A characterization of some alternating group by its order and special conjugacy class sizes

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Abstract. Let G be a group and $N(G)$ be the set of the sizes of conjugacy class of G. Let $m_p(G)$ be the number from $N(G)$ which is not divisible by p and let A_n be the alternating group of degree n. The alternating groups A_5, A_6, A_7, A_8, and A_9 are characterized by their orders and special conjugacy class sizes. So in generality, are the alternating groups characterized by their orders and some special conjugacy class size(s)? In this paper, we show that G is a finite group such that $m_p(G) = m_p(A_n)$ and $m_2(G) = m_2(A_n)$ where $n \in \{p, p+1, p+2\}$, then G is isomorphic to A_n.

Keywords: element order, alternating group, Thompson's problem, conjugacy classes sizes.

1. Introduction

All groups in this paper are finite, and simple groups are non-abelian. For a group G, let $\pi(G)$ denote the set of prime divisors of $|G|$. The prime graph of G is a graph $GK(G)$ with vertex set $\pi(G)$ and two distinct vertices q and q are adjacent by an edge if G has an element of order pq. We denote by $s(G)$ the number of connected components of $GK(G)$. Let $\pi_i = \pi_i(G)$, $i = 1, 2, \cdots, s(G)$, be the connected components of $GK(G)$. For an even order group, let $2 \in \pi_1(G)$. Then $|G|$ can be expressed as a product of $m_1, m_2, \cdots, m_{s(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i$. These m_i's are called the order components of G. Obviously, m_i's are odd components of G with $i \geq 2$. Using [14] and [23], we list the order components for non-abelian finite simple groups L in Tables 1, 2 and 3. This information is used to prove our main theorem. Let A_n

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be the alternating group of degree n. For alternating group A_p with p prime, $p \in \{m_i, i \geq 2\}$. In 1987, J. G. Thompson put forward the following conjecture.

Conjecture 1 ([19, Problem 12.38]). *Let G be a group with trivial center. If L is a simple group satisfying that $N(G) = N(L)$, then $G \cong L$.*

Some authors proved that Thompson's conjecture is valid for groups: $L_n(q)$ [1, 5, 13], $D_n(q)$ [2], ${}^2D_n(q)$ [3], $E_7(q)$ [25], A_{p+3} [18], for simple groups with $s(G) \geq 2$ [7, 8], A_{p+4} [27], all almost sporadic simple groups [21], A_{10} [12], A_{22} [24], A_{26} [17]. Recently, G. Chen and J. Li contributed their interests on the Thompson's conjecture under a weak condition. They successfully characterized some sporadic simple groups and simple K_3-groups (A finite simple group G is called a simple K_n-group if G is simple and $n = |\pi(G)|$) in [16]. Chen et al in [26] showed that simple K_4-groups are also characterized by its order and one special conjugacy class size. For convenience, we denote by $m_p(G)$ the p'-number from $N(G)$. As the development of this topic, we will prove the following.

Main Theorem 1.2. Let G be a finite group and let $n \in \{p, p+1, p+2\}$ where $5 \leq p$ is a prime. Then $G \cong A_n$ if and only if $|G| = |A_n|$, $m_p(G) = m_p(A_n)$ and $m_2(G) = m_2(A_n)$.

We introduce some notation which will be needed in the proof of the main theorem. Let $a \cdot b$ denote the products of an integer a by an integer b. Let G be a group and r a prime. Then we denote the number of the Sylow r-subgroup G_r of G by $n_r(G)$ or n_r. Let S_n be the symmetric group of degree n. Let $\omega(G)$ be the set of element orders of G. Let $x \in \omega(G)$. Let x^G denote the conjugacy classes of G containing x. The other symbols are standard (see [9], for instance).

2. Some preliminary results

In this section, we give some lemmas used to prove the main theorem.

Let $\exp(n, r) = a$ denote that $r^a \mid n$ but $r^{a+1} \nmid n$.

Lemma 1. *Let A_{p+k} be an alternating group of degree $p + k$ where $p + i$ is composite, $i = 1, \cdots, k$, and p is a prime. Then the following hold.*

(1) $\exp(|A_{p+k}|, 2) = \sum_{i=1}^{\infty} [\frac{p+k}{2^i}] - 1$. *In particular,* $\exp(|A_{p+k}|, 2) \leq p + k - 1$.

(2) $\exp(|A_{p+k}|, r) = \sum_{i=1}^{\infty} [\frac{p+k}{r^i}]$ *for each* $r \in \pi(A_{p+k}) \backslash \{2\}$. *Furthermore,* $\exp(|A_{p+k}|, r) < \frac{p+k}{2}$, *where* $3 \leq r \in \pi(A_{p+k})$. *In particular, if* $r > [\frac{p+k}{2}]$, *then* $\exp(|A_{p+k}|, r) = 1$.

