6) and (2.7), we have

$$\Theta(d(x_n,x_{n+1}))\leq[\Theta(d(x_{n-1},x_n)]^k\tag{2.8}$$

which further implies that

$$\begin{aligned}\Theta(d(x_n,x_{n+1}))&\leq[\Theta(d(x_{n-1},x_n)]^k\leq[\Theta(d(x_{n-2},x_{n-1})]^{k^2}\\&\leq...\leq[\Theta(d(x_0,x_1)]^{k^n},\end{aligned}\tag{2.9}$$

for all $n\in\mathbb{N}$. Since $\Theta\in\Omega$, by taking limit as $n\to\infty$ in (2.9) we have,

$$\lim_{n\to\infty}\Theta(d(x_n,x_{n+1}))=1\tag{2.10}$$

which implies that

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0\tag{2.11}$$

by (Θ_2). From the condition (Θ_3), there exist $0 < r < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} = l. \tag{2.12}$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} - l\right| \leq B,$$

for all $n > n_0$. This implies that

$$\frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} \geq l - B = \frac{l}{2} = B,$$

for all $n > n_0$. Then

$$nd(x_n, x_{n+1})^r \leq An[\Theta(d(x_n, x_{n+1})) - 1], \tag{2.13}$$

for all $n > n_0$, where $A = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$B \leq \frac{\Theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r},$$

for all $n > n_0$. This implies that

$$nd(x_n, x_{n+1})^r \leq An[\Theta(d(x_n, x_{n+1})) - 1],$$

for all $n > n_0$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$nd(x_n, x_{n+1})^r \leq An[\Theta(d(x_n, x_{n+1})) - 1], \tag{2.14}$$

for all $n > n_0$. Thus by (2.9) and (2.14), we get

$$nd(x_n, x_{n+1})^r \leq An([(\Theta d(x_0, x_1))]^{r^n} - 1). \tag{2.15}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} nd(x_n, x_{n+1})^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}, \tag{2.16}$$

for all $n > n_1$. Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have,

$$d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \le \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}. \tag{2.17}$$

Since, $0 < r < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ converges. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence in (X, d). The completeness of (X, d) ensures that there exists $u \in X$ such that, $\lim_{n\to\infty} x_n \to u$. Now, we prove that $u \in [Tu]_{\alpha_T(u)}$. We suppose on the contrary that $u \notin [Tu]_{\alpha_T(u)}$, then there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{2n_k+1}, [Tu]_{\alpha_T(u)}) > 0$, for all $n_k \ge n_0$. Since $d(x_{2n_k+1}, [Tu]_{\alpha_T(u)}) > 0$, for all $n_k \ge n_0$, so by (Θ_1), we have

$$\begin{aligned}
&\Theta\left[d(x_{2n_k+1}, [Tu]_{\alpha_T(u)})\right] \le \Theta\left[H([Sx_{2n_k}]_{\alpha_S(x_{2n_k})}, [Tu]_{\alpha_T(u)})\right] \\
&\le [\Theta(d(x_{2n_k}, u))]^k \\
&+ L \min \left\{ \begin{array}{c} d\left(x_{2n_k}, [Sx_{2n_k}]_{\alpha_S(x_{2n_k})}\right), d\left(u, [Tu]_{\alpha_T(u)}\right), \\ d\left(x_{2n_k}, [Tu]_{\alpha_T(u)}\right), d\left(u, [Sx_{2n_k}]_{\alpha_S(x_{2n_k})}\right) \end{array} \right\} \\
&\le [\Theta(d(x_{2n_k}, u))]^k + L \min \left\{ \begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, x_{2n_k+1}\right), d\left(u, [Tu]_{\alpha_T(u)}\right), \\ d\left(x_{2n_k}, [Tu]_{\alpha_T(u)}\right), d\left(u, x_{2n_k+1}\right) \end{array} \right\}.
\end{aligned}$$

Letting $k \to \infty$, in the above inequality and using the continuity of Θ, we have

$$\Theta\left[d(u, [Tu]_{\alpha_T(u)})\right] \le 0.$$

Hence $u \in [Tu]_{\alpha_T(u)}$. Similarly, one can easily prove that $u \in [Su]_{\alpha_S(u)}$. Thus $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$. □

Corollary 9. *Let (X, d) be a complete metric space and let S, T be fuzzy mappings from X into $\mathcal{F}(X)$ and for each $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)}$, $[Ty]_{\alpha_T(x)}$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta\left(H\left([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}\right)\right) \le \Theta(d(x, y))^k,$$

for all $x, y \in X$ with $H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) > 0$. Then there exists some $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Proof. Taking $L = 0$ in Theorem 8. □

