The path graph of the amalgamated graph of C_3 and C_n at an edge or at a vertex

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Abstract. Path graphs were proposed as a generalization of line graphs. The 2-path graph denoted by $P_2(G)$, of a graph G has vertex set the set of all paths of length two. Two such vertices are adjacent in the new graph if their union is a path of length three or a cycle of length three. In this paper we will introduce the path graph of the amalgamated graph of C_3 and C_n at an edge and at a vertex. Also, some new properties of these graphs will be given such as the independence number, domination number and matching number.

Keywords: path graphs, amalgamated graph, independence number, domination number, matching number.

1. Introduction

For any graph G, as a generalization of the line graph Broesma and Hoede, see [3], defind the k-path graphs of G denoted by $P_k(G)$. They studied some properties of these graphs.

Definition 1.1. *The k-path graph of a graph G denoted by $P_k(G)$ has a vertex set the set of all paths of length k in G. Two such vertices are adjacent in $P_k(G)$ if their union is a path or a cycle of length $k+1$.*

In this paper, we will focus our study on the graph $P_2(G)$.

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A characterization of 2-path graphs has been given by Broesma, Hoede and by Huaien Li, see [3] and [4]. Later on, Prisner gave a new characterization of k-path graphs, see [7]. Diameters, centers and distance in path graphs were studied in [2], [5] and [6]. Isomorphisms of path graphs were studied in [1] and [8]. Paths of length 2 in G as well as vertices of $P_2(G)$ will be represented by triples abc, where b is the middle vertex of the path of length 2 in G from a to c and $abc = cba$.

The following two examples explain the definition of path graphs, see [3].

Example 1.1. Let G be the graph obtained from $K_{1,3}$ by subdividing all of its edges once, this graph is denoted by $S(K_{1,3})$. Observe that $P_2(S(K_{1,3})) = C_6$.

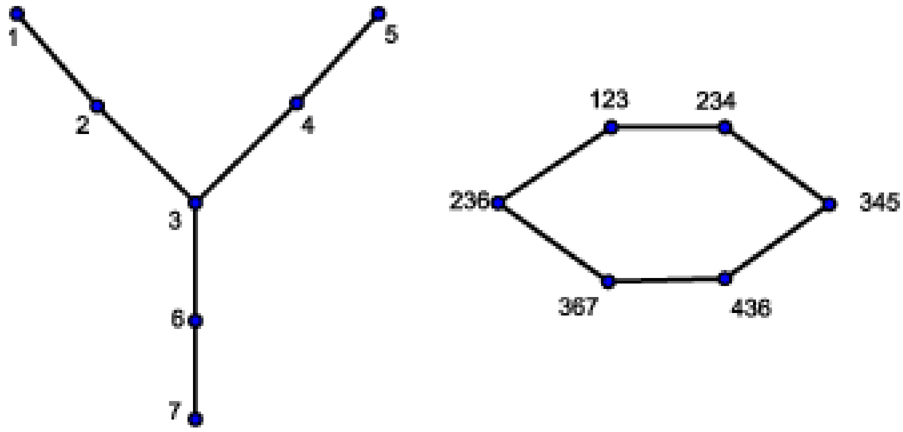

Figure 1. The graph $S(K_{1,3})$ and $P_2(S(K_{1,3}))$

Example 1.2. The graph $S(K_{1,3}) - s$, where s is an end vertex, is denoted by Y. Observe that $P_2(Y) = P_5$. Figure 2 shows the graph Y and $P_2(Y)$.

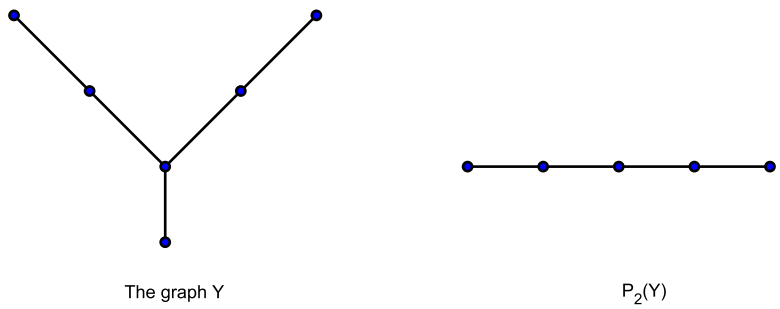

Figure 2. The graph Y and $P_2(Y)$.