Proof. (1) By the definition of Gaussian integer function, we have that

$$\begin{aligned}\exp(|A_{p+k}|,2) &= \sum_{i=1}^{\infty}[\frac{p+k}{2^i}]-1\\ &= ([\frac{p+k}{2}]+[\frac{p+k}{2^2}]+[\frac{p+k}{2^3}]+\cdots)-1\\ &\le (\frac{p+k}{2}+\frac{p+k}{2^2}+\frac{p+k}{2^3}+\cdots)-1\\ &= (p+k)(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots)-1\\ &= p+k-1.\end{aligned}$$

(2) Similarly as (1), we have that

$$\begin{aligned}\exp(|A_{p+k}|,r) &\le (p+k)(\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\cdots)\\ &= \frac{p+k}{r-1}\\ &\le \frac{p+k}{2}\end{aligned}$$

for an odd prime $r \in \pi(A_{p+k})$. If $r > [\frac{p+k}{2}]$, $\exp(|A_{p+k}|,r) = 1$.

The proof is complete. □

Let G be a group whose order is divisible by prime p. The group G is called a C_{pp}-group if the centralizers of a p-element are p-groups (see [4], for instance).

Lemma 2. *Let A_n be an alternating group of degree n, where $n = p, p+1, p+2$. Then the following hold.*

(1) $m_p(A_n) = \begin{cases} \dfrac{(p-1)!}{2}, & n = p;\\ \dfrac{(p-1)!\cdot(p+1)}{2}, & n = p+1;\\ \dfrac{(p-1)!\cdot(p+1)\cdot(p+2)}{2}, & n = p+2.\end{cases}$

(2) *Let $p = 4k+1$. Then* $m_2(A_n) = \begin{cases} \dfrac{p!}{2^{2k}\cdot(2k)!}, & n = p;\\ \dfrac{(p+1)!}{2\cdot 2^{2k}\cdot(2k)!}, & n = p+1;\\ \dfrac{p!}{2^{2k}\cdot(2k)!\cdot 3!}, & n = p+2.\end{cases}$

Let $p = 4k+3$. Then $m_2(A_n) = \begin{cases} \dfrac{p!}{2^{2k}3!\cdot(2k)!}, & n = p;\\ \dfrac{(p+1)!}{2\cdot 2^{2(k+1)}\cdot(2(k+1))!}, & n = p+1;\\ \dfrac{(p+2)!}{2^{2(k+1)}\cdot(2(k+1))!}, & n = p+2.\end{cases}$

Proof. We knew that A_n is a C_{pp}-group. Let's say the cycle type has c_1 1-cycles, c_2 2-cycles, and so on, up to c_k k-cycles, where $1c_1 + 2c_2 + \cdots + kc_k = n$. The number of permutations in the conjugacy classes described by the c_i's is

$$\frac{n!}{\prod_{i=1}^{k} i^{c_i} \prod_{i=1}^{k} c_i!}.$$

Conjugacy classes of permutation in S_n stay the same size in A_n for all cycle types except those cycle type consists of parts that are all odd and distinct.

Table 1. The order components of finite simple groups L with $s(L) = 2$

L	Restrictions of L	m_1	m_2
A_n	$6 < n = p, p+1, p+2$ one of $n, n-2$ is not a prime	$n!/2p$	p
$A_{p-1}(q)$	$(p, q) \neq (3, 2), (3, 4)$	$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - 1)$	$\frac{(q^p-1)}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1) \mid (p+1)$	$q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=2}^{p-1}(q^i - 1)$	$\frac{q^p-1}{q-1}$
${}^2A_{p-1}(q)$		$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{(q^p+1)}{(q+1)(p,q+1)}$
${}^2A_p(q)$	$(q+1) \mid (p+1)$ $(p, q) \neq (3, 3), (5, 2)$	$q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=2}^{p-1}(q^i - 1)$	$\frac{q^p+1}{q+1}$
${}^2A_3(2)$		$2^6 \cdot 3^4$	5
$B_n(q)$	$n = 2^m \geq 4$, q odd	$q^{n^2}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{2}$
$B_p(3)$		$3^{p^2}(3^p + 1) \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p-1}{2}$
$C_n(q)$	$n = 2^m \geq 2$, q odd	$q^{n^2}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$	$q = 2, 3$	$q^{p^2}(q^p + 1) \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$	$p \geq 5, q = 2, 3, 5$	$q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$	$q = 2, 3$	$q^{p(p+1)}(q^p + 1)(q^{p+1} - 1)$ $\prod_{i=1}^{p-1}(q^{2i} - 1)/(2, p-1)$	$\frac{q^p-1}{(2,q-1)}$
${}^2D_n(q)$	$n = 2^m \geq 4$	$q^{n(n-1)} \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$	$n = 2^m + 1 \geq 5$	$2^{n(n-1)}(2^n + 1)(2^{n-1} - 1)$ $\prod_{i=1}^{n-2}(2^{2i} - 1)$	$2^{n-1} + 1$
${}^2D_p(3)$	$5 \leq p \neq 2^m + 1$	$3^{p(p-1)} \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p+1}{4}$
${}^2D_n(3)$	$9 \leq 2^m + 1 \neq p$	$3^{n(n-1)}(3^n + 1)(3^{n-1} - 1)$ $\prod_{i=1}^{n-1}(3^{2i} - 1)/2$	$\frac{3^{n-1}}{2}$
$G_2(q)$	$2 < q \equiv \epsilon \mod 3, \epsilon = \pm 1$	$q^6(q^3 - \epsilon)(q^2 - 1)(q + \epsilon)$	$q^2 - \epsilon q + 1$
${}^3D_4(q)$		$q^{12}(q^6 - 1)(q^2 - 1)(q^4 + q^2 + 1)$	$q^4 - q^2 = 1$
$F_4(q)$	q odd	$q^{24}(q^8 - 1)(q^6 - 1)^2(q^4 - 1)$	$q^4 - q^2 + 1$
${}^2F_4(2)'$		$2^{11} \cdot 3^3 \cdot 5^2$	13
$E_6(q)$		$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)$ $(q^5 - 1)(q^3 - 1)(q^2 - 1)$	$(q^6 + q^3 + 1)/(3, q-1)$
${}^2E_6(q)$	$q > 2$	$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)$ $(q^5 + 1)(q^3 + 1)(q^2 - 1)$	$(q^6 - q^3 + 1)/(3, q+1)$
M_{12}		$2^6 \cdot 3^3 \cdot 5$	5
J_2		$2^7 \cdot 3^3 \cdot 5^2$	7
Ru		$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$	29
He		$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3$	17
McL		$2^7 \cdot 3^6 \cdot 5^3 \cdot 7$	11
Co_1		$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13$	23
Co_3		$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11$	23
Fi_{22}		$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11$	13
HN		$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11$	19

