Some spectral inclusion for strongly continuous semigroups operators

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Abstract. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. In this paper, we show that if there exists $t_0 > 0$ such that $T(t_0)$ is a pseudo B-Fredholm operator, then $T(t)$ is pseudo B-Fredholm for all $t \geq 0$, which is equivalent that $T(t)$ is generalized Drazin invertible for all $t \geq 0$. Also we prove that the spectral inclusion of strongly continuous semigroup hold for pseudo Fredholm, generalized Drazin and pseudo B-Fredholm spectra.

Keywords: C_0-semigroups, direct decomposition, pseudo Fredholm spectrum, generalized Drazin spectrum, pseudo B-Fredholm spectrum.

1. Introduction

Throughout, X denotes a complex Banach space, let us denote by $B(X)$ the algebra of bounded linear operators on X, let A be a closed linear operator on X with domain $D(A) \subseteq X$, we denote by A^*, $N(A)$, $R(A)$, $R^\infty(A) = \bigcap_{n\geq 0} R(A^n)$, $N^\infty(A) = \bigcup_{n\geq 0} N(A^n)$, $K(A)$, $H_0(T)$, $\rho(A)$, $\sigma(A)$, respectively the adjoint, the null space, the range, the hyper-range, the hyper-kernel, the analytic core, the quasi-nilpotent part, the resolvent set and the spectrum of A.

A closed operator A is said to be semi-regular if $R(A)$ is closed and $N(A) \subseteq R^\infty(A)$, see [11]. A closed linear operator A is said to be upper semi-Fredholm if

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$R(A)$ is closed and $dimN(A) < \infty$, and A is lower semi-Fredholm if $codimR(A) < \infty$. If $dimN(A)$ and $codimR(A)$ are both finite then A is called Fredholm operator.

A closed operator A admits a generalized Kato decomposition (GKD) if there exist M, N two closed subspaces of X, A-invariant such that $X = M \oplus N$ and $A = A_{|N} \oplus A_{|M}$, with $A_{|N}$ is a quasi-nilpotent operator and $A_{|M}$ is a semi-regular operator, in this case A is called a pseudo-Fredholm operator (see [9, Definition 1]). The pseudo-Fredholm spectrum is defined by

$$\sigma_{pF}(A) = \{\lambda \in \mathbb{C} \ : A - \lambda \text{ is not pseudo-Fredholm}\}.$$

An operator A is called a pseudo B-Fredholm operator [1], if $A_{|M}$ is a Fredholm operator and $A_{|N}$ is a quasi-nilpotent operator. If $A_{|M}$ is an upper semi Fredholm operator, A is called upper pseudo B-Fredholm. Also if $A_{|M}$ is a lower semi Fredholm operator, A is called lower pseudo B-Fredholm [17].

The pseudo B-Fredholm spectrum, the upper pseudo B-Fredholm spectrum and the lower pseudo B-Fredholm spectrum are defined respectively by:

$$\sigma_{pBF}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not pseudo B-Fredholm}\},$$
$$\sigma_{upBF}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not upper pseudo B-Fredholm}\},$$
$$\sigma_{lpBF}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not lower pseudo B-Fredholm}\}.$$

The concept of generalized Drazin invertible operator has been defined by Koliha. A closed operator A is said to be generalized Drazin invertible, if there exists an operator $S \in B(X)$, $R(S) \subset D(A), R(I - AS) \subset D(A)$, and $SA = AS$, $SAS = S$, $\sigma(A(I - SA)) = \{0\}$, this is equivalent that $A = A_1 \oplus A_2$ where A_1 is an invertible operator and A_2 is a quasi-nilpotent operator [8].

Let E be a subset of X. E is said T-invariant if $T(E) \subseteq E$. If E and F are two closed T-invariant subspaces of X such that $X = E \oplus F$, we say that T is completely reduced by the pair (E, F) and it is denoted by $(E, F) \in Red(T)$. In this case we write $T = T_{|E} \oplus T_{|F}$ and say that T is the direct sum of $T_{|E}$ and $T_{|F}$.

In [3], M D. Cvetković and SČ. Živković-Zlatanović introduced and studied a new concept of generalized Drazin invertibility of bounded operators as a generalization of generalized Drazin invertible operators. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin bounded below if $H_0(T)$ is closed and complemented with a subspace M in X such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is closed which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is bounded below and $T_{|N}$ is quasi-nilpotent, see [3, Theorem 3.6]. An operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin surjective if $K(T)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T)$ and $(K(T), N) \in Red(T)$ which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is surjective and $T_{|N}$ is quasi-nilpotent, see [3, Theorem 3.7].

