A further study on the hyperideals of ordered semihypergroups

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Abstract. In this paper, we first introduce the concepts of prime, weakly prime and semiprime hyperideals in ordered semihypergroups, and give some characterizations of them. Furthermore, we consider the extensions of hyperideals in commutative ordered semihypergroups. As a generalization of the concept of prime hyperideals of ordered semihypergroups, the concept of n-prime hyperideals of ordered semihypergroups is introduced, and related properties are discussed. In particular, we prove that every $(n-1)$-prime hyperideal of ordered semihypergroups is n-prime for any positive integer $n \geq 3$. Moreover, we investigate the relationship between n-prime hyperideals and extensions of hyperideals, and prove that a hyperideal I of a commutative ordered semihypergroup is n-prime if and only if any extension of I is $(n-1)$-prime $(n \geq 3)$. Finally, we prove that if I is a semiprime hyperideal of a commutative ordered semihypergroup S, then I is the intersection of all extensions of I. Especially, if I is also n-prime $(n \geq 3)$, then I can be expressed as the intersection of all $(n-1)$-prime hyperideals of S containing it.

Keywords: ordered semihypergroup, prime hyperideal, semiprime hyperideal, n-prime hyperideal, extension of hyperideal.

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1. Introduction

The algebraic hyperstructure is a natural generalization of the classical algebraic structures which was first introduced by Marty [18] in 1934. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. After the pioneering work of Marty, algebraic hyperstructures have been intensively studied, both from the theoretical point of view and especially for their applications in other fields such as Euclidean and non-Euclidean geometries, graphs and hypergraphs, fuzzy sets, automata, cryptography, artificial intelligence, codes, probabilities, lattices and so on (see [4]). Recently, algebraic hyperstructures have been developed by many researchers. A lot of papers and several books have been written on algebraic hyperstructure theory, see [7, 8, 10, 14, 19, 20]. There are some books on the general theory of algebraic hyperstructures: one by Corsini [3] on the basic theory of hypergroups, another by Vougiouklis [27], mostly on representations of hypergroups and on H_v-structures, which are hyperstructures satisfying conditions weaker than the classic ones.

Semihypergroups have been found useful for dealing with problems in different areas of algebraic hyperstructures. Many authors studied different aspects of semihypergroups, for instance, Anvariyeh et al. [1], Davvaz [5], Davvaz and Poursalavati [9], Hasankhani [12], Hila and Abdullah [15] and Leoreanu [17], also see [11, 21, 31]. It is now natural to investigate the existing subsystems of other algebraic hyperstructures. In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have several applications in the theory of sequential machines, formal languages, computer arithmetics and error-correcting codes. There are several results which have been added to the theory of ordered semigroups by Kehayopulu, Davvaz, Satyanarayana, Xie, and many other researchers. For more details, the reader is referred to [16, 23, 28, 29]. A theory of hyperstructures on ordered semigroups can be developed. In [13], Heidari and Davvaz applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. In particular, they defined and studied the hyperideals of an ordered semihypergroup. Also see [2, 6, 24]. In [30], Yaqoob et al. also defined the partially ordered left almost semihypergroups, and studied related properties.

It is well known that hyperideals of a semihypergroup with special properties always play an important role in the study of semihypergroups structure. Motivated by the study of hyperideals in hyperrings and semihypergroups, and also motivated by Davvaz's works in ordered hyperstructures, we attempt in the present paper to study hyperideals of ordered semihypergroups in detail. The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper. In Section 3, we introduce the concepts of prime, weakly prime and semiprime hyperideals in ordered semihypergroups, and give some characterizations of them. In Section 4, we consider the extensions of hyperideals in commutative ordered semihypergroups. In addition, we define n-prime hyperideals and n-semiprime hyperideals of ordered semihypergroups, and investigate their related properties. In particular, we show that for any positive integer $n \geq 2$, n-prime hyperideals of an ordered semihypergroup are a generalization of prime hyperideals. We also prove that every $(n-1)$-prime hyperideal of ordered semihypergroups is n-prime for any positive integer $n \geq 3$. Moreover, we investigate the relationship between extensions of hyperideals and n-prime hyperideals, and prove that a hyperideal I of a commutative ordered semihypergroup is n-prime if and only if any extension of I is $(n-1)$-prime $(n \geq 3)$. Especially, we prove that a semiprime, n-prime hyperideal $(n \geq 3)$ of a commutative ordered semihypergroup S can be expressed as the intersection of all $(n-1)$-prime hyperideals of S containing it. As an application of the results of this paper, the corresponding results in ordinary semihypergroups can be also obtained by moderate modification.