Case 1. $n = p$

In this case, $n = p = p \cdot 1$ and so the cycle types are odd and so by [22], the conjugacy classes of a p-element of A_p split into two classes and so $m_p(A_p) = \frac{p!}{2p} = \frac{(p-1)!}{2}$.

Case 2. $n = p + 1$.

In this case, $n = p + 1 = 1 \cdot 1 + p \cdot 1$ and so the cycle types are odd and so by [22], the conjugacy classes of a p-element of A_{p+1} split into two classes and so $m_p(A_{p+1}) = \frac{p!(p+1)}{2p} = \frac{(p-1)! \cdot (p+1)}{2}$.

Case 3. $n = p + 2$.

Then $n = p+2 = 1\cdot2+p\cdot1$ and so $m_p(A_{p+2}) = \frac{p!(p+1)(p+2)}{2p} = \frac{(p-1)!\cdot(p+1)\cdot(p+2)}{2}$. Similar as the case $m_p(A_n)$, we can compute the $m_2(A_n)$. □

Lemma 3 ([11, pp. 85, Theorem 80])**.** *For any prime p, $(p-1)! \equiv -1 (mod\, p)$.*

Table 2. The order components of finite simple groups L with $s(L) = 3$

L	Restrictions of L	m_1	m_2	m_3
A_n	$6 < n = p, p-2$ are primes	$\frac{n!}{2n(n-2)}$	p	$p-2$
$A_1(q)$	$4 \mid q+1$	$q+1$	q	$\frac{q-1}{2}$
$A_1(q)$	$4 \mid q-1$	$q-1$	q	$\frac{q+1}{2}$
$A_1(q)$	$2 \mid q$	q	$q+1$	$q-1$
$A_2(2)$		8	3	7
$^2A_5(2)$		$2^{15}\cdot3^6\cdot5$	7	11
$^2D_p(3)$	$5 \le p = 2^m+1$	$2\cdot3^{p(p-1)}(3^{p-1}+1)$ $\prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{3^{p-1}}{2}$	$\frac{3^p+1}{4}$
$^2D_{p+1}(2)$	$n \ge 2, p = 2^m-1$	$2^{p(p-1)}(2^p-1)$ $\prod_{i=1}^{p-1}(2^{2i}-1)$	2^p+1	$2^{p+1}+1$
$G_2(q)$	$q \equiv 0 \mod 3$	$q^6(q^2-1)^3$	q^2-q+1	q^2+q+1
$^2G_2(q)$	$q = 3^{2m+1} > 3$	$q^3(q^2-1)$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$
$F_4(q)$	q even	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1
$^2F_4(q)$	$q = 2^{2m+1} > 2$	$q^{12}(q^4-1)(q^3+1)$	$q^2-\sqrt{2q^3}+$ $q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+$ $q+\sqrt{2q}+1$
$E_7(2)$		$2^{36}\cdot3^{11}\cdot5^2\cdot7^3\cdot11$ $13\cdot17\cdot19\cdot43$	73	127
$E_7(3)$		$2^{23}\cdot3^{63}\cdot5^2\cdot7^3\cdot11^2$ $13^2\cdot19\cdot37\cdot41\cdot61\cdot73\cdot547$	757	1093
M_{11}		$2^4\cdot3^2$	5	11
M_{23}		$2^7\cdot3^2\cdot5\cdot7$	11	23
M_{24}		$2^{10}\cdot3^3\cdot5\cdot7$	11	13
J_3		$2^7\cdot3^5\cdot5$	17	19
HS		$2^9\cdot3^2\cdot5^3$	7	11
Suz		$2^{13}\cdot3^7\cdot5^2\cdot7$	11	13
Co_2		$2^{18}\cdot3^6\cdot5^3\cdot7$	11	23
Fi_{23}		$2^{18}\cdot3^{13}\cdot5^2\cdot7\cdot11\cdot13$	17	23
F_3		$2^{15}\cdot3^{10}\cdot5^3\cdot7^2\cdot13$	19	31
F_2		$2^{24}\cdot3^{13}\cdot5^6\cdot7^2\cdot$ $11\cdot13\cdot17\cdot19\cdot23$	31	47