The generalized Drazin invertible spectrum, generalized Drazin bounded below and surjective of $T \in \mathcal{B}(X)$ are defined respectively by

$\sigma_{gD}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not generalized Drazin } \}$,
$\sigma_{gD\mathcal{M}}(T) = \{\lambda \in \mathbb{C}, \ T - \lambda I \text{ is not generalized Drazin bounded below}\}$;
$\sigma_{gD\mathcal{Q}}(T) = \{\lambda \in \mathbb{C}, \ T - \lambda I \text{ is not generalized Drazin surjective}\}$.

We have:

$$\sigma_{gD}(T) = \sigma_{gD\mathcal{M}}(T) \cup \sigma_{gD\mathcal{Q}}(T).$$

A family $(T(t))_{t\geq 0}$ of operators on X is called a strongly continuous semigroup of operators if:

1. $T(0) := I$,
2. $T(s+t) := T(s)T(t) \quad$ for all $s, t \geq 0$
3. $\lim_{t\downarrow 0} T(t)x := x$, for every $x \in X$.

The linear operator A defined in the domain:

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} exists\}$$

by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt} |_{t=0} \quad \text{for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$, we note that the domain of A is dense in X and A is a closed operator.

In [2], [5] and [12], the authors proved that: $e^{t\sigma(A)} \subset \sigma(T(t))$ and $e^{t\nu(A)} \subseteq \nu(T(t)) \subseteq e^{t\nu(A)} \cup \{0\}$ where $\nu \in \{\sigma_p, \sigma_r\}$, point spectrum and residual spectrum.

After than Engle et al. [5] give a condition for a strongly continuous semigroup that satisfies this equality for spectrum and approximative spectrum, they proved that:

$$\sigma_{ap}(T(t)) \setminus \{0\} = e^{t\sigma_{ap}(A)}, t \geq 0,$$

and

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, t \geq 0,$$

where $T(t)$ is a eventually norm-continuous semigroup.

A. Elkoutri and M. A. Taoudi [4] proved that:

$$e^{t\nu(A)} \subseteq \nu(T(t)), \text{ for all } t \geq 0,$$

where $\nu(.) \in \{\sigma_\gamma(.); \sigma_{\gamma e}(.); \sigma_\pi(.); \sigma_F(.)\}$ the semi regular spectrum, essentially semi regular spectrum, upper semi-Fredholm and Fredholm spectrum, respectively.

In [14] we gave conditions of a strongly continuous semigroup that satisfies:

$$e^{t\sigma_\nu(A)} \subseteq \sigma_\nu(T(t)) \subseteq e^{t\sigma_\nu(A)} \cup \{0\},$$

for $\sigma_\nu(A)$ the semi regular spectrum, essentially semi regular spectrum, upper semi-Fredholm and Fredholm spectrum and proved that the first inclusion is true for B-Fredholm spectrum. In the same direction we proved that this inclusion is hold for Drazin invertible spectrum and quasi-Fredholm spectrum see [15]. The main objective of this article is to continue in the same direction and development of spectral theory for a C_0-semigroup and its generator. In section 2 we will give some proposition for the decomposition of strongly continuous semigroup and we prove that if there exists t_0 such that $T(t_0)$ is upper pseudo B-Fredholm (res.lower pseudo B-Fredholm, pseudo B-Fredholm) operator then $T(t)$ is upper pseudo B-Fredholm (resp.lower pseudo B-Fredholm, pseudo B-Fredholm) for all $t \geq 0$, same thing for left and right generalized Drazin invertible, B-Fredholm operator and for Drazin invertible.

In section 3 we prove that the spectral inclusion of strongly continuous semigroup hold for the pseudo-Fredholm spectrum, pseudo-B-Fredholm and generalized Drazin spectrum. Also, we will prove under the condition of a C_0-semigroup, that The following assertions are equivalents:

(i) A is pseudo-Fredholm;

(ii) A is generalized Drazin invertible;

(iii) A is pseudo B-Fredholm.

2. Decomposition of strongly continuous semigroup.

Let $T(t)$ be a strongly continuous semigroup and A its infinitesimal generator. In the first we will gives the following definition and some properties necessary for proof the subsequent results.