Definition 1.2. *Any two graphs G and H are said to be amalgamated at an edge or at a vertex if G and H have exactly one edge or one vertex in common respectively.*

We need the following result about path graphs, see [3].

Theorem 1.1. *For a vertex abc of $P_2(G)$, deg(abc) = deg(a) + deg(c) -2. Note that deg(a) and deg(c) are degrees in G, wheareas deg(abc) is the degree of the vertex abc of $P_2(G)$.*

2. The path graph of the amalgamated graph of C_3 and C_n at an edge

First, we give some examples to show how the path graph of the amalgamated graph of C_3 and C_i at an edge look like, for i=3, 4, 5 . Then from these examples we will deduce the path graph of the amalgamated graph of C_3 and C_n at an edge and some of its properties for any n.

Example 2.1. Consider the graph G_1, the amalgamated graph of C_3 and C_3 at an edge. We represent this graph and its path graph in Figure 3.

We have $V(P_2(G_1)) = 1, 2, 3, ..., 8$ and $E(P_2(G_1))=\{e_1, e_2, e_3, ..., e_{12}\}$.

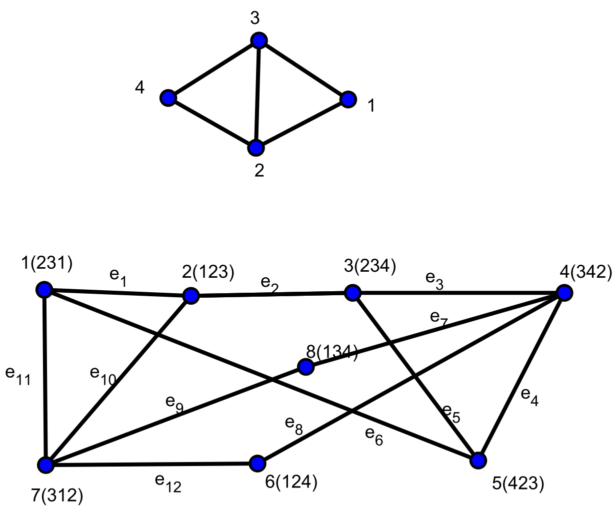

Figure 3. G_1 and $P_2(G_1)$.

Notice that $|V(G_1)|=4$ and $|V(P_2(G_1)|=8$. Also we can see that $|E(P_2(G_1)|=$ 12.

To find the domination number of the path graph of G_1. Observe that S=\{4, 7\} is a minimum dominating set and thus $\gamma(P_2(G_1))=2$.

To find the independent number, observe that the sets X={1, 3, 6, 8}, Y={2, 4} and Z= {5, 7} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_1))=|X|=4$.

In the graph $P_2(G_1)$, we have the following maximal matching sets, A={e_1, e_3, e_9}, B={e_2, e_6, e_7, e_{12}}, C={e_5, e_8, e_{10}} and D={e_2, e_4, e_{11}}. Observe that the matching number of $P_2(G_1)$ equals 4.

Notice that $P_2(G_1)$ has a Hamiltonion path which is 6, 7, 8, 4, 3, 2, 1, 5.

Example 2.2. Consider the graph G_2, the amalgamated graph of C_3 and C_4 at an edge. We represent this graph and its path graph in Figure 4. Observe that $V(P_2(G_2))=\{ 1, 2, 3, ..., 8, 9\}$ and $E(P_2(G_2))=\{e_1, e_2, e_3, ..., e_{12}, e_{13}\}$.