Table 3. The order components of finite simple groups L with $s(L) > 3$

L	Restrictions of L	m_1	m_2	m_3	m_4	m_5	m_6
$A_2(4)$		2^6	3	5	7		
$^2B_2(q)$	$q = 2^{2m+1} > 2$	q^2	$q-1$	$q-\sqrt{2q}+1$	$q^2+\sqrt{2q}+1$		
$^2E_6(2)$		$2^{36}\cdot3^9\cdot5^2\cdot7^2\cdot11$	13	17	19		
$E_8(q)$	$q \equiv 2,3 \mod 5$	$q^{120}(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)$ $(q^{10}-1)(q^8-1)$ $(q^4+1)(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1		
M_{22}		$2^7\cdot3^2$	5	7	11		
J_1		$2^3\cdot3\cdot5$	7	11	19		
ON		$2^9\cdot3^4\cdot5\cdot7^3$	11	19	31		
LyS		$2^8\cdot3^7\cdot5^6\cdot7\cdot11$	31	37	67		
Fi'_{24}		$2^{21}\cdot3^{16}\cdot5^2\cdot7^3\cdot11\cdot13$	17	23	29		
F_1		$2^{46}\cdot3^{20}\cdot5^9\cdot7^6\cdot11^2\cdot13^3$ $17\cdot19\cdot23\cdot29\cdot31\cdot47$	41	59	71		
$E_8(q)$	$q \equiv 0,1,4 \mod 5$	$q^{120}(q^{18}-1)(q^{14}-1)$ $(q^{12}-1)^2(q^{10}-1)^2$ $(q^8-1)^2(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4+1	$\frac{q^{10}+1}{q^2+1}$	
J_4		$2^{21}\cdot3^3\cdot5\cdot7\cdot11^3$	23	29	31	37	43

Lemma 4 ([10, Theorem 9.3.1])**.** *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $(m,n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1}\cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:*

(1) *$q_i^{\beta_i} \equiv 1 (mod\, p_j)$ for some p_j.*

(2) *The order of some chief factor of G is divided by $q_i^{\beta_i}$.*

Lemma 5 ([3, Lemma 1.2] and [20, Lemma 7]). *Let $x, y \in G$, $(|x|, |y|) = 1$, and $xy = yx$. Then*

(1) $C_G(xy) = C_G(x) \cap C_G(y)$;

(2) $|x^G|$ *divides* $|(xy)^G|$;

(3) *If* $|x^G| = |(xy)^G|$, *then* $C_G(x) \leq C_G(y)$

Lemma 6 ([15, Lemma 1]). *If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $\frac{n+1}{2} < p_i < n$. Here*

(1) $s(n) = 6$ *for* $n \geq 48$;
(2) $s(n) = 5$ *for* $42 \leq n \leq 47$;
(3) $s(n) = 4$ *for* $38 \leq n \leq 41$;
(4) $s(n) = 3$ *for* $18 \leq n \leq 37$;
(5) $s(n) = 2$ *for* $14 \leq n \leq 17$;
(6) $s(n) = 1$ *for* $6 \leq n \leq 13$.

In particular, for every natural number $n > 6$, there exists a prime p such that $\frac{n+1}{2} < p < n - 1$, and for every natural number $n > 3$, there exists an odd prime number p such that $n - p < p < n$.

Lemma 7 ([15, Lemmas 3 and 6]). *Let P be a finite simple group. Then the following results hold.*

(1) *If $GK(P)$ is disconnected graph $GK(P)$. Then $m_i(P) = 1$ for $2 < i < t(P)$. Let n_i stand for the only element of $m_i(P)$ for $i > 1$. Then P, and m_i for $2 < i < t(P)$ are such as in Tables 1-3, where p is an odd prime number.*

(2) *If P is not isomorphic to ${}^2G_2(q)$, then, for every i, there is at most one prime number $s \in \pi_i(P)$ such that $(r + 1)/2 < s < r$.*

(3) *If P is isomorphic to ${}^2G_2(q)$, then there are at most three prime numbers $s \in \pi(P)$ such that $(r + 1)/2 < s < r$.*

(4) *For every prime number s satisfying the inequality $(r + 1)/2 < s < r$, the order of the factor group $Aut(P)/P$ is not divisible by s.*

A finite group G is 2-Frobenius group if G has a normal series $1 \leq H \leq K \leq G$ such that K and G/K are Frobenius groups with Kernels H and K/H, respectively.