Subspace semigroups [5]. If Y is a closed subspace of X such that $T(t)Y \subseteq Y$ for all $t \geq 0$, (i.e., if Y is $(T(t))_{t\geq 0}$-invariant), then the restrictions $T(t)_| := T(t)_{|Y}$ form a strongly continuous semigroup $(T(t)_|)_{t\geq 0}$, called the subspace semigroup, on the Banach space Y.

The part of A in Y is the operator $A_|$ defined by

$$A_{|y} := Ay$$

with domain

$$D(A_|) := \{y \in D(A) \cap Y : Ay \in Y\}.$$

In other words, $A_|$ is the "maximal" operator induced by A on Y and, as will be seen, coincides with the generator of the semigroup $(T(t)_|)_{t\geq 0}$ on Y.

Proposition 2.1 ([5]). *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X and assume that the restricted semigroup $(T(t)_|)_{t\geq 0}$ is strongly continuous on some $(T(t))_{t\geq 0}$-invariant Banach space $Y \hookrightarrow X$. Then the generator of $(T(t)|)_{t\geq 0}$ is the part $(A_|, D(A_|))$ of A in Y.*

Lemma 2.1 ([6, Lemma 332]). *If A is a closed linear operator $(X \to X')$ with $\beta(A) < \infty$, then A has closed range.*

It is clear that, if $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ two C_0-semigroups with generators A and B respectively, then for all $t \geq 0$, $R(t) = T(t) \oplus S(t)$ is a C_0-semigroups its generator is $R = A \oplus B$ [16], in the following proposition we prove the converse.

Now we denote by $T(t)_{|X_s}$ the restrictions of $T(t)$ on X_s and $T(t)_{|X_u}$ the restrictions of $T(t)_{t\geq 0}$ on X_u.

Proposition 2.2. *Let$(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. If there exist X_s, X_u two closed $(T(t))_{t\geq 0}$-invariants subspaces of X, such that $X = X_s \oplus X_u$ then $T(t)_{|X_s}$ and $T(t)_{|X_u}$ are strongly continuous semigroups, furthermore the generator of a strongly continuous semigroup $T(t) = T(t)_{|X_s} \oplus T(t)_{|X_u}$ is $A = A_{|X_s \cap D(A)} \oplus A_{|X_u \cap D(A)}$ defined in $D(A) = D(A) \cap X_s \oplus D(A) \cap X_u$.*

Proof. According to the definition of subspace semigroup and X_s, X_u are a closed $(T(t))_{t\geq 0}$-invariants subspaces of X, then X_s and X_u are a Banach spaces therefore the strongly continuity of $(T(t)_{|X_s})_{t\geq 0}$ and $(T(t)_{|X_u})_{t\geq 0}$ are automatic.

Moreover the existence of

$$y = \lim_{t \downarrow 0} \frac{1}{t}(T(t)x_s - x_s) \in X,$$

for some $x_s \in X_s$ implies that $y \in X_s$, therefore the generator of $(T(t)_{|X_s})_{t\geq 0}$ is $A_{|X_s \cap D(A)}$ with domain $D(A) \cap X_s$, the same for the generator of $(T(t)_{|X_u})_{t\geq 0}$ is $A_{|X_u \cap D(A)}$ with domain $D(A) \cap X_u$ and $A_{|X_s \cap D(A)} \oplus A_{|X_u \cap D(A)}$ is a generator of a strongly continuous semigroup $T(t) = T(t)_{|X_s} \oplus T(t)_{|X_u}$ with domain $D(A) = D(A) \cap X_s \oplus D(A) \cap X_u = D(A) \cap (X_s \oplus X_u) = D(A) \cap X$. □

Remark 1. We recall that the C_0-semigroup $(T(t))_{t\geq 0}$ is nilpotent if there exists $t_0 > 0$, such that $T(t) = 0$ for $t \geq t_0$. It is clear that if there exists $t_0 > 0$, such that $T(t_0)$ is nilpotent operator then $T(t)$ is nilpotent for all $t \geq 0$.

Also, we recall that the C_0-semigroup $(T(t))_{t\geq 0}$ is quasi-nilpotent if $\{0\} = \sigma(T(t))$, then if there exists $t_0 > 0$, such that $T(t_0)$ is quasi-nilpotent operator then $T(t)$ is quasi-nilpotent for all $t \geq 0$.