2. Preliminaries and some notations

Recall that a *hypergroupoid* $(S, \circ)$ is a nonempty set S together with a hyperoperation, that is a map $\circ : S \times S \to P^*(S)$, where $P^*(S)$ denotes the set of all the nonempty subsets of S. The image of the pair (x, y) is denoted by $x \circ y$. If $x \in S$ and A, B are nonempty subsets of S, then $A \circ B$ is defined by $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. Also $A \circ x$ is used for $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$. A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$ (see [3]). A semihypergroup $(S, \circ)$ is called a *hypersemilattice* if $x \in x \circ x$ and $x \circ y = y \circ x$ for all $x, y \in S$ (see [22]).

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation "$\leq$" such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. In the following, we shall extend the concept of ordered semigroups to the hyper version, and introduce the concept of ordered semihypergroups from [13].

Definition 2.1. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [13]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in P^*(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

Definition 2.2. An element e in an ordered semihypergroup $(S, \circ, \leq)$ is called *identity* if $a \in a \circ e \cap e \circ a$ for any $a \in S$.

Definition 2.3. Let $(S, \circ, \leq)$ be ordered semihypergroup. Then S is called *commutative* if $a \circ b = b \circ a$ for any $a, b \in S$.

Clearly, every ordered semigroup can be regarded as an ordered semihypergroup. In the following we give two examples of ordered semihypergroups.

Example 2.4. Let $(S, \leq)$ be a partially ordered set. If for every $x, y \in S$, we define $x \circ y = \{x, y\}$, then $(S, \circ, \leq)$ is a commutative ordered semihypergroup.

Example 2.5 ([13]). Let $(S, \cdot, \leq)$ be an ordered semigroup. If for every $x, y \in S$, we define $x \circ y = < x, y >$, where $< x, y >$ is the ideal of S generated by $\{x, y\}$, then $(S, \circ, \leq)$ is an ordered semihypergroup.

Let S be an ordered semihypergroup. For $\emptyset \neq H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

By a *subsemihypergroup* of an ordered semihypergroup S we mean a nonempty subset A of S such that $A \circ A \subseteq A$. A nonempty subset A of an ordered semihypergroup S is called a *left* (resp. *right*) *hyperideal* of S if (1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right hyperideal of S, then it is called a (*two-sided*) *hyperideal* of S (see [13]). We denote by $L(A)$ (resp. $R(A), I(A)$) the left (resp. right, two-sided) hyperideal of S generated by A ($\emptyset \neq A \subseteq S$). One can easily prove that $L(A) = (A \cup S \circ A]$, $R(A) = (A \cup A \circ S]$ and $I(A) = (A \cup S \circ A \cup A \circ S \cup S \circ A \circ S]$. In particular, if $A = \{a\}$, then we write $L(a), R(a), I(a)$ instead of $L(\{a\}), R(\{a\}), I(\{a\})$, respectively. If S is commutative, then $I(a) = (a \cup S \circ a] = (a \cup a \circ S]$.

Lemma 2.6. *Let S be an ordered semihypergroup. Then the following statements hold:*

(1) $A \subseteq (A], \quad \forall A \subseteq S$.

(2) *If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.*

(3) $(A] \circ (B] \subseteq (A \circ B]$ *and* $((A] \circ (B]] = (A \circ B], \quad \forall A, B \subseteq S$.

(4) $((A]] = (A], \quad \forall A \subseteq S$.

(5) *For every hyperideal T of S, we have $(T] = T$.*

(6) *If A, B are hyperideals of S, then $(A \circ B]$ is a hyperideal of S.*

(7) *For every $a \in S$, $(S \circ a \circ S]$ is a hyperideal of S.*

(8) *If T is a hyperideal of S and A, B are two nonempty subsets of S such that $A \preceq B \subseteq T$, then $A \subseteq T$.*

(9) *For any two nonempty subsets A, B of S such that $A \preceq B$, we have $C \circ A \preceq C \circ B$ and $A \circ C \preceq B \circ C$ for any nonempty subset C of S.*

Proof. Straightforward.