Lemma 8 ([23]). *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*

(1) *G is a Frobenius group or 2-Frobenius group;*

(2) *G has a normal series $1 \leq H \leq K \leq G$ such that $\pi(G/K) \cup \pi(H) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.*

Lemma 9 ([6]). *Let G be a Frobenius group of even order with kernel K and complement H. Then $s(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$ and the following assertions hold:*

(1) K *is nilpotent;*
(2) $|K| \equiv 1(mod\,|H|)$.

3. The proof of Main Theorem

In this section, we give the main theorem's proof.

Proof of the main theorem.

Proof. By [16] and [26], the alternating groups A_5, A_7, A_8, A_9 and A_{10} are valid for the main theorem. So in the following, we assume that $p \geq 11$.

We will prove the theorem by the following lemmas.

Lemma 10. *G is insoluble.*

Proof. By Lemma 1 and hypotheses, we have that $|G_p| = |A_n|_p = p$. It's known that p is the greatest prime divisor of $|A_n|$. Assume that G is soluble. We consider three cases.

Case 1. $n = p$.

By Lemma 2, $m_p(A_p) = \frac{(p-1)!}{2}$. On the other hand, $|G| = p \cdot m_p(G) = \frac{p!}{2}$ and so G has a maximal subgroup of $m_p(A_p)$ (actually, the maximal subgroup is a Hall $\pi(G)\backslash\{p\}$-subgroup). Then by Lemma 4, $\frac{(p-1)!}{2} \equiv 1(\text{mod } p)$. On the other hand, by Lemma 3 $(p-1)! \equiv -1(\text{mod } p)$. It follows that $p = 3$ contradicting $p \geq 11$.

Case 2. $n = p + 1$.

By Lemma 2, $m_p(A_{p+1}) = \frac{(p-1)!\cdot(p+1)}{2}$. Note that $|G| = p{\cdot}m_p(A_{p+1}) = \frac{(p+1)!}{2}$. So G has a maximal subgroup of order $m_p(A_{p+1})$ (actually, the maximal subgroup is a Hall $\pi(G)\backslash\{p\}$-subgroup). Then by Lemma 4, $\frac{(p-1)!\cdot(p+1)}{2} \equiv \frac{(p-1)!}{2} \equiv 1(\text{mod } p)$. But by Lemma 3 $(p-1)! \equiv -1(\text{mod } p)$. It follows that $p = 3$ contradicting $p \geq 11$.

Case 3. $n = p + 2$.

By Lemma 2, $m_p(A_{p+2}) = \frac{(p-1)!\cdot(p+1)(p+2)}{2}$. Similarly as the proof of Case 1 or 2, we have $\frac{(p-1)!\cdot(p+1)(p+2)}{2} \equiv (p-1)!(p+2) \equiv 1(\text{mod } p)$. It follows from Lemma 3, that $(p+2) \equiv -1(\text{mod } p)$ so $p = 3$, a contradiction. □

Lemma 11. *p is the odd component of G. In particular $s(G) \geq 2$.*

Proof. We knew that A_n with $n \in \{p, p+1, p+2\}$ is C_{pp}-group. In fact, we show that G is also a C_{pp}-group. Assume the contrary, then there exists an element x of G of order r such that $r \cdot p \mid |C_G(x)|$. Let y be an element of $C_G(x)$ having order p. Then $xy = yx$. Since $|G_p| = p$, then G_p ia abellian and $|y^G|$ is a p'-number. But by Lemma 5, $|y^G| \mid |(xy)^G|$. It follows that $[|x^G|, |y^G|] \mid |(xy)^G|$. But $m_p(G)$ is the only maximal p'-number of conjugacy classes sizes of G since $|G| = n!$ and $|G| = p \cdot m_p(G)$. Thus p divides $|x^G|$. Hence $pr \cdot m_p(G)$ divides $|(xy)^G|$ Therefore $|(xy)^G| \geq prm_p(G) \geq 2|G|$, a contradiction. □

By Lemma 11, $s(G) \geq 2$, and so, by Lemma 8, G has one of the following structures:

(1) G is a Frobenius group or 2-Frobenius group;

(2) G has a normal series $1 \leq K \leq H \leq G$ such that $\pi(G/H) \cup \pi(K) \subseteq \pi_1$ and H/K is a non-abelian simple group. In particular, K is nilpotent, $G/H \lesssim \mathrm{Out}(H/K)$ and the odd order components of G are the odd order components of H/K.

So in the following, we consider case by case.

Lemma 12. *G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. Suppose the contrary. we first consider when G is a Frobenius group with kernel K and complement H. By Lemma 9, $\{\pi(K), \pi(H)\} = \{\pi((p-1)!), \{p\}\}$. The following two cases are considered:

(1) If $\pi(H) = \{p\}$, then $p \geq 11$ and $\pi(K) = \pi((p-1)!)$. By Lemma 6, there is a prime r such that $\frac{p+1}{2} < r < p$. Since K is nilpotent and G_r is of order r, then Lemma 9(2) implies $p = |H| \mid |G_r| - 1 = r - 1 < p - 1$, a contradiction.