Now we will proof the following property that depends of the decomposition of strongly continuous semigroup.

Proposition 2.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup.*

1. *If there exists $t_0 > 0$ such that $T(t_0)$ is upper pseudo B-Fredholm then $T(t)$ is upper pseudo B-Fredholm for all $t \geq 0$.*
2. *If there exists $t_0 > 0$ such that $T(t_0)$ is lower pseudo B-Fredholm then $T(t)$ is lower pseudo B-Fredholm for all $t \geq 0$.*

3. *If there exists $t_0 > 0$ such that $T(t_0)$ is pseudo B-Fredholm then $T(t)$ is pseudo B-Fredholm for all $t \geq 0$.*

Proof. 1. If there exists $t_0 > 0$ such that $T(t_0)$ is upper pseudo-B-Fredholm then there exist two closed $T(t_0)$-invariants subspaces $X_1, X_2 \subset X$ such that $T(t_0) = T(t_0)_{|X_1} \oplus T(t_0)_{|X_2}, T(t_0)_{|X_1}$ is upper semi Fredholm and $T(t_0)_{|X_2}$ is quasi-nilpotent. Since $T(t_0)_{|X_1}$ is upper semi Fredholm then $\alpha(T(t_0)_{|X_1}) < \infty$ and $R(T(t_0)_{|X_1})$ is closed. We show that $\alpha(T(t)_{|X_1}) < \infty$ and $R(T(t)_{|X_1})$ is closed for all $t \geq 0$. Since $\alpha(T(t_0)_{|X_1}) < \infty$ then 0 is an eigenvalue with finite multiplicity of $T(t_0)$. As proof [12, Theorem 6.6], let $x \in X_1, x \neq 0$ be an eigenvector associated to 0. Putting $t_1 = t_0/2$, then $T(t_0)x = T(t_1)T(t_1)x = 0$, hence 0 is an eigenvalue of $T(t_1)$. Proceeding by induction, we define a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ such that 0 is an eigenvalue of $T(t_n)$, for all $n \in \mathbb{N}$.

For $n \geq 0$, we define the sets

$$\digamma_n = N(T(t_n)_{|X_1}) \bigcap \{x \in X_1 : ||x|| = 1\}.$$

Clearly, the inclusion $N(T(s)_{|X_1}) \subseteq N(T(t)_{|X_1})$, for $s \geq t$ implies that $(\digamma_n)_n$ is a decreasing sequence (in the sense of the inclusion) of nonempty compact subsets of X_1. Thus $\bigcap_{n=0}^{\infty} \digamma_n \neq \emptyset$. If $x \in \bigcap_{n=0}^{\infty} \digamma_n$ then

$$(**) \qquad \|T(t_n)x - x\| = \|x\| = 1 \text{ for all } n \geq 1$$

Since $t_n \to 0$ as $n \to \infty$, (**) contradicts the strong continuity of $(T(t)_{|X_1})_{t\geq 0}$.

This shows that $N(T(t_0)_{|X_1}) = \{0\}$, that is, $(T(t_0)_{|X_1})$ is injective and $\alpha(T(t_0)_{|X_1}) = 0$. Let $0 < t \leq t_0$. The inclusion $N(T(t)_{|X_1}) \subseteq N(T(t_0)_{|X_1})$ implies that $\alpha(T(t)_{|X_1}) = 0$. Assume now that $t > t_0$ and $x \in N(T(t)_{|X_1})$, then there exists an integer n such that $nt_0 > t$ and therefore $T(nt_0)x = T(nt_0 - t)T(t)x = 0$. Hence, we have $x = 0$ and consequently $N(T(t)_{|X_1}) = \{0\}$ for all $t > t_0$, therefore $(T(t)_{|X_1})$ is injective and $\alpha(T(t)_{|X_1}) = 0$ for all $t \geq 0$.