Lemma 2.7. *Let S be an ordered semihypergroup and $\{A_i \mid i \in I\}$ a family of hyperideals of S. Then $\bigcup_{i \in I} A_i$ is a hyperideal of S and $\bigcap_{i \in I} A_i$ is also a hyperideal of S if $\bigcap_{i \in I} A_i \neq \emptyset$.*

Proof. Straightforward.

For the sake of simplicity, throughout this paper, we denote $A^n = A \circ A \circ \cdots \circ A$ (n-copies).

Lemma 2.8. *Let S be an ordered semihypergroup. Then the following statements are equivalent:*

(1) *$(A^2] = A$ for every hyperideal A of S.*
(2) *$A \cap B = (A \circ B]$ for all hyperideals A, B of S.*
(3) *$I(a) \cap I(b) = (I(a) \circ I(b)]$ for any $a, b \in S$.*
(4) *$I(a) = ((I(a))^2]$ for all $a \in S$.*
(5) *$a \in (S \circ a \circ S \circ a \circ S]$ for all $a \in S$.*

Proof. (1) $\Longrightarrow$ (2). Let A and B be hyperideals of S. Then, by Lemma 2.6, $(A \circ B] \subseteq (A \circ S] \subseteq (A] = A$ and $(A \circ B] \subseteq (S \circ B] \subseteq (B] = B$, from which we can conclude that $(A \circ B] \subseteq A \cap B$. On the other hand, by Lemma 2.7, $A \cap B$ is a hyperideal of S. Then, by (1), we have

$$A \cap B = ((A \cap B)^2] = ((A \cap B) \circ (A \cap B)] \subseteq (A \circ B].$$

Thus $A \cap B = (A \circ B]$.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are clear.

(4) $\Longrightarrow$ (5). Let $a \in S$. By hypothesis and Lemma 2.6, we have

$$(I(a))^2 = ((I(a))^2] \circ I(a) = ((I(a))^2] \circ (I(a)] \subseteq ((I(a))^3].$$

Then, we have

$$(I(a))^3 = (I(a))^2 \circ I(a) = (I(a))^2 \circ (I(a)] \subseteq ((I(a))^3] \circ (I(a)] \subseteq ((I(a))^4].$$

Further, it can be shown that $(I(a))^4 \subseteq ((I(a))^5]$. Thus

$$\begin{aligned} I(a) &= ((I(a))^2] \subseteq (((I(a))^3]] = ((I(a))^3] \subseteq (((I(a))^4]] = ((I(a))^4] \\ &\subseteq (((I(a))^5]] = ((I(a))^5] \subseteq (S \circ I(a)] \subseteq (I(a)] = I(a), \end{aligned}$$

which implies that $I(a) = ((I(a))^5]$. On the other hand, we have

$$\begin{aligned} (I(a))^3 &= (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]^3 \\ &\subseteq ((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)^2] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq (S \circ a \cup S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq ((S \circ a \cup S \circ a \circ S) \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)] \subseteq (S \circ a \circ S]. \end{aligned}$$

Then,

$$\begin{aligned} (I(a))^4 &\subseteq (S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq (S \circ a \circ S \circ a \cup S \circ a \circ S \circ a \circ S], \end{aligned}$$

and we have

$$\begin{aligned}(I(a))^5 &\subseteq (S \circ a \circ S \circ a \cup S \circ a \circ S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \\ &\subseteq (S \circ a \circ S \circ a \circ S].\end{aligned}$$

Therefore, $a \in I(a) = ((I(a))^5] \subseteq ((S \circ a \circ S \circ a \circ S]] = (S \circ a \circ S \circ a \circ S]$.

(5) $\Longrightarrow$ (1). Let A be a hyperideal of S. Then $(A^2] = (A \circ A] \subseteq (A \circ S] \subseteq (A] = A$. Conversely, let $x \in A$. Then, by (5) and Lemma 2.6, we have

$$\begin{aligned} x &\in (S \circ x \circ S \circ x \circ S] \subseteq (S \circ A \circ S \circ A \circ S] \\ &= ((S \circ A) \circ S \circ (A \circ S)] \subseteq (A \circ S \circ A] \subseteq (A \circ A] = (A^2],\end{aligned}$$

which means that $A \subseteq (A^2]$. This completes the proof.