(2) If $\pi(K) = \{p\}$, then $\pi(H) = \pi((p-1)!)$. Lemma 6, there is a prime r with that $\frac{p+1}{2} < r < p$ and $|H_r| = r$. Hence $[K]H_r$ is a Frobenius group and so, $|H_r| \mid |K| - 1 < p - 1$. It follows that $r < \frac{p-1}{2}$, a contradiction.

Let G be a 2-Frobenius group. Then G is soluble contradicting to Lemma 10. □

Lemma 13. *Let G be a finite group and $r \in \pi(G)$. If $r^2 \nmid |G|$, then G has a normal series $1 \leq K \leq H \leq G$, such that H/K is a simple group and $r \in \pi(H/K)$.*

Proof. Since G is a finite group, G has a chief series. So let $G_0 \leq G_1 \leq G_2 \cdots \leq G_l = G$ be a chief series of G. There exists some t, such that $1 \leq t \leq l$ and $r \in \pi(G_t) \setminus \pi(G_{t-1})$. Let $H = G_t$ and $K = G_{t-1}$, then $1 \leq K \leq H \leq G$ is a normal series of G and H/K is a chief factor of G. Therefore H/K is a minimal normal subgroup of G/K. We know that the chief factors are characteristically simple. Also every characteristically simple group is a simple group or a product of isomorphic simple groups. So H/K is a simple group or a product of isomorphic simple groups. If $r^2 \nmid |G|$, then by Lemma 1, $|G_r| = r$ and $r > [\frac{n}{2}]$. By Lemma 7(4), $r \nmid \mathrm{Out}(H/K)$. So we have $r \mid |K|$. By our assumption, $p \geq 11$ and so there is a Hall $\{p, r\}$-subgroup L with $p \neq r$. So L is cyclic and hence there is an element of order $p \cdot r$, contradicting Lemma 11. It follows that $r \in \pi(H/K)$. □

Lemma 14. *H/K is not isomorphic to any sporadic simple groups.*

Proof. Suppose the contrary. By Lemma 13, for any prime r such that $\frac{p+1}{2} < r < p$, then $r \in \pi(H/K)$. By Tables 1, 2 and 3, H/K is not isomorphic to one of the following groups: M_{12}, Ru, He, McL, Co_1, Co_3, HN, M_{11}, M_{23}, M_{24}, J_3, Co_2, Fi_{23}, F_2, F_3, J_1, ON, LyS, $Fi_{24}^{'}$, F_1 and J_4. If $K/H \cong J_2$, then $|K/H| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, so $p = 7$ and hence, $5^2 \mid |H/K| \nmid |A_n|$ contradicting Lemma 1, If K/H is isomorphic to McL or HS, then $p = 11$ and hence, $5^3 \mid |H/K| \nmid |A_n|$, a contradiction. If H/K is isomorphic to Szu or Fi_{22}, then $p = 13$ and hence, $|H/K| \nmid 2|A_n|$, contradiction. Finally, $H/K \cong M_{22}$, then $p = 11$. Since $|G| = |K||H/K||G/H|$, then $|H|_3 = 3^3$. By Lemma 11, $3 \cdot p \notin \omega(G)$. It follows that the Sylow 11-subgroup acts fixed freely on the set of elements of order 3 and so $11 \mid 3^2 - 1$, a contradiction. □

Lemma 15. *H/K is not isomorphic to any finite simple groups of Lie type.*

Proof. By Lemma 10 G is insoluble. If $r \in \pi(G)$ and $r^2 \nmid |G|$, then by Lemma 13, $r \mid |H/K|$. By Lemma 11, p is the odd component of G and so is of H/K. By Lemma 7

(1) If P is not isomorphic to ${}^2G_2(q)$, then, for every i, there is at most one prime number $s \in \pi_i(P)$ such that $(r + 1)/2 < s < r$.

(2) If P is isomorphic to ${}^2G_2(q)$, then there are at most three prime numbers $s \in \pi(P)$ such that $(r + 1)/2 < s < r$.

(3) $s(G) \geq 2$ and $p \in \{m_i\}$ for $i \geq 2$

In the following, we consider three cases.

(1) $s(G) = 2$.

1.1. $H/K \cong A_{p'-1}(q')$ with $(p', q') \neq (3, 2), (3, 4)$. We have

$$p = \frac{q'^{p'} - 1}{(q' - 1)(p', q' - 1)}.$$

By Lemma 6, $p \leq 13$ and so $(p', q') = (3, 3)$. Thus $|A_2(3)| = 3^3 \cdot 2^3 \cdot 2 \cdot 13$, which contradicting Lemma 13.

1.2. $H/K \cong A_{p'}(q')$ with $q' - 1 \mid p' + 1$. Then $p = \frac{q'^{p'}-1}{q'-1}$. By Lemma 6, $p \leq 13$ and and so $(p', q') = (3, 3)$. Thus $|A_3(3)| = 3^3 \cdot 2^4 \cdot 5 \cdot 2^3 \cdot 2 \cdot 13$, which contradicting Lemma 13.