It remains to show that $R(T(t)_{|X_1})$ is closed for all $t \geq 0$. Assume that $T(t_0)_{|X_1}$ is upper semi Fredholm , then $\alpha(T(t_0)_{|X_1}) < \infty$ and $\beta(T(t_0)_{|X_1}) = \infty$ (if $\beta(T(t_0)_{|X_1}) < \infty$, as proof (2) then $\beta(T(t)_{|X_1}) < \infty$ for all $t \geq 0$ according to lemma 2.1 $R(T(t_0)_{|X_1})$ is closed). Let $T^*(t_0)$ be the dual operator of $T(t_0)$. Obviously, $(T^*(t_0)_{|X_1^*})$ is lower semi Fredholm and consequently $\beta(T^*(t_0)_{|X_1^*}) < \infty$. Hence $\beta(T^*(t)_{|X_1^*}) < \infty$ for all $t \geq 0$. Now applying lemma 2.1 we infer that $R(T^*(t))$ is closed in X_1^*, for all $t \geq 0$. This together with the closed graph theorem of Banach [19, page 205] implies that $R(T(t))$ is closed in X_1 for all $t \geq 0$. Therefore $T(t)_{|X_1}$ is upper semi Fredholm for all $t \geq 0$. Also we have $T(t_0)_{|X_2}$ is quasi-nilpotent implies that $T(t)_{|X_1}$ is quasi-nilpotent for all $t \geq 0$, therefore $T(t)$ is upper pseudo B-Fredholm for all $t \geq 0$.

2. To prove this item, we will proceed by duality. Let $(T^*(t))_{t\geq 0}$ be the dual semigroup of $(T(t))_{t\geq 0}$. Since $\beta(T(t)_{|X_1}) = \alpha(T^*(t)_{|X_1^*})$, then it suffices to show that $\alpha(T^*(t)_{|X_1^*}) = 0$ for all $t \geq 0$. By hypothesis, we have $\alpha(T^*(t_0)_{|X_1^*}) <$

∞. Let x^* be an element of $N(T^*(t_0)_{|X_1^*})$. Arguing as above, we construct a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ such that 0 is an eigenvalue of $T^*(t_n)$, for all $n \in \mathbb{N}$ and we define the sets

$$\daleth_n = N(T^*(t_n)_{|X_1^*}) \bigcap \{x^* \in X_1^* : ||x^*|| \neq 1\}.$$

Clearly, the inclusion $N(T^*(s)_{|X_1^*}) \subseteq N(T^*(t)_{|X_1^*})$, for $s \geq t$, imply that $(\daleth_n)_n$ is a decreasing sequence (in the sense of the inclusion) of nonempty compact subsets of X_1^*. Thus

$$\bigcap_{n=0}^{\infty} \daleth_n \neq \emptyset.$$

If $x^* \in \bigcap_{n=0}^{\infty} \daleth_n$ then

(***) $\quad | < T^*(t_n)x^* - x^*, x > | = | < x^*, x > | \neq 0 \;\; \forall n \geq 1,$ for all $x \in X_1$.

Using the fact that $(T^*(t))_{t\geq 0}$ is continuous in the *weak** topology at $t = 0$, we conclude that

(****) $\quad \lim_{t\to 0} | < T^*(t_n)x^* - x^*, x > | = 0, \qquad$ for all $x \in X_1$.

Combining (***) and (****), we obtain $< x^*, x >= 0$ for all $x \in X_1$. This shows that $x^* = 0$ and therefore $\alpha(T^*(t_0)) = 0$. By the same argument as above, we show that $\alpha(T^*(t)_{|X_1^*}) = 0$ for all $t \geq 0$.

Assume now that $T(t_0)_{|X_1}$ is lower semi Fredholm, then $\beta(T(t_0)) < \infty$ and $\alpha(T(t_0)_{|X_1}) = \infty$ (if $\alpha(T(t_0)_{|X_1}) < \infty$ the proof is contained in (1)). It follows From the above that $\beta(T(t)_{|X_1}) < \infty$ for all $t \geq 0$. Again using 2.1 we see that $R(T(t)_{|X_1})$ is closed in X_1 for all $t \geq 0$, which completes the proof of (2).

3. It follows from (1) and (2). □

The proof of the following Theorem produces directly from proof of Proposition 2.3.

Note that $(T(t))_{t\geq 0}$ is upper(lower) pseudo B-Fredholm if $(T(t))_{t\geq 0}$ is upper(lower) pseudo B-Fredholm for all $t \geq 0$.