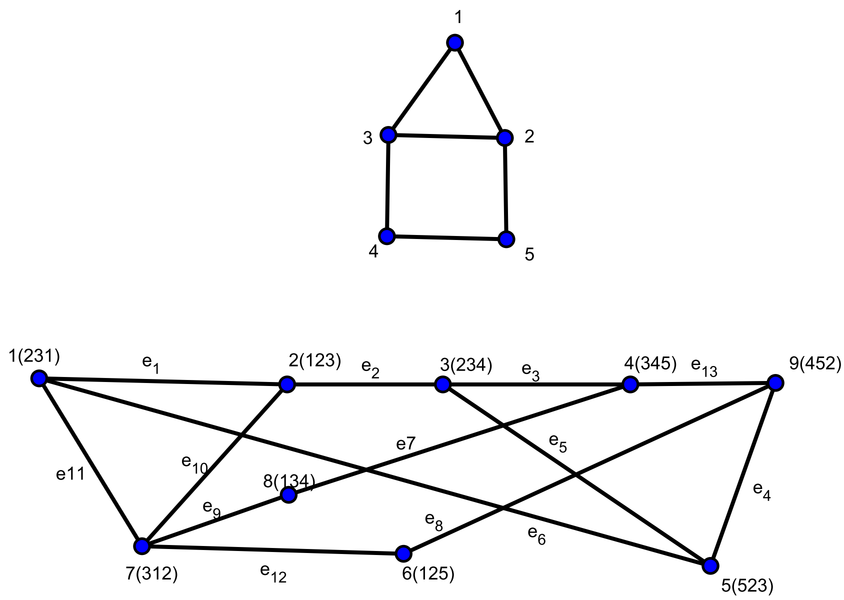

Figure 4. G_2 and $P_2(G_2)$.

Notice that $|V(G_2)|=5$ and $|V(P_2(G_2))|=9$. Also we can see that $|E(P_2(G_2))|=$ 13.

One can easily check that the set {4, 5, 7} is a minimum dominating set and hence $\gamma(P_2(G_2))=3$.

The sets X={1, 3, 8, 9}, Y={2, 4, 5, 6} and Z= {3, 7, 9} are maximal independent sets. Observe that X is a maximum independent set and so $\alpha(P_2(G_1))=|X|=4$.

In the graph $P_2(G_2)$, we have the following maximal matching sets, A={e_1, e_3, e_4, e_9}, B={e_4, e_7, e_{10}}, C={e_2, e_6, e_{12}, e_{13}} and D={e_5, e_7, e_8, e_{11}}. Observe that the matching number equals 4.

Notice that $P_2(G_2)$ has a Hamiltonion path which is 6, 9, 5, 3, 2, 1, 7, 8, 4.

Example 2.3. Consider the graph G_3, *the amalgamated graph of C_3 and C_5 at an edge. We represent this graph and its path graph in Figure* 5.

We have $V(P_2(G_3)) = \{1, 2, 3, ..., 14\}$ and $E(P_2(G_3)) = \{e_1, e_2, e_3, ..., e_{14}\}$.

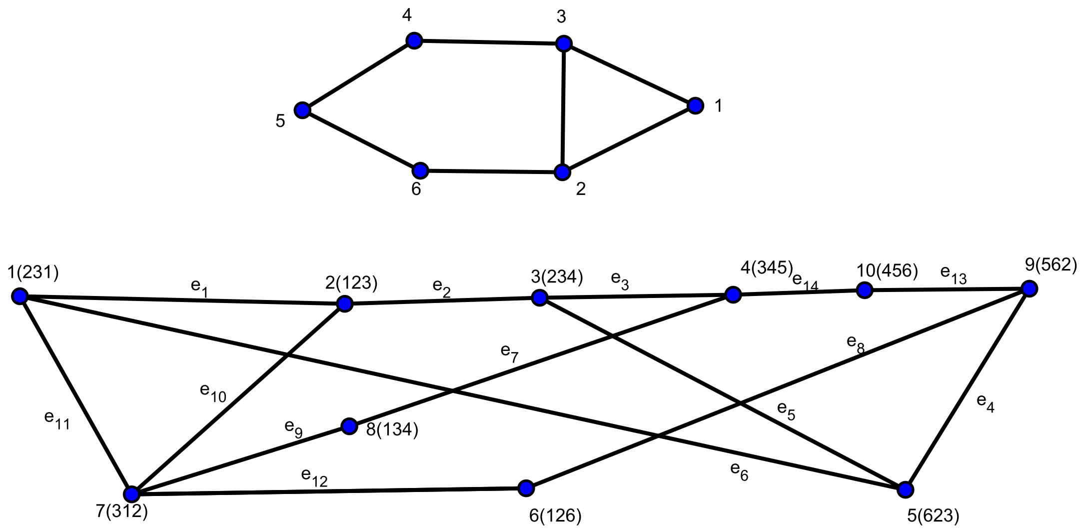

Figure 5. G_3 and $P_2(G_3)$.

