Invo-clean rings associated with central polynomials

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Abstract. Let R be an associative ring with identity and let $C(R)$ be the center of a ring R and let $g(x)$ be a fixed polynomial in $C(R)[x]$. We defined R to be $g(x)$-invo clean if every element in R can be written as a sum of an involution and a root of $g(x)$. In this paper, we investigate conditions on a ring to be $g(x)$-invo clean ring. Some properties and several examples are given.
Keywords: clean rings, $g(x)$-invo clean rings, invo clean rings.

1. Introduction

Let R be an associative ring with identity. Following [6], we define an element r of a ring R to be clean if there is an idempotent $e \in R$ and a unit $u \in R$ such that $r = u + e$. A clean ring is defined to be one in which every element is clean. Clean rings were first introduced by Nicholson [6] as a class of exchange rings.

The invo-clean rings was introduced by Danchev [2]. He defined and completely described the structure of invo-clean rings having identity.

Camillo and Simon [1] , defined $g(x)$-clean rings. An element $r \in R$ is called $g(x)$-clean if $r = s + u$ where $g(s) = 0$ and u is a unit of R and R is a $g(x)$-clean ring if every element is $g(x)$-clean. The $(x^2 - x)$-clean rings are precisely the clean rings. In Fan and Yang [3], authors studied more properties of g(x)-clean rings. Among many conclusions, they proved that if $g(x) \in (x-a)(x-b)C(R)[x]$. where $a, b \in C(R)$ with $(b - a)$ unit in R, then R is a clean ring if and only if R is $(x - a)(x - b)$-clean. For the study of clean rings and their generalizations, we refer to [4], [5], [7].

In this paper, we introduce the notion of $g(x)$-invo clean ring. A ring R is said to be $g(x)$-invo clean ring if any element in R can be written as a sum of

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involution and a root of $g(x)$. Clearly, invo-clean rings are (x^2-x)-invo clean rings.

Throughout this paper, we assume that all rings are associative with identity and all modules are unitary. As usual, $U(R)$ denotes the set of all units of R, $Inv(R)$ the subset of $U(R)$ consisting of all involutions (i.e.; $v \in Inv(R)$ then $v^2 = 1$) of $R, Id(R)$ the set of all idempotents of R and $Nil(R)$ the set of all nilpotents, $C(R)$ denotes the center of R and $g(x)$ be a fixed polynomial with coefficients in $C(R)$.

2. $g(x)$-Invo clean rings

In this section, we define $g(x)$ -invo clean rings, we give some properties of $g(x)$-invo clean ring and present several examples.

Definition 2.1. Let R be a ring and let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called $g(x)$-invo clean if $r = v + s$ where $g(s) = 0$ and v is an involution of R i.e., $v^2 = 1$. We say that R is $g(x)$-invo clean if every element in R is $g(x)$-invo clean.

Clearly, Every $(x^2 - x)$-invo clean ring is invo clean.

Example 2.2. $\mathbb{Z}_7$ is ($x^6 - 1$)-invo clean ring which is not invo-clean ring**.**

Example 2.3. The ring $M_2(\mathbb{Z}_2)$ is $\left(x^3 - x\right)$ -invo clean ring.

Proposition 2.4. *Every $g(x)$ -invo clean ring is $g(x)$ -clean ring.*

Proof. Suppose R is a $g(x)$ -invo clean ring and let $r \in R$. Then $r = v + s$ where v is involution and $g(s) = 0$. But every involution is unit. Thus, R is $g(x)$ -clean ring.
The converse of Proposition 2.4 is not true in general. For example, we can see that $M_2(\mathbb{Z}_2)$ is $\left(x^6-1\right)$ -clean ring which not $\left(x^6-1\right)$-invo clean, since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ cannot be written as a sum of involution and a root of $\left(x^6-1\right)$. □

Let R and S be rings and $\Psi : C(R) \to C(S)$ be a ring epimorphism with $\Psi(1_R) = 1_S$. For $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, we let $g^*(x) = \sum_{i=0}^{n} \Psi(a_i)x^i \in C(S)[x]$. In particular, If $g(x) \in \mathbb{Z}[x]$, then $g^*(x) = g(x)$.