Theorem 2.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup.*

1. *A C_0-semigroup $T(t)$ is upper pseudo B-Fredholm if and only if $T(t)$ is generalized Drazin bounded below.*
2. *A C_0-semigroup $T(t)$ is a lower pseudo B-Fredholm if and only if $T(t)$ is generalized Drazin surjective.*
3. *A C_0-semigroup $T(t)$ is pseudo B-Fredholm if and only if $T(t)$ is generalized Drazin invertible.*

Proposition 2.4. *Let $t_0 > 0$ and let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X.*

1. *If $T(t_0)$ is a B-Fredholm operator, then $T(t)$ is a B-Fredholm operator for all $t \geq 0$.*
2. *If $T(t_0)$ is Drazin invertible, then $T(t)$ is Drazin invertible for all $t \geq 0$.*
3. *If $T(t_0)$ is a generalized Drazin invertible operator, then $T(t)$ is a generalized Drazin invertible operator for all $t \geq 0$.*

Proof. 1. Suppose that $T(t_0)$ is a B-Fredholm operator, then there exist two closed subspaces $X_1, X_2 \subset X$ $T(t)$-invariants, such that

$$X = X_1 \oplus X_2, \; T(t_0) = T(t_0)_{|X_1} \oplus T(t_0)_{|X_2}.$$

$T(t_0)_{|X_1}$ is a Fredholm operator and $T(t_0)_{|X_2}$ is nilpotent. Moreover as a C_0-semigroup $T(t_0)_{|X_1}$ is a Fredholm operator, then according to proof (3) of Proposition 2.3, we have $T(t)_{|X_1}$ is a Fredholm operator for all $t \geq 0$ and also from remake 1 $T(t)_{|X_2}$ is nilpotent for all $t \geq 0$. This show that $T(t)$ is a B-Fredholm operator, for all $t \geq 0$.

2. Suppose that $T(t_0)$ is Drazin invertible, then there exist two closed subspaces $X_1, X_2 \subset X$ $T(t)$-invariants, such that

$$X = X_1 \oplus X_2, \; T(t_0) = T(t_0)_{|X_1} \oplus T(t_0)_{|X_2}.$$

$T(t_0)_{|X_1}$ is an invertible operator and $T(t_0)_{|X_2}$ is nilpotent. As a C_0-semigroup $T(t_0)_{|X_1}$ is an invertible operator, according to [5, Proposition page 80], we have $T(t)_{|X_1}$ is an invertible operator for all $t \geq 0$ and also $T(t)_{|X_2}$ is nilpotent for all $t \geq 0$. This show that $T(t)$ is a Drazin invertible operator, for all $t \geq 0$.

3. By the same argument of (2) □

3. Spectrum inclusion for C_0-semigroup

To continue the development of a spectral theory for semigroups and their generators, we will give a technique to prove that the inclusion spectral is holds for $\sigma_{pF}, \sigma_{lgD}, \sigma_{rgD}, \sigma_{gD}$ and σ_{pBF}. For this we begin with proved the following result which will be used to prove the following theorem.

Proposition 3.1. *Let $(T(t))_{t\geq}$ a C_0-semigroup and A its generator. If $e^{\lambda t} - T(t)$ is quasi-nilpotent for some $\lambda \in \mathbb{C}$, then $\lambda - A$ is quasi-nilpotent.*

Proof. We have $e^{\lambda t} - T(t)$ is quasi-nilpotent for some $\lambda \in \mathbb{C}$, then $\sigma(e^{\lambda t} - T(t)) = \{0\}$, since $e^{t\sigma(A)} \subseteq \sigma(T(t)) = \{e^{\lambda t}\}$, this implies that $\sigma(A) \subseteq \{\lambda\}$ therefore $\sigma(\lambda - A) \subseteq \{0\}$. Hence $\lambda - A$ is quasi-nilpotent. □

Theorem 3.1. *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ we have the spectral inclusion*

$$e^{t\nu(A)} \subseteq \nu(T(t)), \quad t \geq 0.$$

Where $\nu(.) \in \{\sigma_{pF}(.); \sigma_{pBF}(.)\}$.

Proof. Pseudo-Fredholm spectrum. Let $t_0 > 0$ be fixed and suppose that $(e^{\lambda t_0} - T(t_0))$ is pseudo-Fredholm, for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then there exist two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces X_1, X_2 of X such that $X = X_1 \oplus X_2$, $(e^{\lambda t_0} - T(t_0))_{|X_1}$ is a semi regular operator and $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent.

From [4, Theorem 2.1] this implies that $(\lambda - A)_{|(D(A)\cap X_1)}$ is a semi regular operator and according to proposition 3.1, we have $(\lambda - A)_{|(D(A)\cap X_2)}$ is quasi-nilpotent, then $(\lambda - A)$ is pseudo-Fredholm.

