Some results on K-frames

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Abstract. In this paper we present some results on K-frames when $K \in B(H)$ is an injective closed range operator. Also we give a condition on K-frames $\{f_n\}_{n\in N}$ and $\{g_n\}_{n\in N}$ so that $\{f_n + g_n\}_{n\in N}$ is again a K-frame for H. Finally, Schatten class operators are also discussed in terms of K-frames.

Keywords: K-frames, Schatten class operators.

1. Introduction

Frames in Hilbert spaces were introduced by R.J. Duffin and A.C. Schaffer. Later Daubechies, Grossmann and Meyer gave a strong place to frames in harmonic analysis. Frame theory plays an important role in signal processing, sampling theory, coding and communications and so on. Frames were introduced as a better replacement to orthonormal basis. We refer [2] for an introduction to frame theory.

K-frames were introduced by L. Gavruta, to study atomic systems with respect to bounded linear operators. K-frames are more general than classical frames. In K-frames the lower bound only holds for the elements in the range of K.

Some basic definitions and results related to frames and K-frames are contained in section 2. In section 3 we have included some new results on K-frames. Section 4 contains our main results relating K-frames and operators in Schatten classes.

Throughout this paper, H is a separable Hilbert space and we denote by $B(H)$, the space of all linear bounded operators on H. For $K \in B(H)$, we denote $R(K)$ the range of K. Also, $GL(H)$ denote the set of all bounded linear operators which have bounded inverses.

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2. Preliminaries

For a separable Hilbert space H, a sequence $\{f_n\}_{n\in N} \subset H$ is said to be a **frame** ([2]) for H if there exist $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$.If $A = B$, we say that $\{f_n\}_{n\in N}$ is a tight frame in H. Let $K \in B(H)$. We say that $\{f_n\}_{n\in N} \subset H$ is a K**-frame** ([3]) for H if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$.

If $\{f_n\}_{n\in N} \subset H$ is an ordinary frame for H, then $\{Kf_n\}_{n\in N}$ is a K-frame for H ([5]). If $T \in B(H)$ and $\{f_n\}_{n\in N}$ is a K-frame for H, then $\{Tf_n\}_{n\in N}$ is a TK-frame for H ([5]). If $\{f_n\}_{n\in N} \subset H$ is a K-frame for H, then $\{K^N f_n\}_{n\in N}$ is a K^N-frame for H where $N \geq 1$ is a fixed integer ([5]). $\mathcal{F}_K(H) \subset \mathcal{F}_M(H)$ if and only if $R(K) \supset R(M)$ where $\mathcal{F}_K(H), \mathcal{F}_M(H)$ denote the set of all K-frames and M-frames on H ([4]). Also, we use the result: $T \in B(H)$ is an injective and closed range operator if and only if there exists a constant $c > 0$ such that $c\|x\|^2 \leq \|Tx\|^2$, for all $x \in H$ ([6]),in the proof of our main results.

3. K-frames

In this section we present our results on K-frames.

Theorem 3.1. *Let $K \in B(H)$ be an injective and closed range operator.If $\{f_n\}_{n\in N}$ is a frame for $R(K)$, then $\{K^*f_n\}_{n\in N}$ is a frame for H and hence $\{KK^*f_n\}_{n\in N}$ is a K-frame for H.*

Proof. Let $\{f_n\}_{n\in N}$ be a frame for $R(K)$.Then there exist constants $A, B > 0$ such that, for all $x \in R(K)$,

$$A\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2.$$

Also,by our assumption, there exists $c > 0$ such that $c\|x\|^2 \leq \|Kx\|^2$, for all $x \in H$. For $x \in H, Kx \in R(K)$, and we get

$$A\|Kx\|^2 \leq \sum_{n=1}^{n=\infty} |\langle Kx, f_n\rangle|^2 \leq B\|Kx\|^2.$$

Therefore,

$$Ac\|x\|^2 \leq A\|Kx\|^2 \leq \sum_{n=1}^{n=\infty} |\langle Kx, f_n\rangle|^2 \leq B\|Kx\|^2 \leq B\alpha^2\|x\|^2,$$

for all $x \in H$ and for some $\alpha > 0$, i.e.