Notice that $|V(G_3)|=6$ and $|V(P_2(G_3))|=10$. Also we can see that $|E(P_2(G_3))|= 14$.

One can check that S=$\{4, 5, 7, 10\}$ is a minimum dominating set and thus $\gamma(P_2(G_3))=4$.

Observe that the sets X=$\{1, 3, 6, 8, 10\}$, Y=$\{2, 4, 9\}$ and Z= $\{4, 5, 7\}$ are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_3))=|X|=5$.

In the graph $P_2(G_3)$, we have the following maximal matching sets, A=$\{e_1, e_3, e_4, e_9\}$, B=$\{e_2, e_6, e_7, e_{12}, e_{13}\}$, C=$\{e_5, e_8, e_{11}, e_{14}\}$ and D=$\{e_4, e_{10}, e_{14}\}$. Observe that the matching number of $P_2(G_3)$ equals 5.

Notice that $P_2(G_3)$ has a Hamiltonion path which is 6, 9, 5, 3, 2, 1, 7, 8, 4, 10.

We follow the same way as given in the last three examples to get the general form of the path graph of the amalgamated graph of C_3 and C_n. Now we give the following theorem that gives the number of edges and vertices of the path graph of the amalgamated graph of C_3 and C_n at an edge. This graph shown in Figure 6.

Theorem 2.1. *Let G be the amalgamated graph of C_3 and C_n at an edge and $|V(G)| = n+1$. Then $|V(P_2(G)| = n+1+4$ and $|E(P_2(G) |= n+1+8$.*

Proof. The graph G has exactly two vertices of degree three each one of them give rise to three vertices of $P_2(G)$. The remaining $(n+1-2)$ vertices of G are of degree 2. Each one of these gives rise to one vertex of P_2 (G). Thus $|V(P_2(G)|=6+n+1-2=n+1+4$.

The graph G has only one vertex of degree two that is adjacent to two vertices of degree three. This gives two edges in $P_2(G)$. The graph G has two vertices of degree three and each one of these two vertices is adjacent to two vertices of degree two. Each one of these vertices contributes four to $E(P_2(G))$. There are two vertices of degree two and each one of them is adjacent to a vertex of degree three and a vertex of degree two.This contributes three to the edges of $P_2(G)$. The remaining $(n+1-5)$ vertices of G are of degree two and each one of them is adjacent to two vertices of degree 2. Thus every vertex of the remaining $(n+1-5)$ vertices increase $|E(P_2(G))|$ by one. Hence $|E(P_2(G))| = 2+4+4+3+(n+1-5) = n+1+8$. □

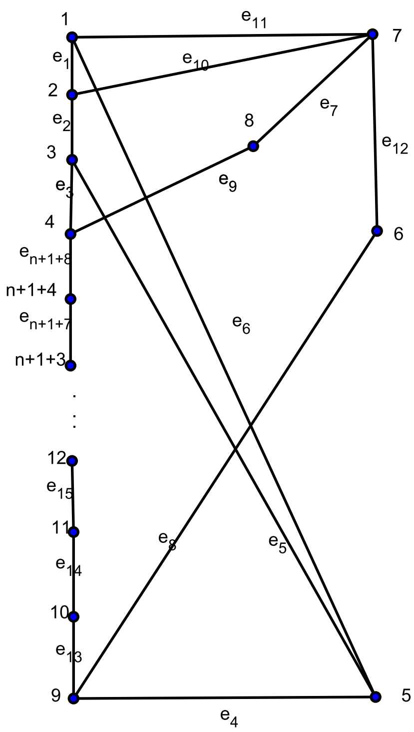

Figure 6. The path graph of the amalgamated graph of C_3 and C_n at an edge.

Let G be the amalgamated graph of C_3 and C_n for $n \geq 6$.