Proposition 2.5. *Let $\theta : R \to S$ be a ring epimorphism. If R is $g(x)$ -invo clean, then S is $g^*(x)$ -invo clean.*

Proof. Let $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ and consider $g^*(x) = \sum_{i=0}^{n} \theta(a_i)x^i \in C(S)[x]$. For every $\beta \in S$, there exist $r \in R$ such that $\theta(r) = \beta$. Since R is $g(x)$ -invo clean, there exists $s \in R$ and $v \in Inv(R)$ such that $r = v + s$ and $g(s) = 0$. Then $\beta = \theta(r) = \theta(v + s) = \theta(v) + \theta(s)$ with $\theta(v) \in Inv(S)$, and $g^*(\theta(s)) = \sum_{i=0}^{n} \theta(a_i)(\theta(s))^i = \sum_{i=0}^{n} \theta(a_i)\theta(s^i) = \sum_{i=0}^{n} \theta(a_i s^i) = \theta(\sum_{i=0}^{n} a_i s^i)$ $= \theta(g(s)) = \theta(0) = 0$. Therefore, S is $g^*(x)$ -invo clean. □

Proposition 2.6. *Let R be an $g(x)$-nil clean with $n^2 = -2n$ for every $n \in Nil(R)$. Then R is $g(x)$ -invo clean.*

Proof. Suppose R is a $g(x)$-nil clean and let $r \in R$. Then $r - 1 = n + s$ where $n \in Nil(R)$ and $g(s) = 0$. Thus $r = (1 + n) + s$. Indeed $1 + n$ is an involution. Therefore R is $g(x)$ -invo clean. □

Proposition 2.7. *If R an $g(x)$-invo clean ring and I is an ideal of R, then $R = R/I$ is $g^*(x)$ -invo clean.*

Proof. Let R be an $g(x)$ -invo clean ring and $\theta : R \to R/I$ defined by $\theta(r) = r = r + I$. Then θ is an epimorphism. By Proposition 2.5 R/I is $g(x)$-invo clean. □

Proposition 2.8. *Let $R_1, R_2, \ldots, R_k$ be rings and $g(x) \in \mathbb{Z}[x]$. Then $R = \prod_{i=1}^{k} R_i$ is $g(x)$-invo clean if and only if R_i is $g(x)$-invo clean for all $i \in \{1, 2, \ldots, n\}$.*

Proof. $\Rightarrow$) : For each $i \in \{1, 2, \ldots, n\}$, R_i is a homomorphic image of $\prod_{i=1}^{k} R_i$ under the projection homomorphism. Hence, R_i is $g(x)$ -invo clean by Proposition 2.5.
$\Leftarrow$) : Let $(x_1, x_2, \ldots, x_k) \in \prod_{i=1}^{k} R_i$. For each i, write $x_i = v_i + s_i$ where $v_i \in Inv(R_i)$, $g(s_i) = 0$. Let $v = (v_1, v_2, \ldots, v_k)$ and $s = (s_1, s_2, \ldots, s_k)$. Then, it is clear that $v \in Inv(R)$ and $g(s) = 0$. Therefore, R is $g(x)$-invo clean. □

Theorem 2.9. *Let R be a ring and let $R[t]$ be the rings of polynomial in an indeterminate t with coefficients in R and let $f(t) = a_0 + a_1 t + \ldots + a_n t^n \in R[t]$. If $f(t)$ is an involution then a_0 is an involution in R and $a_1, \ldots, a_n$ are nilpotents.*

Proof. Assume $f(t)$ is a unit then a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotents. Since $Inv(R) \subseteq U(R)$, the statement holds. □