Pseudo B-Fredholm. Let $t_0 > 0$ be fixed and suppose that $(e^{\lambda t_0} - T(t_0))$ is pseudo B-Fredholm, for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then there exist X_1, X_2 two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces of X, such that

$$X = X_1 \oplus X_2, e^{\lambda t_0} - T(t_0) = (e^{\lambda t_0} - T(t_0))_{|X_1} \oplus (e^{\lambda t_0} - T(t_0))_{|X_2},$$

$(e^{\lambda t_0} - T(t_0))_{|X_1}$ is a Fredholm operator and $(e^{\lambda t_0} - T(t_0)\big)_{|X_2}$ is quasi-nilpotent. From [13] this implies that $(\lambda - A)_{|(D(A)\cap X_1)}$ is a Fredholm operator and according to proposition 3.1, we have $(\lambda - A)_{|(D(A)\cap X_2)}$ is quasi-nilpotent, then $(\lambda - A)$ is pseudo B- Fredholm. □

By the same argument we can proof the following theorem.

Theorem 3.2. *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ we have the spectral inclusion*

$$e^{t\nu(A)} \subseteq \nu(T(t)), \quad t \geq 0.$$

Where $\nu(.) \in \{\sigma_{upBF}(.); \sigma_{lpBF}(.)\}$.

Theorem 3.3. *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ we have the spectral inclusion*

$$e^{t\nu(A)} \subseteq \nu(T(t)), \quad t \geq 0.$$

Where $\nu(.) \in \{\sigma_{gD\mathcal{M}}(.); \sigma_{gD\mathcal{Q}}(.); \sigma_{gD}(.)\}$.

Proof. Generalized Drazin bounded below:

Suppose that $(e^{\lambda t_0} - T(t_0))$ is generalized Drazin bounded below, then, there exist (X_1, X_2) two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces of X, such that

$$X = X_1 \oplus X_2, e^{\lambda t_0} - T(t_0) = (e^{\lambda t_0} - (T(t_0))_{|X_1} \oplus (e^{\lambda t_0} - T(t_0))_{|X_2},$$

$(e^{\lambda t_0} - T(t_0))_{|X_1}$ is bounded below, and $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent. From [5], this implies that $(\lambda - A)_{|(D(A)\cap X_1)}$ is bounded below, and according to proposition 3.1, we have $(\lambda - A_{|(D(A)\cap X_2)})$ is quasi-nilpotent, then $(\lambda - A)$ is generalized Drazin bounded below.

Generalized Drazin surjective. Suppose that $(e^{\lambda t_0} - T(t_0))$ is generalized Drazin surjective, then there exist (X_1, X_2) two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces of X, such that

$$X = X_1 \oplus X_2, e^{\lambda t_0} - T(t_0) = (e^{\lambda t_0} - T(t_0))_{|X_1} \oplus (e^{\lambda t_0} - T(t_0))_{|X_2},$$

$(e^{\lambda t_0} - T(t_0))_{|X_1}$ is surjective and $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent. As we have

$$X_1 = R(e^{\lambda t_0} - T(t_0))_{|X_1}) \subseteq R((\lambda - A)_{|(D(A)\cap X_1)}),$$

then $(\lambda - A)_{|(D(A)\cap X_1)}$ is surjective. According to proposition 3.1, we have $(\lambda - A)_{|(D(A)\cap X_2)}$ is quasi-nilpotent, then $(\lambda - A)$ is right generalized Drazin inverse.

Generalized Drazin inverse. Suppose that $(e^{\lambda t_0} - T(t_0))$ is generalized Drazin inverse then there exist (X_1, X_2) two closed $(e^{\lambda t_0} - T(t_0))$-invariant subspaces of X, such that

$$X = X_1 \oplus X_2, e^{\lambda t_0} - T(t_0) = (e^{\lambda t_0} - T(t_0))_{|X_1} \oplus (e^{\lambda t_0} - T(t_0))_{|X_2},$$

$(e^{\lambda t_0} - T(t_0))_{|X_1}$ is invertible and $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent.

As X_1, X_2 two subspaces closed of X then X_1, X_2 are a Banach spaces and from [5, 18] and [12, Theorem 2.3], we have $(e^{\lambda t_0} - T(t_0))_{|X_1}$ is invertible this implies that $(\lambda - A)_{|(D(A)\cap X_1)}$ is invertible, and according to proposition 3.1, $(e^{\lambda t_0} - T(t_0))_{|X_2}$ is quasi-nilpotent, we have $(\lambda - A)_{|(D(A)\cap X_2)}$ is quasi-nilpotent, then $(\lambda - A)$ is generalized Drazin inverse. □

In the end of this paper we prove the following theorem.