Corollary 10. *Let (X, d) be a complete metric space and let S be fuzzy mapping from X into $\mathcal{F}(X)$ and for each $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that*

$[Sx]_{\alpha_S(x)}$, $[Sy]_{\alpha_S(y)}$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta \in \Omega$, $k \in (0,1)$ and $L \geq 0$ such that

$$\Theta\left(H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_S(y)}\right)\right) \leq \Theta(d(x,y))^k + LM(x,y)$$

for all $x, y \in X$ with $H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_S(y)}\right) > 0$, where

$$M(x,y) = \min\left\{d\left(x, [Sx]_{\alpha_S(x)}\right), d\left(y, [Sy]_{\alpha_S(y)}\right), d\left(x, [Sy]_{\alpha_S(y)}\right), d\left(y, [Sx]_{\alpha_S(x)}\right)\right\}.$$

Then there exists some $u \in X$ such that $u \in [Su]_{\alpha_S(u)}$.

Proof. Taking $S = T$ in Theorem 8. □

Corollary 11. *Let (X,d) be a complete metric space and let S be a fuzzy mapping from X into $\mathcal{F}(X)$ and for each $x \in X$, there exist $\alpha_S(x), \alpha_T(y) \in (0,1]$ such that $[Sx]_{\alpha_S(x)}$, $[Sy]_{\alpha_S(y)}$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta \in \Omega$ and $k \in (0,1)$ such that*

$$\Theta\left(H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_T(y)}\right)\right) \leq \Theta(d(x,y))^k,$$

for all $x, y \in X$ with $H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_T(y)}\right) > 0$. Then there exists some $u \in [Su]_{\alpha_S(u)}$.

Proof. Taking $S = T$ and $L = 0$ in Theorem 8. □

Now we state a common fixed point result for two multivalued mappings.

Theorem 12. *Let (X,d) be a complete metric space and let $F, G : X \to CB(X)$ be multivalued mappings. Assume that there exist some $\Theta \in \Omega$, $k \in (0,1)$ and $L \geq 0$ such that*

$$\Theta(H(Fx, Gy)) \leq \Theta(d(x,y))^k + LM(x,y),$$

for all $x, y \in X$ with $H(Fx, Gy) > 0$, where

$$M(x,y) = \min\{d(x, Fx), d(y, Gy), d(x, Gy), d(y, Fx)\}.$$

Then there exists some $u \in Fu \cap Gu$.

Proof. Consider a mapping $\alpha : X \to (0,1]$ and a pair of fuzzy mappings $S, T : X \to \mathcal{F}(X)$ defined by

$$S(x)(t) = \begin{cases} \alpha(x), & \text{if } t \in Fx, \\ 0, & \text{if } t \notin Fx \end{cases}$$

and

$$T(x)(t) = \begin{cases} \alpha(x), & \text{if } t \in Gx, \\ 0, & \text{if } t \notin Gx. \end{cases}$$

Then$[Sx]_{\alpha(x)} = \{t : S(x)(t) \geq \alpha(x)\} = Fx$ and $[Tx]_{\alpha(x)} = \{t : T(x)(t) \geq \alpha(x)\} = Gx$. Thus, Theorem 8 can be applied to obtain $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)} = Fu \cap Gu$. □

Corollary 13. *Let (X, d) be a complete metric space and let $F, G : X \to CB(X)$ be multivalued mappings. Assume that there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta(H(Fx, Gy)) \leq \Theta(d(x, y))^k,$$

for all $x, y \in X$ with $H(Fx, Gy) > 0$. Then there exists some $u \in Fu \cap Gu$.

Proof. Taking $L = 0$ in Theorem 12. □

Corollary 14. *Let (X, d) be a complete metric space and let $G : X \to CB(X)$ be multivalued mappings. Assume that there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $L \geq 0$ such that*

$$\Theta(H(Gx, Gy)) \leq \Theta(d(x, y))^k + LM(x, y),$$

for all $x, y \in X$ with $H(Gx, Gy) > 0$, where

$$M(x, y) = \min\{d(x, Gx), d(y, Gy), d(x, Gy), d(y, Gx)\}.$$

Then there exists some $u \in X$ such that $u \in Gu$.