$$E\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, K^* f_n\rangle|^2 \leq F\|x\|^2,$$

for all $x \in H$ where $E = Ac > 0, F = B\alpha^2 > 0$. Therefore, $\{K^* f_n\}_{n\in N}$ is a frame for H and hence $\{KK^* f_n\}_{n\in N}$ is a K-frame for H. □

Corollary 3.2. *Let $K \in B(H)$ be an injective and closed range operator and $\{f_n\}_{n\in N} \subset H$ be such that $\{(K^{-1})^* f_n\}_{n\in N}$ is a frame for $R(K)$.Then $\{f_n\}_{n\in N}$ is a frame for H.*

Theorem 3.3. *Suppose $\{f_n\}_{n\in N}$ is a K-frame for H where K^* is an injective and closed range operator. Then there exist constants $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^*x\|^2,$$

for all $x \in H$.

Proof. Since $\{f_n\}_{n\in N}$ is a K-frame for H, there exist constants $C, D > 0$ such that

$$C\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq D\|x\|^2,$$

for all $x \in H$. Since $K^* \in B(H)$ is an injective and closed range operator, there exist $d > 0$ such that

$$d\|x\|^2 \leq \|K^*x\|^2,$$

for all $x \in H$. Therefore, for all $x \in H$,

$$C\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq D\|x\|^2 \leq (D/d)\|K^*x\|^2,$$

for all $x \in H$ there exist $A = C, B = D/d > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^*x\|^2.$$

□

Corollary 3.4. *Suppose $\{f_n\}_{n\in N}$ is a K-frame for H where K^* is an injective and closed range operator. Then $\{f_n\}_{n\in N}$ is a frame for H.*

Definition 3.5. *A sequence $\{f_n\}_{n\in N} \subset H$ is said to be a $2K$-frame for H if there exist $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^*x\|^2,$$

for all $x \in H$.

Theorem 3.6. *Let $\{f_n\}_{n\in N}$ be a K-frame for H with bounds A_1, B_1 and $\{g_n\}_{n\in N}$ be a $2K$-frame for H with bounds A_2, B_2 such that $0 < B_2 < A_1$. Then $\{f_n+g_n\}_{n\in N}$ is a K-frame for H with frame bounds A_1-B_2 and $B_1+B_2\|K^*\|^2$.*

Proof. By definition of K-frame and $2K$-frame, we have

$$A_1\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B_1\|x\|^2$$

and

$$A_2\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2 \leq B_2\|K^*x\|^2,$$

for all $x \in H$. Consider,

$$\sum_{n=1}^{n=\infty} |\langle x, f_n + g_n\rangle|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 + \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2 \tag{1}$$

$$\leq B_1\|x\|^2 + B_2\|K^*x\|^2 \tag{2}$$

$$\leq (B_1 + B_2\|K^*\|^2)\|x\|^2, \tag{3}$$

for all $x \in H$.

Consider,

$$\sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 = \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n - g_n\rangle|^2 \tag{4}$$

$$\leq \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n\rangle|^2 + \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2. \tag{5}$$

This implies that,

$$A_1\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n\rangle|^2 + B_2\|K^*x\|^2$$

$$i.e. \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n\rangle|^2 \geq (A_1 - B_2)\|K^*x\|^2$$

where $A_1 - B_2 > 0$.This completes the proof. □

4. K-frames and operators in Schatten classes

Definition 4.1 ([7]). *Let T be a compact operator on a separable Hilbert space H. Given $0 < p < \infty$, we define the* ***Schatten*** *p* ***- class*** *of H, denoted by $S_p(H)$ or simply S_p, to be the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to l^p.$S_p(H)$ is a two sided ideal in $B(H)$.*

Following two theorems by H. Bingyang, L.H. Khoi and K. Zhu gives a characterization for Schatten p-class operators in terms of frames.