From the sketch of the graph $P_2(G)$, we can get the following results

1. A minimum dominating set of $P_2(G)$ is S ={4, 5, 7, 10, 12, 14, ..., n+1+3} if n+1 is odd, and S={ 4, 5, 7, 11, 13, 15, 17, ..., n+1+3} if n+1 is even. Hence $\gamma(G)=|S| = \lfloor \frac{|V(P_2(G))|}{2} \rfloor - 2$.

2. $P_2(G)$ has a Hamiltonion path which is 10, 11, 12, ..., n+1+4, 4, 8, 7, 1, 2, 3, 5, 9, 6.

3. To find the independence number of $P_2(G)$, we have two cases to consider. If n+1 is even, then we have the following maximal Independent sets, X={1, 3, 6, 8, 10, 12, 14, ..., n+1+2, n+1+4}, Y={2, 4, 9, 11, 13, ..., n+1-2, n, n+1+1, n+1+3} and Z={4, 5, 7, 10, 12 14, 16, ..., n+1, n+1+2}. The set X is a maximum independent set. So,

$$\alpha(P_2(G)) = |X| = \lfloor \frac{V(P_2(G))}{2} \rfloor$$

If n+1 is odd, then we have the following maximal independent sets, X={1, 3, 8, 9, 11, 13, ..., n+1+2, n+1+4}, Y={2, 4, 5, 6, 10, 12, 14, ..., n, n+1+1, n+1+3} and Z={4, 5, 7, 10, 12, 14, 16, ..., n+1+1, n+1+3}. The set Y is a maximum independent set. So,

$$\alpha(P_2(G)) = |Y| = \lfloor \frac{|V(P_2(G))|}{2} \rfloor.$$

4. In the graph $P_2(G)$, if n+1 is even we have the following maximal matching sets, A={e_1, e_3, e_{12}, e_{13}, e_{15}, e_{17}, ..., e_{n+1+3}, e_{n+1+5}, e_{n+1+7}}, B={e_2, e_4, e_{11}, e_{14}, e_{16}, e_{18}, ..., e_{n+1+2}, e_{n+1+4}, e_{n+1+6}, e_{n+1+8}}, C={e_2,e_6,e_7, e_8, e_{14}, e_{16}, e_{18}, ..., e_{n+1+4}, e_{n+1+6}, e_{n+1+8}} and D={e_1, e_5, e_9, e_{13}, e_{15}, e_{17}, ..., e_{n+1+3}, e_{n+1+5}, e_{n+1+7}}.

If n+1 is odd, then we have the following maximal matching sets, A={e_1, e_3, e_{12}, e_{13}, e_{15}, e_{17}, ..., e_{n+1+4}, e_{n+1+6}}, B={ e_2, e_4, e_9, e_{11}, e_{14}, e_{16}, e_{18}, ..., e_{n+1+3}, e_{n+1+5}, e_{n+1+7}}, C={e_2, e_6, e_7, e_8, e_{14}, e_{16}, e_{18}, ..., e_{n+1+3}, e_{n+1+5}, e_{n+1+7}} and D={e_1, e_5, e_9, e_{13}, e_{15}, e_{17}, ..., e_{n+1+2}, e_{n+1+4}, e_{n+1+6}}.

Observe that the matching number of $P_2(G)$ equals $\lceil \frac{|E(P_2(G))|-10}{2} \rceil$+3.

3. The path graph of the amalgamated graph of C_3 and C_n at a vertex

In this section, we will introduce the path graph of the amalgamated graph of C_3 and C_n at a vertex. Then some properties of this graph will be studied.

First we give some examples to show how the path graph of the amalgamated graph of C_3 and C_i at a vertex look like, for i=3, 4 and 5.

Example 3.1. Consider the graph G_3^*, the amalgamated graph of C_3 and C_3 at a vertex. This graph and its path graph are represented in Figure 7.

We have $V(P_2(G_3^*))$ = {a, b, c, d, e, f, g, h, i, k} and $E(P_2(G_3^*))$ ={e_1, e_2, e_3, ..., e_{13}}.