Proposition 2.10. *Let R be a commutative ring, then the ring of polynomials $R[t]$ is not invo clean (not $(x^2 - x)$ -invo clean).*

Proof. Let t be an invo clean, then we may write $t = a_0 + a_1 t + \ldots + a_n t^n + e$ where $e \in Id(R[t]) = Id(R)$ and $a_0 \in Inv(R), a_1, \ldots, a_n \in Nil(R)$. Hence, $1 = a_1 \in J(R)$. Which a contradiction. Hence, $R[t]$ is not invo-clean. □

Let R be a commutative ring and M an R -module. The idealization $R(M)$ of R and M is the ring $R(M) = R \oplus M$ with multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. Note that if $(r, m) \in R(M)$, then $(r, m)^k = (r^k, k r^{k-1} m)$ for any $k \in \mathbb{N}$.

Lemma 2.11. *Let R be a commutative ring with $char(R) = 2$ and M an R -module. Then (v, m) is an involution in $R(M)$ if and only if v is involution in R.*

Proof. $\Rightarrow)$: Let $(v,m) \in R(M)$ then $(v,m)^2 = (v^2, 2vm) = (1,0)$. So, $v^2 = 1$. Thus, v is involution.
$\Leftarrow)$: Let v be an involution, and $(v,m) \in R(M)$. Then $(v,m)^2 = (1,0)$. Hence, (v,m) is an involution of $R(M)$. $\square$

We recall that R logically embeds into $R(M)$ via $r \to (r,0)$. Therefore any polynomial $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ can be written as $g(x) = \sum_{i=0}^{n} (a_i, 0)\, x^i \in R(M)[x]$ and conversely.

Proposition 2.12. *Suppose R is a commutative ring with $Char(R) = 2$ and M an R -module. So the idealization $R(M)$ of R and M is $g(x)$-invo clean if and only if R is $g(x)$-invo clean.*

Proof. $\Rightarrow)$: Since $R \simeq R(M)/(0 \oplus M)$ is a homomorphic image of $R(M)$. Hence R is $g(x)$ -invo clean by Proposition 2.5.
$\Leftarrow)$: Let $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $r \in R$. Write $r = v + s$ where $v \in Inv(R)$ and $g(s) = 0$. Then for $m \in M$, $(r,m) = (v,m) + (s,0)$ where $(v,m) \in Inv(R(M))$ and
$g(s,0) = a_0(1,0) + a_1(s,0) + a_2(s,0)^2 + \ldots + a_n(s,0)^n$
$= a_0(1,0) + a_1(s,0) + a_2(s^2,0) + \ldots + a_n(s^n,0)$
$= (a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n, 0) = (g(s),0) = (0,0)$. Therefore, $R(M)$ is $g(x)$-invo clean. $\square$

3. $(x^2 + cx + d)$-invo clean rings

We consider some types of $(x^2 + cx + d)$ -invo clean rings.

Theorem 3.1. *Let R be a ring and $a, b \in C(R)$ and $g(x) \in (x-a)(x-b)$ where $b - a \in Inv(R)$. Then R is invo-clean if and only if R is $(x-a)(x-b)$-invo clean.*

Proof. $\Rightarrow)$: Since R is invo-clean and $r \in R$ then $\frac{r-a}{b-a} = v + e$ where $v \in Inv(R)$ and $e \in Id(R)$ then $r = v(b-a) + e(b-a) + a$, $b - a \in C(R)$ and $C(R)$ is a subring of R. Since $(e(b-a)+a-a)(e(b-a)+a-b) = (eb-ea)(eb-ea+a-b) =$
$e^2b^2 - e^2ba + eab - eb^2 - e^2ab + e^2a^2 - e^2a^2 - ea^2 + eab = 0$, it follows $e(b-a)+a$ is root of $(x-a)(x-b)$. Since $v(b-a) \in Inv(R)$ by $(v(b-a))^2 = v(b-a)v(b-a) = v^2(b-a)^2 = 1.1 = 1$, it follow that $v(b-a) \in Inv(R)$. Then R is $(x-a)(x-b)$-invo clean.
$\Leftarrow)$: Let $r \in R$. Since R is $(x-a)(x-b)$-invo clean, $r(b-a)+a = v+e$ where e is root of $(x-a)(x-b)$ and $v \in Inv(R)$. Thus, $r = \frac{e-a}{b-a} + \frac{v}{b-a}$. Clearly, $\frac{v}{b-a} \in Inv(R)$ and $\frac{e-a}{b-a}$ is an idempotent since $\left(\frac{e-a}{b-a}\right)^2 = \frac{e-a}{b-a}$. Hence R is invo-clean. $\square$