The reader is referred to [4, 28] for notation and terminology not defined in this paper.

3. Prime hyperideals of ordered semihypergroups

In this section we introduce and characterize the prime, weakly prime and semiprime hyperideals in ordered semihypergroups. Some properties of them are investigated.

Definition 3.1. Let T be a nonempty subset of an ordered semihypergroup S. Then T is called *prime* if for all nonempty subsets A, B of S such that $A \circ B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$. Equivalently, if for any element a, b of S such that $a \circ b \subseteq T$, we have $a \in T$ or $b \in T$.

Definition 3.2. Let T be a nonempty subset of an ordered semihypergroup S. Then T is called *weakly prime* if for all hyperideals A, B of S such that $A \circ B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$.

Definition 3.3. Let T be a nonempty subset of an ordered semihypergroup S. Then T is called *semiprime* if for any nonempty subset A of S such that $A \circ A \subseteq T$, we have $A \subseteq T$. Equivalently, if for any element a of S such that $a \circ a \subseteq T$, we have $a \in T$.

One can easily observe that the prime subsets of an ordered semihypergroup are weakly prime and semiprime. However, the converse is not true, in general, as shown in the following example.

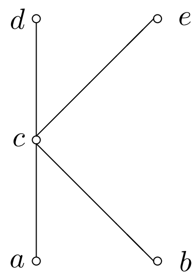
Example 3.4. We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	$\{e\}$
d	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{d\}$	$\{e\}$
e	$\{a,b\}$	$\{a,b\}$	$\{c\}$	$\{c\}$	$\{e\}$

$$\leq := \{(a,a),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),\\ (c,d),(c,e),(d,d),(e,e)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a,c),(b,c),(c,d),(c,e)\}.$$



Then $(S,\circ,\leq)$ is an ordered semihypergroup ([26]), and the sets $\{a,b\}$, $\{a,b,c,e\}$ and S are all hyperideals of S. We can easily verify that the nonempty subset $\{a,b,c\}$ of S is weakly prime and semiprime, but it is not prime. In fact, since $e \circ d = \{c\} \subseteq \{a,b,c\}$, but $e \notin \{a,b,c\}$ and $d \notin \{a,b,c\}$.

In the following we shall characterize the prime, weakly prime and semiprime hyperideals of ordered semihypergroups.

Theorem 3.5. *Let S be an ordered semihypergroup and T a hyperideal of S. Then the following statements are equivalent:*

(1) *T is prime.*

(2) *If A is a left hyperideal, B a right hyperideal of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

(3) *If $a,b \in S$ such that $L(a) \circ R(b) \subseteq T$, then $a \in T$ or $b \in T$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1). Let $a,b \in S$ be such that $a \circ b \subseteq T$. Then, since T is a hyperideal of S, we have $L(a) \circ R(b) = (a \cup S \circ a] \circ (b \cup b \circ S] \subseteq (a \circ b \cup S \circ a \circ b \cup a \circ b \circ S \cup S \circ a \circ b \circ S] \subseteq (T \cup S \circ T \cup T \circ S \cup S \circ T \circ S] \subseteq (T] = T$. Thus, by hypothesis, $a \in T$ or $b \in T$. Hence T is prime.

Theorem 3.6. *Let S be an ordered semihypergroup and T a hyperideal of S. Then the following statements are equivalent:*

(1) *T is weakly prime.*

(2) *If $a,b \in S$ such that $(a \circ S \circ b] \subseteq T$, then $a \in T$ or $b \in T$.*

(3) *If $a,b \in S$ such that $I(a) \circ I(b) \subseteq T$, then $a \in T$ or $b \in T$.*

(4) *If A,B are two right hyperideals of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

(5) *If A,B are two left hyperideals of S such that $A \circ B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

(6) *If A is a right hyperideal, B a left hyperideal of S such that $A\circ B\subseteq T$, then $A\subseteq T$ or $B\subseteq T$.*