Proof. Taking $F = G$ in Theorem 12. □

Corollary 15. *Let (X, d) be a complete metric space and let $G : X \to CB(X)$ be multivalued mappings. Assume that there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta(H(Gx, Gy)) \leq \Theta(d(x, y))^k,$$

for all $x, y \in X$ with $H(Gx, Gy) > 0$. Then there exists some $u \in X$ such that $u \in Gu$.

Proof. Taking $F = G$ and $L = 0$ in Theorem 12. □

Theorem 16. *Let (X, d) be a complete metric linear space and let $S, T : X \to W(X)$ be fuzzy mappings. Suppose that there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $L \geq 0$ such that*

$$\Theta(d_\infty(S(x), T(y))) \leq \Theta(p(x, y))^k + LM(x, y),$$

for all $x, y \in X$ with $d_\infty(S(x), T(y)) > 0$, where

$$M(x, y) = \min\{p(x, S(x)), p(y, T(y)), p(x, T(y)), p(y, S(x))\}.$$

Then there exists some $u \in X$ such that $\{u\} \subset S(u)$ and $\{u\} \subset T(u)$.

Proof. Let $x \in X$, then by Lemma 6 there exists $y \in X$ such that $y \in [Sx]_1$. Similarly, we can find $z \in X$ such that $z \in [Tx]_1$. It follows that for each $x \in X$, $[Sx]_{\alpha(x)}$, $[Tx]_{\alpha(x)}$ are nonempty, closed and bounded subsets of X. As $\alpha(x) = \alpha(y) = 1$, by the definition of a d_∞-metric for fuzzy sets, we have

$$H\left([Sx]_{\alpha(x)}, [Ty]_{\alpha(x)}\right) \leq d_\infty(S(x), T(y)),$$

for all $x, y \in X$. From (Θ_1), we have

$$\begin{aligned}\Theta\left(H\left([Sx]_{\alpha(x)}, [Ty]_{\alpha(x)}\right)\right) &\leq \Theta\left(d_\infty(S(x), T(y))\right) \\ &\leq [\Theta\left(p\left(x, y\right)\right)]^k + LM(x, y)\end{aligned}$$

where

$$M(x, y) = \min\left\{p\left(x, S(x)\right), p\left(y, T(y)\right), p\left(x, T(y)\right), p\left(y, S(x)\right)\right\},$$

for all $x, y \in X$. Since $[Sx]_1 \subseteq [Sx]_\alpha$ for each $\alpha \in (0, 1]$. Therefore $d(x, [Sx]_\alpha) \leq d(x, [Sx]_1)$ for each $\alpha \in (0, 1]$. It implies that $p(x, S(x)) \leq d(x, [Sx]_1)$. Similarly, $p(x, T(x)) \leq d(x, [Tx]_1)$. Furthermore this implies that for all $x, y \in X$,

$$\Theta\left(H\left([Sx]_1, [Ty]_1\right)\right) \leq [\Theta\left(d\left(x, y\right)\right)]^k + LM(x, y)$$

where

$$M(x, y) = \min\left\{d\left(x, [Sx]_1\right), d\left(y, [Ty]_1\right), d\left(x, [Ty]_1\right), d\left(y, [Sx]_1\right)\right\}.$$

Now, by Theorem 8, we obtain $u \in X$ such that $u \in [Su]_1 \cap [Tu]_1$, i.e., $\{u\} \subset T(u)$ and $\{u\} \subset S(u)$. □

Corollary 17. *Let (X, d) be a complete metric linear space and let $S, T : X \to W(X)$ be fuzzy mappings. Suppose that there exist some $\Theta \in \Omega$ and $k \in (0, 1)$ such that*

$$\Theta(d_\infty\left(S(x), T(y)\right)) \leq \Theta(p(x, y))^k,$$

for all $x, y \in X$ with $d_\infty\left(S(x), T(y)\right) > 0$. Then there exists some $u \in X$ such that $\{u\} \subset S(u)$ and $\{u\} \subset T(u)$.