1.3. $H/K \cong^2 A_{p'-1}(q')$. Then

$$p = \frac{q'^{p'} + 1}{(q' + 1)(q', p' + 1)}.$$

Similarly, $p \leq 13$ and so $q' = 4, p' = 3$. Hence $|{}^2A_2(4)| = 2^{12} \cdot 3 \cdot 5 \cdot 3 \cdot 13$. But $11 \mid |H/K|$, a contradiction.

1.4. $H/K \cong^2 A_3(2)$. Then $p = 5$ and so $2^6 \nmid 2|A_n|$ by Lemma 1, a contradiction.

1.5. $H/K \cong B_n(q')$ with $n = 2^m \geq 4$, q' odd. Then $p = (q'^n + 1)/2$. Thus $q' = 5$ and $n = 2 \ngeq 4$, contradicting Lemma 1.

1.6. $H/K \cong B_{p'}(3)$. We have $p = \frac{3^{p'}-1}{2}$ and so $p = 3$. It follows that $3^9 \mid |A_n|$, a contradictino.

1.7. $H/K \cong C_n(q')$ with $n = 2^m \geq 2$, q' odd. Then $p = \frac{q'^n+1}{(2,q-1)}$ and so $q' = 5, n = 2$. It follows that $5^4 \mid |A_n|$, a contradiction.

1.8. $H/K \cong C_{p'}(q')$ with $q' = 2, 3$. Then $p = \frac{q'^{p'}-1}{(2,q'-1)}$ and so $p' = 3, q' = 3$. Whence, $3^9 \mid |A_n|$, a contradiction.

1.9. $H/K \cong D_{p'}(q')$ with $p' \geq 5, q = 2, 3, 5$. Then $p = \frac{q'^{p'}-1}{q'-1}$ and so $p' = 3 \ngeqq 5$, a contradiction.

1.10. $H/K \cong D_{p'+1}(q')$ with $q' = 2, 3$. $p = \frac{q'^{p'}-1}{(2,q'-1)}$ and so $p' = 3, q' = 3$. Whence, $3^{12} \mid |A_n|$, a contradiction.

1.11. $H/K \cong^2 D_n(q')$ with $n = 2^m \geq 4$. Then $p = \frac{q'^n+1}{(2,q'+1)}$. Then $p = \frac{q'^n+1}{(2,q'+1)}$ and so $q' = 5, n = 2$. Since $|^2D_2(5)| = 5^2 \cdot 2^2 \cdot 13$, then $11 \nmid |A_n|$ contradicting Lemma 13.

1.12. $H/K \cong^2 D_n(2)$ with $n = 2^m + 1 \geq 5$, $H/K \cong^2 D_n(3)$ with $9 \leq 2^m + 1 \neq p'$ and $H/K \cong F_4(q')$ with q' odd. There is no prime number r such that $7 \leq r = 2^{n-1} + 1 \leq 13$, $7 \leq r = \frac{3^{n-1}+1}{2} \leq 13$ and $7 \leq r = q'^4 - q'^2 + 1 \leq 13$.

1.13. $H/H \cong^2 D_p(3)$ with $5 \leq p' \neq 2^m = 1$. Since $p = \frac{3^{p'}+1}{4}$, then $p' = 3$. Therefore $3^6 \mid |A_n|$, a contradiction.

1.14. $H/K \cong^3 D_4(q')$. Then $p = q'^4 - q'^2 + 1$ and so $q' = 2$ since $7 \leq p \leq 13$. But $11 \nmid |^3D_4(2)| = 2^{12} \cdot 3^2 \cdot 7 \cdot 3 \cdot 3 \cdot 7 \cdot 13$ contradicting Lemma 13.

1.15. $H/K \cong G_2(q')$ with $2 < q' \equiv \epsilon \mod 3, \epsilon = \pm 1$. Then $p = q'^2 - \varepsilon q' + 1$ and so $q' = 4, \varepsilon = 1$. It follows that $2^{24} \mid 2|A_n|$, a contradiction.

1.16. $H/K \cong^2 F_4(2)'$. Then $p = 13$ and $11 \mid |^2F_4(2)'|$, a contradiction.

1.17. $H/K \cong E_6(q')$. Then $p = \frac{q'^6+q'^3+1}{(3,q'-1)} > 13$ and so we rule out this case.

(2) $s(G) = 3$.

2.1. $H/K \cong A_1(q')$ with $4 \mid q' + 1$. By Lemmas 7(2) and 13, $7 \leq p \leq 13$. Thus $p = q'$ or $p = \frac{q'-1}{2}$.

If the former, then $p = q' = 7$ and $p = q' = 11$. If $p = q' = 7$, then $5 \nmid |A_n|$, a contradiction. If $p = q' = 11$, then $7 \nmid |A_n|$, a contradiction.

If the latter, then $q' = 23 \nleq 13$, a contradiction.