Proof. (1) $\Rightarrow$ (2). Let $a,b\in S$ be such that $(a\circ S\circ b]\subseteq T$. Then, by Lemma 2.6, we have

$$\begin{aligned}(S\circ a\circ S]\circ(S\circ b\circ S] &= (S\circ a\circ S\circ S\circ b\circ S]\subseteq(S\circ(a\circ S\circ b)\circ S]\\ &\subseteq (S\circ T\circ S]\subseteq(T]=T.\end{aligned}$$

Since $(S\circ a\circ S],(S\circ b\circ S]$ are hyperideals of S and T is weakly prime, we have $(S\circ a\circ S]\subseteq T$ or $(S\circ b\circ S]\subseteq T$. Say $(S\circ a\circ S]\subseteq T$, then, by Lemma 2.6 and the proof of Lemma 2.8, we have

$$\begin{aligned}((I(a))^2]\circ I(a) &= ((I(a))^2]\circ(I(a)]\subseteq((I(a))^3]\\ &\subseteq ((S\circ a\circ S]]=(S\circ a\circ S]\subseteq(T]=T.\end{aligned}$$

Since T is weakly prime and $((I(a))^2]$ is a hyperideal of S, we have $((I(a))^2]\subseteq T$ or $I(a)\subseteq T$. If $I(a)\subseteq T$, then $a\in I(a)\subseteq T$. Let $((I(a))^2]\subseteq T$. Then, by Lemma 2.6(1), $(I(a))^2\subseteq T$. Since T is weakly prime, we have $I(a)\subseteq T$ and $a\in T$. Similarly, say $(S\circ b\circ S]\subseteq T$, we have $b\in T$.

(2) $\Rightarrow$ (3). Let $a,b\in S$ be such that $I(a)\circ I(b)\subseteq T$. Then, by Lemma 2.6, we have

$$(a\circ S\circ b]\subseteq((a]\circ(S\circ b]]\subseteq(I(a)\circ I(b)]\subseteq(T]=T.$$

By (2), we have $a\in T$ or $b\in T$.

(3) $\Rightarrow$ (4). Suppose that A,B are right hyperideals of S, $A\circ B\subseteq T$ and $A\nsubseteq T$. Then we prove that $B\subseteq T$. In fact, let $a\in A, a\notin T$ and $b\in B$. Then, we have:

$$I(a)=(a\cup S\circ a\cup a\circ S\cup S\circ a\circ S]\subseteq(A\cup S\circ A\cup A\circ S\cup S\circ A\circ S]=(A\cup S\circ A],$$

$$I(b)=(b\cup S\circ b\cup b\circ S\cup S\circ b\circ S]\subseteq(B\cup S\circ B\cup B\circ S\cup S\circ B\circ S]=(B\cup S\circ B].$$

Thus we have

$$\begin{aligned}I(a)\circ I(b) &\subseteq (A\cup S\circ A]\circ(B\cup S\circ B]\subseteq((A\cup S\circ A)\circ(B\cup S\circ B)]\\ &\subseteq (A\circ B\cup S\circ A\circ B\cup A\circ S\circ B\cup S\circ A\circ S\circ B]\\ &= (A\circ B\cup S\circ A\circ B]\subseteq(T\cup S\circ T]=(T]=T.\end{aligned}$$

Since $a\notin T$, by (3), we have $b\in T$. Hence $B\subseteq T$.

(3) $\Rightarrow$ (5). Similar to the proof of (3) $\Rightarrow$ (4), we omit it.

(3) $\Rightarrow$ (6). Let A be a right hyperideal, B a left hyperideal of S such that $A\circ B\subseteq T$ and $A\nsubseteq T$. Then we wish to show that $B\subseteq T$. To do this, let $a\in A, a\notin T$ and $b\in B$. Since $I(a)\subseteq(A\cup S\circ A], I(b)\subseteq(B\cup B\circ S]$, we have

$$\begin{aligned}I(a)\circ I(b) &\subseteq (A\cup S\circ A]\circ(B\cup B\circ S]\subseteq((A\cup S\circ A)\circ(B\cup B\circ S)]\\ &\subseteq (A\circ B\cup S\circ A\circ B\cup A\circ B\circ S\cup S\circ A\circ B\circ S]\\ &\subseteq (T\cup S\circ T\cup T\circ S\cup S\circ T\circ S]=(T]=T.\end{aligned}$$

By (3), $a \in T$ or $b \in T$. Since $a \notin T$, we have $b \in T$. Thus $B \subseteq T$.