Theorem 4.2 ([1]). *Suppose T is a compact operator on H and $2 \leq p < \infty$.Then the following conditions are equivalent:*

(*a*) *$T \in S_p$;*

(*b*) *$\{\|Te_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H;*

(*c*) *$\{\|Tf_n\|\}_{n\in N} \in l^p$ for every frame $\{f_n\}_{n\in N}$ in H.*

Theorem 4.3 ([1]). *Suppose T is a compact operator on H and $0 \leq p \leq 2$.Then the following conditions are equivalent:*

(*a*) *$T \in S_p$;*

(*b*) *$\{\|Te_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N}$ in H;*

(*c*) *$\{\|Tf_n\|\}_{n\in N} \in l^p$ for some frame $\{f_n\}_{n\in N}$ in H.*

At first we focus on the case where $2 \leq p < \infty$.

Theorem 4.4. *Suppose T is a compact operator on H and $K \in B(H)$. If T is in the Schatten class S_p, then $\{\|Tf_n\|\}_{n\in N} \in l^p$ for every K-frame $\{f_n\}_{n\in N}$ in H, where $2 \leq p < \infty$.*

Proof. Suppose $T \in S_p$,$2 \leq p < \infty$.

Let $\{f_n\}_{n\in N}$ be a K-frame for H and $\{e_n\}_{n\in N}$ be an orthonormal basis for H. Then $\{h_n\}_{n\in N} = \{f_n\}_{n\in N} \bigcup \{e_n\}_{n\in N}$ is a frame for H and $\{\|Th_n\|\}_{n\in N} \in l^p$,$2 \leq p < \infty$. Therefore $\{\|Tf_n\|\}_{n\in N} \in l^p$,$2 \leq p < \infty$ and the result is proved. □

Theorem 4.5. *Suppose T is a compact operator on H and $K \in B(H)$.If $\{\|Tf_n\|\}_{n\in N} \in l^p$ for every K-frame $\{f_n\}_{n\in N}$ in H, then $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H, where $2 \leq p < \infty$.*

Proof. Let $\{e_n\}_{n\in N}$ be an orthonormal basis for H. Then $\{Ke_n\}_{n\in N}$ is a K-frame for H. Therefore by our assumption $\{\|TKe_n\|\}_{n\in N} \in l^p$,$2 \leq p < \infty$.Hence $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H. □

Theorem 4.6. *Suppose T is a compact operator on H and $K \in GL(H)$ and $2 \leq p < \infty$.Then the following are equivalent:*

(a) *T is in the Schatten class S_p;*

(b) $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for every K-frame* $\{f_n\}_{n\in N}$ *in* H.

Proof. Clearly, (a) implies (b) holds by Theorem 4.4. Now suppose (b) holds. Then $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H. This implies that $TK \in S_p$. Using the fact that S_p is a two- sided ideal in $B(H)$, $TKK^{-1} \in S_p$, i.e. $T \in S_p$. This completes the proof. □

Now we move onto the case where $0 < p \leq 2$.

Theorem 4.7. *Let T be a compact operator on H and $K \in B(H)$. Suppose* $\{\|Te_n\|\}_{n\in N}\in l^p$ *for some orthonormal basis* $\{e_n\}_{n\in N} \subset H$. *Then* $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for some K-frame* $\{f_n\}_{n\in N}$ *for H, where* $0 < p \leq 2$.

Proof. Suppose $\{\|Te_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N} \subset H$. Then $T \in S_p$, which implies that $TK \in S_p$ for any $K \in B(H)$. By Theorem 4.3, $\{\|TKe_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N}$ in H. Now take $f_n = Ke_n$, so that $\{f_n\}_{n\in N}$ is a K-frame for H and hence the theorem holds. □

Theorem 4.8. *Let T be a compact operator on H and $K \in B(H)$,where K^* is an injective closed range operator. If* $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for some K-frame* $\{f_n\}_{n\in N}$ *for H,then $T \in S_p$, where $0 < p \leq 2$.*

Proof. By Corollary 3.4,if K^* is an injective closed range operator,then every K-frame is a frame and then applying Theorem 4.3, we get $T \in S_p$. □

5. Acknowledgement

The first author acknowledges the financial support of University Grants Commission.

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Accepted: 20.12.2018