Theorem 3.4. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$.*

If $\lim_{t\to\infty} \frac{1}{t^n}\|T(t)\| = 0$, for some $n \in \mathbb{N}$, the following assertions are equivalents:

1. *A is pseudo-Fredholm;*
2. *A is generalized Drazin invertible;*
3. *A is pseudo B-Fredholm.*

Proof. $(1) \Rightarrow (2)$: Since A is pseudo-Fredholm then there exist $(X_1 \cap D(A), X_2 \cap D(A))$ two closed A-invariant subspaces of $D(A)$, such that

$$D(A) = X_1 \cap D(A) \oplus X_2 \cap D(A); \quad A = (A_{|D(A)\cap X_1}) \oplus (A_{|D(A)\cap X_2}),$$

$(A_{|(D(A)\cap X_2)})$ is quasi-nilpotent and $(A_{|(D(A)\cap X_1)})$ is a semi regular operator.

Since $A_{|(D(A)\cap X_1)}$ is a semi regular operator, therefore $R(A_{|(X_1\cap D(A))})$ is closed and $N(A_{|(X_1\cap D(A))}) \subseteq R^\infty(A_{|(X_1\cap D(A))}) \subseteq R^\infty(A)$.

Let $y \in N(A_{|(X_1 \cap D(A))})$ then there exists $x \in (X_1 \cap D(A^n))$ such that $y = A^n x$.

We integrate by parts in the formal :

$$T(t)x - x = \int_0^t T(s)Axds, \qquad \text{for all } x \in (X_1 \cap D(A^n)), \text{ and for all } t \geq 0.$$

We obtain,

$$T(t)x = x + tAx + \frac{t^2}{2!}A^2x + \int_0^t \frac{(t-s)^2}{2!}T(s)A^3xds.$$

We repeat these operations we obtain:

$$T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}T(s)A^nxds.$$

Hence,

$$T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + y\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}ds.$$

$$T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + \frac{t^n}{n!}y.$$

Dividing by $t^n > 0$:

$$\frac{1}{t^n}T(t)x = \frac{1}{t^n}\sum_{k=0}^{n-1} \frac{t^k}{k!}A^kx + \frac{1}{n!}y.$$

As $\lim_{t\to\infty}\frac{1}{t^n}\|T(t)\| = 0$ and $\lim_{t\to\infty}\frac{1}{t^n}\sum_{k=0}^{n-1}\frac{t^k}{k!}A^kx = 0$ for all $0 \leq k \leq n-1$, then $y = 0$, yields $N(A_{|X_1\cap D(A)}) = \{0\}$.

On the other hand, let $(T(t)')_{t\geq 0}$ with generator A' the adjoint semigroup of $(T(t))_{t\geq 0}$. Since $A_{|(X_1\cap D(A))}$ is semi regular, then $A'_{|(X'_1\cap D(A'))}$ is also semi regular see [10, Proposition 1.6]. By using the formula [18, Proposition 1.2.2],

$$T(t)'x' - x' = weak^* \int_0^t T(s)'A'x'ds, \text{ for all } x' \in (X'_1 \cap D(A')), \text{ and for all } t \geq 0.$$

In the same manner as above we can show that: $N(A'_{|(X'_1\cap D(A'))}) = \{0\}$. This is equivalent to $\overline{R(A_{|(X_1\cap D(A))})} = (X_1 \cap D(A))$.

Since $R(A_{|(X_1\cap D(A))})$ is closed therefore $R(A_{|(X_1 \bigcap D(A))}) = (X_1 \cap D(A))$. Then $A_{|(X_1\cap D(A))}$ is surjective then $A_{|(X_1\cap D(A))}$ invertible and as $A_{|(X_2\cap D(A))}$ is quasi-nilpotent consequently A is generalized Drazin invertible.

(2) $\Rightarrow$ (1): Obvious.

Since the class of generalized Drazin invertible operator is a subclass of pseudo B-Fredholm operator and the class of pseudo B-Fredholm operator is a subclass of pseudo-Fredholm operator, hence (1) and (2) and (3) are equivalent.

□

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