Corollary 3.2. *Let R be a ring. Then R is invo-clean if and only if R is $(x^2 + x)$-invo clean.*

Proof. In the previous Theorem 3.1 but $a = 0$ and $b = -1$. □

Proposition 3.3. *Let R be a ring with $2 \in Inv(R)$ and $k \in N$. Then the following are equivalent:*

(1) R is invo clean
(2) R is $\left(x^2 - 2x\right)$-invo clean
(3) R is $\left(x^2 + 2x\right)$-invo clean
(4) R is $\left(x^2 - 2^{2k}x\right)$-invo clean
(5) R is $\left(x^2 + 2^{2k}x\right)$-invo clean
(6) R is $\left(x^2 - 1\right)$-invo clean
(7) R is For every $r \in R$, r can be expressed as $r = v+s$ with $v, s \in Inv(R)$.

Proof. (1) ⇒ (2) Since R is invo clean and $r \in R$, $\frac{r}{2} = v + s$ with $v \in Inv(R)$ and $s^2 = s$, then $r = 2v+2s$ with $2v \in Inv(R)$ and $(2s)^2 - 2(2s) = 4s^2 - 4s = 0$. Hence, R is $\left(x^2 - 2x\right)$-invo clean.

(2) ⇒ (1) Since R is $\left(x^2 - 2x\right)$-invo clean, $2r = v + s$ where $v \in Inv(R)$ and s is a root of $\left(x^2 - 2x\right)$. Then, $r = \frac{v}{2} + \frac{s}{2}$, where $\frac{v}{2}$ is an invo of R and $\left(\frac{s}{2}\right)^2 = \frac{(s)(s-2+2)}{(2)^2} = \frac{s.2}{(2)^2} = \frac{s}{2}$. So, R is invo clean. Correspondingly, we may prove (3) ⇒ (1).

(2) ⇒ (3) R is $\left(x^2 - 2x\right)$-invo clean and let $r \in R$, $-r = v + s$ such that $v \in Inv(R)$ and $s^2 - 2s = 0$. Then, $r = (-v) + (-s)$ with $-v \in Inv(R)$ and $(-s)^2 + 2(-s) = s^2 - 2s = 0$. Thus, R is $\left(x^2 + 2x\right)$-invo clean.

(1) ⇔ (4) By Theorem 3.1, let $a = 0$ and $b = 2^{2k}$, Then, R is $\left(x^2 - 2^{2k}x\right)$-invo clean.

(1) ⇔ (5) Can be proved by (1) ⇔ (4) and (2) ⇒ (3).

(1) ⇒ (6) Since R is invo clean and $r \in R$ then $r = v+s$ where $v, s \in Inv(R)$ and $s^2 = s$. Then s is a root of $x^2 - 1$ by (7). Then $\left(x^2 - 1\right)$-invo clean.

(7) ⇒ (6) Let $r \in R$ we write $r = v + s$ with $v, s \in Inv(R)$ and $s^2 = 1$, then s is a root of $x^2 - 1$ and $v \in Inv(R)$. Then, $\left(x^2 - 1\right)$ is invo clean ring.

(6) ⇒ (7) If R is $\left(x^2 - 1\right)$-invo clean, then for every $r \in R$ there exist $v, s \in Inv(R)$ such that $r = v + s$. □

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