2.2. $H/K \cong A_1(q')$ with $4 \mid q-1$. Then $p = q'$ or $p = \frac{q'+1}{2}$. If the former, then $p = q' = 13$ and so $7 \nmid |A_n|$, a contradiction. If the latter, then $p = 13, q' = 25$ and so $7, 11 \nmid |A_n|$, a contradiction.

2.2. $H/K \cong A_1(q')$ with $2 \mid q'$. Then $p = q' + 1$ or $p = q' - 1$. If the former, there is no solution since $7 \leq p \leq 13$. If the latter, $q' = 8$ and so $5 \nmid |A_n|$, a contradiction.

2.3. $H/K \cong A_2(2)$. Then $p = 7$ and so $5 \nmid |A_2(2)|$, a contradiction.

2.4. $H/K \cong E_7(2)$ or $H/K \cong E_7(3)$. Then the primes are larger than 13 and so we rule out these cases.

2.5. $H/K \cong^2 A_5(2)$. Then $p = 11$. In this case, $2^{15} \nmid 2|A_n|$, a contradiction.

2.6. $H/K \cong^2 D_{p'}(3)$ with $5 \leq p' = 2^m + 1$. Then $p = \frac{3^{p'-1}+1}{2}$ or $\frac{3^{p'}+1}{4}$. By Lemmas 7(2) and 13, $7 \leq p \leq 13$. Thus the equations have no solution. Similar, we can rule out "$H/K \cong^2 D_{p'+1}(2)$ with $m \geq 2, p' = 2^m - 1$".

2.7. $H/K \cong G_2(q')$ with $q' \equiv 0 \mod 3$. Then $p = q'^2 - q' + 1$ or $p = q'^2 + q' + 1$ and so $q' = 3$. But $5 \nmid |A_n|$, a contradiction.

2.8. $H/K \cong^2 G_2(q')$ with $q' = 3^{2m+1} > 3$. Then $p = q' - \sqrt{3q} + 1$ or $p = q' + \sqrt{3q} + 1$. By Lemmas 7(3) and 13, $p \leq 37$. It follows that $q' = 27$. Thus $11, 31 \nmid |A_n|$, a contradiction.

2.9. $H/K \cong^2 F_4(q')$ with $q' = 2^{2m+1} > 2$ and $H/K \cong F_4(q')$ with q' even. In both cases, there is no solution.

(3) $s(G) > 3$.

3.1. $H/K \cong A_2(4)$. Then $|K/H| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and so $p = 7$, $m_7(A_2(4)) = 2^6 \cdot 3^2 \cdot 5$.

If $n = 7$, then $2^6 \mid 7!$, a contradiction.

If $n = 8$, then by [9, pp. 24], $m_2(A_2(4)) = 3^2 \cdot 5 \cdot 7$. Since in this case, $|H/K| = |A_8|$, then $|G| = |A_8|$. But $m_2(A_8) = 3 \cdot 5 \cdot 7$, a contradiction by Lemma 2.4 of [1].

If $n = 9$, then $|G| = |A_9| = 2^6 \cdot 3^4 \cdot 5 \cdot 7$ by hypotheses, and so $m_2(A_9) = 3^3 \cdot 5 \cdot 7$, $m_7(A_9) = 2^6 \cdot 3^4 \cdot 5$. By Lemma 8(2) and [9], $A_2(4) \leq G/K \leq \mathrm{Aut}(A_2(4))$. If $G/K \cong A_2(4)$, then $|K| = 3^2$ and $Z(G) = 3^2$. It follows that there is an element of order $3 \cdot 7$, contradicting to Lemma 11. If $G/K \cong \mathrm{Aut}(A_2(4))$, then since $|\mathrm{Out}(A_2(4))| = |2 \times S_3| = 2^2 \cdot 3$, order consideration rules out.

3.2. $H/K \cong^2 E_6(2)$. Then $p = 19 \nleq 13$, a contradiction.

3.2. $H/K \cong^2 B_2(q')$ with $q' = 2^{2m+1} > 2$. Then $p = q' - 1, q' - \sqrt{2q'} + 1$ or $q' + \sqrt{2q'} + 1$ and so, $q' = 8$. But $11 \nmid |A_n|$, a contradiction.

3.3. $H/K \cong E_8(q')$. In these cases, $p > 13$ and so , we rule out these cases.

This completes the proof of the lemma. □

Lemma 16. *G is isomorphic to A_n with $n \in \{p, p+1, p+2\}$.*

Proof. By Lemmas 12, 14 and 15, $H/K \cong A_n$. Order consideration, we have that $n = p, p+1, p+2$.

If $n = p$, then $A_p \leq H/K \leq S_p$. If $H/K \cong A_p$, then order consideration gets the desired results. If $H/K \cong S_p$, then we rule out this case by group order.

Similarly, we can conclude that $G \cong A_{p+1}$ if $n = p+1$; $G \cong A_{p+2}$ if $n = p+2$.

The Lemma is proved. □

This completes the proof of Main Theorem. □

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