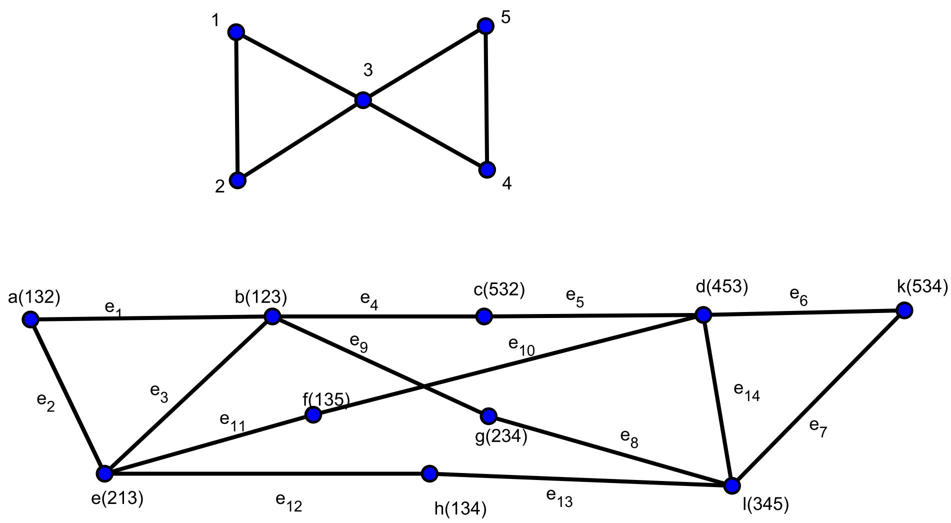

Figure 7. The graph of G_3^* and $P_2(G_3^*)$ at an edge.

Notice that $|V(G_3^*)|=5$ and $|V(P_2(G_3^*)|=10$. Also we can see that $|E(P_2(G_3^*)|=14$.

The set S={e, l, d} is a minimum dominating set of $P_2(G_3^*)$ and thus $\gamma(P_2(G_3^*))=3$.

Observe that the sets X={a, c, k, g, f, h}, Y={b, d, h} and Z= {e, l, c} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_3^*))=|X|=6$.

In the graph $P_2(G_3^*)$, we have the following maximal matching sets, A={e_1, e_5, e_7, e_{11}}, B={ e_2, e_4, e_6, e_8}, C={e_5, e_7, e_9, e_{12}} and D={e_3, e_{10}, e_{13}}. Observe that the matching number of $P_2(G_3^*)$ equals 4.

Example 3.2. Consider the graph G_4^*, the amalgamated graph of $C_3$and C_4 *at a vertex. This graph and its path graph are represented in Figure* 8.

We have $V(P_2(G_4^*))$ = {a, b, c, d, e, f, g, h, i, k, i_1} and $E(P_2(G_4^*))$ ={e_1, e_2, e_3, ...,e_{13}, $\acute{e}_1$, $\acute{e}_2$}.

Notice that $|V(G_4^*)|=6$ and $|V(P_2(G_4^*)|=11$. Also we can see that $|E(P_2(G_4^*)|=15$.

The set S={e, l, d} is a minimum dominating set of $P_2(G_4^*)$ and thus $\gamma(P_2(G_4^*))=3$.

Observe that the sets X={a, c, k, g, f, h, i_1}, Y={b, d, h} and Z= {e, l, c} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_4^*))=|X|=7$.

In the graph $P_2(G_4^*)$ we have the following maximal matching sets, A={e_1, e_5, e_7, e_{11}}, B={ e_2, e_4, e_6, e_8}, C={e_5, e_7, e_9, e_{12}} and D={e_3, e_{10}, e_{13}}. Observe that the matching number of $P_2(G_4^*)$ equals 4.