In the following, we suppose that $\widehat{T}$ (for details, see [[28], [29]]) is the set-valued mapping induced by fuzzy mappings $T : X \to \mathcal{F}(X)$, i.e.,

$$\widehat{T}x = \left\{y : T(x)(t) = \max_{t \in X} T(x)(t)\right\}.$$

Proof. Taking $L = 0$ in Theorem 16. □

Corollary 18. *Let (X,d) be a complete metric space and let $S,T : X \to \mathcal{F}(X)$ be fuzzy mappings such that for all $x \in X$, $\widehat{S}(x), \widehat{T}(x)$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta \in \Omega$, $k \in (0,1)$ and $L \geq 0$ such that*

$$\Theta\left(H\left(\widehat{S}(x), \widehat{T}(y)\right)\right) \leq \Theta(d(x,y))^k + LM(x,y),$$

for all $x, y \in X$ with $H(\widehat{S}(x), \widehat{T}(y)) > 0$, where

$$M(x,y) = \min\left\{d\left(x, \widehat{S}(x)\right), d\left(y, \widehat{T}(y)\right), d\left(x, \widehat{T}(y)\right), d\left(y, \widehat{S}(x)\right)\right\}.$$

Then there exists a point $x^* \in X$ such that $S(x^*)(x^*) \geq S(x^*)(x)$ and $T(x^*)(x^*) \geq T(x^*)(x)$, for all $x \in X$.

Proof. By Theorem 12, there exists $x^* \in X$ such that $x^* \in \widehat{S}x^* \cap \widehat{T}x^*$. Then by Lemma 7, we have

$$S(x^*)(x^*) \geq S(x^*)(x) \quad \text{and} \quad T(x^*)(x^*) \geq T(x^*)(x),$$

for all $x \in X$. □

Corollary 19. *Let (X,d) be a complete metric space and let $S,T : X \to \mathcal{F}(X)$ be fuzzy mappings such that for all $x \in X$, $\widehat{S}(x), \widehat{T}(x)$ are nonempty, closed and bounded subsets of X. Assume that there exist some $\Theta \in \Omega$ and $k \in (0,1)$such that*

$$\Theta\left(H\left(\widehat{S}(x), \widehat{T}(y)\right)\right) \leq \Theta(d(x,y))^k,$$

for all $x, y \in X$ with $H(\widehat{S}(x), \widehat{T}(y)) > 0$. Then there exists a point $x^* \in X$ such that $S(x^*)(x^*) \geq S(x^*)(x)$ and $T(x^*)(x^*) \geq T(x^*)(x)$ for all $x \in X$.

Proof. Taking $L = 0$ in Corollary 18. □

Example 20. Let $X = [0,1]$ and define $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(x,y) = |x - y|.$$

Then (X,d) is a complete metric space. Define a pair of mappings $S,T : X \to \mathcal{F}(X)$, for $\alpha \in [0,1]$ as follows:

For $x \in X$, we have

$$S(x)(t) = \begin{cases} \alpha, & \text{if } 0 \leq t \leq \frac{x}{30}, \\ \frac{\alpha}{2}, & \text{if } \frac{x}{30} < t \leq \frac{x}{20}, \\ \frac{\alpha}{3}, & \text{if } \frac{x}{20} < t \leq \frac{x}{10}, \\ \frac{\alpha}{5}, & \text{if } \frac{x}{10} < t \leq 1 \end{cases},$$

and

$$T(x)(t)=\begin{cases}\alpha, & \text{if } 0\leq t\leq \frac{x}{15},\\ \frac{\alpha}{3}, & \text{if } \frac{x}{15}<t\leq \frac{x}{10},\\ \frac{\alpha}{4}, & \text{if } \frac{x}{10}<t\leq \frac{x}{5},\\ \frac{\alpha}{7}, & \text{if } \frac{x}{5}<t\leq 1\end{cases},$$

such that

$$\begin{aligned}[Tx]_{\alpha} &= \left[0,\frac{x}{15}\right],\\ [Sx]_{\alpha} &= \left[0,\frac{x}{30}\right].\end{aligned}$$

Let $\Theta(t)=e^{\sqrt[k]{t}}$. Then there exists some $k=\frac{1}{\sqrt{15}}\in(0,1)$ and $L=0$ such that

$$\Theta\left(H\left([Sx]_{\alpha_{S(x)}},[Ty]_{\alpha_{T(y)}}\right)\right)\leq \Theta(d(x,y))^{k}+LM(x,y),$$

for all $x,y\in X$ with $H([Sx]_{\alpha_{S(x)}},[Ty]_{\alpha_{T(y)}})>0$, where

$$M(x,y)=\min\left\{d\left(x,[Sx]_{\alpha_{S(x)}}\right),d\left(y,[Ty]_{\alpha_{T(y)}}\right),d\left(x,[Ty]_{\alpha_{T(y)}}\right),\right.$$
$$\left.d\left(y,[Sx]_{\alpha_{S(x)}}\right)\right\}$$

is satisfied to obtain $0\in[S0]_{\alpha}\cap[T0]_{\alpha}$.

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Accepted: 19.12.2018