(4) $\Rightarrow$ (1), (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1) are clear. This completes the proof.

Theorem 3.7. *Let S be an ordered semihypergroup and T a hyperideal of S. Then T is weakly prime if and only if for all hyperideals A, B of S such that $(A \circ B] \cap (B \circ A] \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$.*

Proof. Let T be weakly prime, A, B hyperideals of S and $(A \circ B] \cap (B \circ A] \subseteq T$. By Lemma 2.6(6), $(A \circ B]$ and $(B \circ A]$ are hyperideals of S, and we have

$$(A \circ B] \circ (B \circ A] \subseteq (A \circ B] \cap (B \circ A] \subseteq T,$$

Since T is weakly prime, we have $(A \circ B] \subseteq T$ or $(B \circ A] \subseteq T$. Say $(A \circ B] \subseteq T$, then by Lemma 2.6(1) we have $A \circ B \subseteq T$, and we deduce that $A \subseteq T$ or $B \subseteq T$. Similarly, say $(B \circ A] \subseteq T$, we have $B \subseteq T$ or $A \subseteq T$.

Conversely, let A, B be hyperideals of S such that $A \circ B \subseteq T$. Then $(A \circ B] \cap (B \circ A] \subseteq (A \circ B] \subseteq (T] = T$. Thus, by hypothesis, we have $A \subseteq T$ or $B \subseteq T$. Therefore, T is weakly prime.

Theorem 3.8. *Let S be an ordered semihypergroup. Then the hyperideals of S are weakly prime if and only if they form a chain under inclusion and one of the five equivalent conditions of Lemma 2.8 holds in S.*

Proof. Suppose that the hyperideals of S are weakly prime. Let A, B be hyperideals of S. Then, by hypothesis and Lemma 2.6(6), $(A \circ B]$ is a weakly prime hyperideal of S. Since $A \circ B \subseteq (A \circ B]$, we have

$$A \subseteq (A \circ B] \subseteq (S \circ B] \subseteq (B] = B$$

or

$$B \subseteq (A \circ B] \subseteq (A \circ S] \subseteq (A] = A.$$

Also, since $A^2 \subseteq (A^2]$ and $(A^2]$ is a hyperideal of S, we have $A \subseteq (A^2]$. On the other hand, by Lemma 2.6, $(A^2] = (A \circ A] \subseteq (A \circ S] \subseteq (A] = A$. Thus $(A^2] = A$.

Conversely, let A, B, T be hyperideals of S such that $A \circ B \subseteq T$. By hypothesis, we have $A \subseteq B$ or $B \subseteq A$. Say $A \subseteq B$, then, by Lemma 2.8, $A = A \cap B = (A \circ B] \subseteq (T] = T$. Similarly, say $B \subseteq A$, we have $B \subseteq T$. Therefore, T is weakly prime.

Definition 3.9. Let $(S, \circ, \leq)$ be an ordered semihypergroup. S is called *intra-regular* if, for each element a of S, there exist $x, y \in S$ such that $a \preceq x \circ a \circ a \circ y$. Equivalently, $a \in (S \circ a \circ a \circ S], \forall a \in S$.

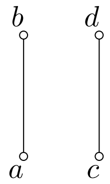
Example 3.10. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{a\}$	$\{a, b\}$	$\{a, c\}$	S
b	$\{b\}$	$\{b\}$	$\{b, d\}$	$\{b, d\}$
c	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, b), (c, d)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. Moreover, S is intra-regular. Indeed, for any $x \in S$, we have $x \in x^2 = x^4 \subseteq S \circ x \circ x \circ S$, which implies that $x \in (S \circ x \circ x \circ S]$.

In order to characterize the prime hyperideals of an ordered semihypergroup, we need the following lemmas.