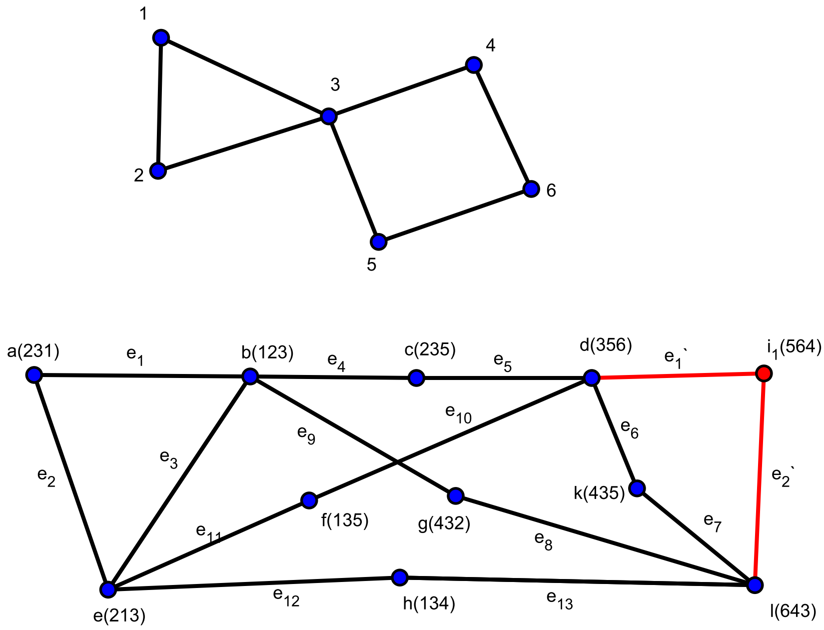

Figure 8. The graph of G_4^* and $P_2(G_4^*)$ at an edge.

Example 3.3. Consider the graph G_5^*, the amalgamated graph of $C_3$and C_5 at a vertex. We represent this graph and its path graph in Figure 9.

We have $V(P_2(G_5^*)) = \{a, b, c, d, e, f, g, h, i, k, i_1, i_2\}$ and $E(P_2(G_5^*)) =\{e_1, e_2, e_3, ..., e_{13}, \acute{e}_1, \acute{e}_2, \acute{e}_3\}$.

Notice that $|V(G_5^*)|$=7 and $|V(P_2(G_5^*)|$=12. Also we can see that $|E(P_2(G_5^*)|$= 16.

The set S=$\{$ e, l, d$\}$ is a minimum dominating set of $P_2(G_5^*)$ and thus $\gamma(P_2(G_5^*))$=3.

Observe that the sets X=$\{a, c, k, g, f, h, i_1\}$, Y=$\{b, d, h, i_2\}$ and Z= $\{e, l, c, i_1\}$ are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_5^*))$=|X|=7.

In the graph $P_2(G_5^*)$ we have the following maximal matching sets, A=$\{e_1, \acute{e}_2, e_5, e_7, e_{11}\}$, B=$\{ e_2, \acute{e}_2, e_4, e_6, e_8\}$, C=$\{\acute{e}_2, e_5, e_7, e_9, e_{12}\}$ and D=$\{\acute{e}_2, e_3, e_{10}, e_{13}\}$. Observe that matching number of $P_2(G_5^*)$ equals 5.

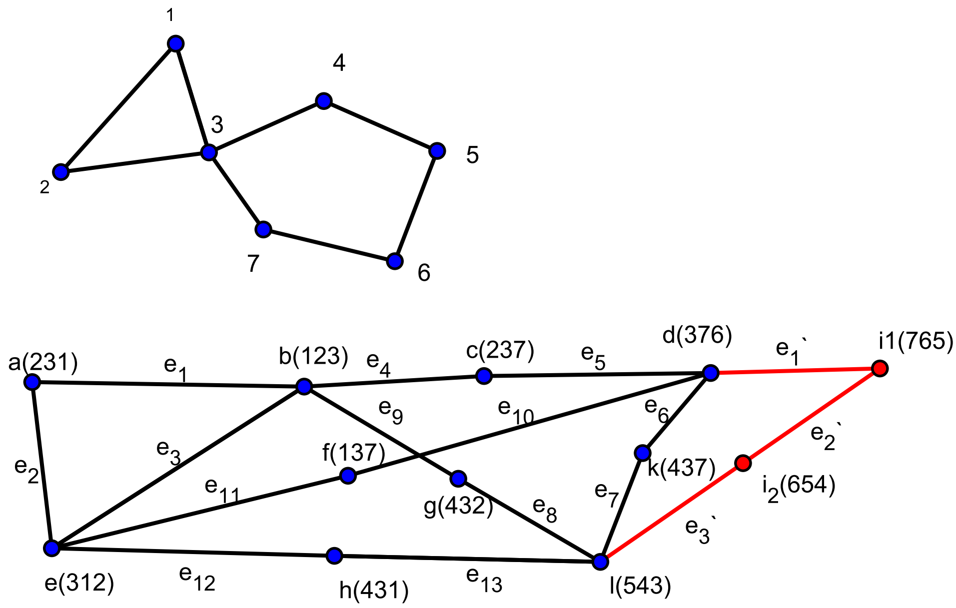

Figure 9. The graph of G_5^* and $P_2(G_5^*)$ at an edge.

We follow the same way as the last three examples to get the general form of the amalgamated graph of C_3 and C_n at a vertex. We denote the amalgamated graph of C_3 and C_n at a vertex by G_n^*. This graph is shown in Figure 10.

Notice that if $|V(G_n^*)|$=n+2 then, $|V(P_2(G_n^*)|$=n+7, also we can see that for all n$\geq$4, $|E(P_2(G_i^*)|= n+11$.

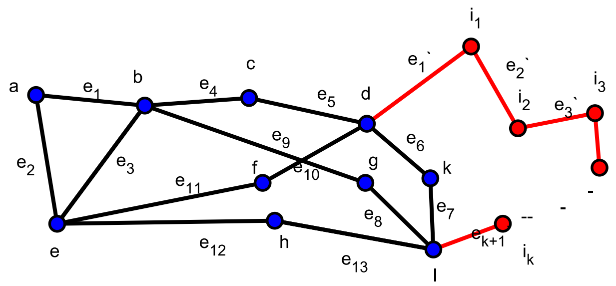

Figure 10. The path graph of the amalgamated graph of C_3 and C_n at a vertex.

From the sketch of the graph $P_2(G_n^*)$, we can get the following results

A minimum dominating set of $P_2(G_n^*)$ is S={e, l, d}$\cup${i_3, i_5, i_7, ..., i_{k-3}, i_{k-1}}. Hence $\gamma(P_2(G_n^*))=3+\gamma(P_{k-2})=3+\lceil\frac{k-2}{3}\rceil$, ($P_{k-2}$ is the path i_2, i_3, ..., i_{k-1}).

Now, we want to find independent number of the graph $P_2(G_n{}^*)$. First denote the path $i_1,i_2,i_3,...,i_k$ by P_k. Observe that if k is odd, then the sets X={a, c, k, g, f, h, i_1, i_3,..., i_k}, Y={b, d, h, i_2, i_4,...,i_{k-1}} and Z={e, l, c, i_1, i_3, ...,i_{k-2}} are maximal independent sets. The set X is a maximum independent set. If k is even, then the sets X={a, c, k, g, f, h, i_1, i_3,..., i_{k-1}}, Y={b, d, h, i_2, i_4,...,i_k} and Z={e, l, c, i_1, i_3, ..., i_{k-1}} are maximal independent sets. The set X is a maximum independent set. So, $\alpha(P_2(G_n^*))=6+\alpha(P_k) =6+\lceil\frac{K}{2}\rceil$.

In the graph $P_2(G_n^*)$ if k is even we have the following maximal matching sets, A={e_1, e_5, e_{11}, e_7, $é_2$, $é_4$,....,$é_k$}, B={ e_4, e_2, e_8, e_6,$é_2$, $é_4$,....,$é_k$}, C={e_9, e_{12}, e_5, e_7, $é_2$, $é_4$, ..., $é_k$} and D={e_3, e_{13}, e_{10}, $é_2$, $é_4$, ...$é_k$}. If k is odd, then $P_2(G_n^*)$ has the following maximal matching sets, A={e_1, e_5, e_{11}, e_7, $é_2$, $é_4$,...., $é_{k-1}$}, B={ e_4, e_2, e_8, e_6,$é_2$, $é_4$,....,$é_{k-1}$}, C={e_9, e_{12}, e_5, e_7, $é_2$, $é_4$, ..., $é_{k-1}$} and D={e_3, e_{13}, e_{10}, $é_2$, $é_4$, ...$é_{k-1}$}. Observe that the matching number of $P_2(G_n^*)$=|A|=$4+\lceil\frac{k-1}{3}\rceil$.

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