Lemma 3.11. *If the hyperideals of an ordered semihypergroup S are semiprime, then the following statements hold:*

(1) *$I(x) = (S \circ x \circ S]$ for any $x \in S$.*

(2) *For any $x, y \in S$, $I(x) \cap I(y) = I(z)$ for some $z \in x \circ y$.*

Proof. (1) Let $x \in S$. Since $x^2 \circ x^2 = x^2 \circ x \circ x \subseteq (S \circ x \circ S]$ and $(S \circ x \circ S]$ is a hyperideal of S, by hypothesis, we have $x^2 \subseteq (S \circ x \circ S]$ and $x \in (S \circ x \circ S]$. Hence $I(x) \subseteq (S \circ x \circ S]$. The reverse inclusion is immediate. Thus $I(x) = (S \circ x \circ S]$.

(2) Let $x, y, z \in S$ such that $z \in x \circ y$. Then, we have

$$z \in x \circ y \subseteq I(x) \circ S \subseteq I(x)$$

and

$$z \in x \circ y \subseteq S \circ I(y) \subseteq I(y),$$

from which we can conclude that $I(z) \subseteq I(x), I(z) \subseteq I(y)$, and we have $I(z) \subseteq I(x) \cap I(y)$. To prove the inverse inclusion, let $t \in I(x) \cap I(y)$. By (1), $t \in (S \circ x \circ S]$ and $t \in (S \circ y \circ S]$. Then there exist $a, b, c, d \in S$ such that $t \preceq a \circ x \circ b$ and $t \preceq c \circ y \circ d$. Thus, by Lemma 2.6(9), $t \circ t \preceq c \circ y \circ d \circ a \circ x \circ b$. On the other hand, $y \circ d \circ a \circ x \subseteq (S \circ (x \circ y) \circ S]$. Indeed, by Lemma 2.6(1) we have

$$(y \circ d \circ a \circ x)^2 = y \circ d \circ a \circ x \circ y \circ d \circ a \circ x \subseteq S \circ (x \circ y) \circ S \subseteq (S \circ (x \circ y) \circ S].$$

Since the hyperideal $(S \circ (x \circ y) \circ S]$ is semiprime, we have $y \circ d \circ a \circ x \subseteq (S \circ (x \circ y) \circ S]$. Furthermore, since $(S \circ (x \circ y) \circ S]$ is a hyperideal of S, we have $c \circ y \circ d \circ a \circ x \circ b \subseteq (S \circ (x \circ y) \circ S]$. Thus, by Lemma 2.6(8), we have $t \circ t \subseteq (S \circ (x \circ y) \circ S]$. Also, since $(S \circ (x \circ y) \circ S]$ is semiprime, we have $t \in (S \circ (x \circ y) \circ S]$, and there exists $z \in x \circ y$ such that $t \in (S \circ z \circ S] = I(z)$. It implies that $I(x) \cap I(y) \subseteq I(z)$. Therefore, $I(x) \cap I(y) = I(z)$ for some $z \in x \circ y$.

Lemma 3.12. *Let S be an ordered semihypergroup. Then S is intra-regular if and only if the hyperideals of S are semiprime.*

Proof. Suppose that S is intra-regular. Let T be a hyperideal of S and $a \in S$ such that $a \circ a \subseteq T$. Then, since S is intra-regular, we have

$$a \in (S \circ a \circ a \circ S] \subseteq (S \circ T \circ S] \subseteq (T] = T,$$

which means that T is semiprime.

Conversely, assume that the hyperideals of S are semiprime. Let $a \in S$. We denote by $I(a^2)$ the hyperideal of S generated by a^2. Since $a^2 \subseteq I(a^2)$, by hypothesis we have

$$a \in I(a^2) = (a^2 \cup S \circ a^2 \cup a^2 \circ S \cup S \circ a^2 \circ S].$$

Then $a \leq t$ for some $t \in a^2 \cup S \circ a^2 \cup a^2 \circ S \cup S \circ a^2 \circ S$. If $t \in a^2$, then $a \in (a^2] \subseteq ((a^2] \circ (a^2]] = (a^4] \subseteq (S \circ a^2 \circ S]$. If $t \in S \circ a^2$, then $a \in (S \circ a^2] \subseteq (S \circ (S \circ a^2] \circ a] \subseteq ((S] \circ (S \circ a^2] \circ (a]] = (S \circ (S \circ a^2) \circ a] \subseteq (S \circ a^2 \circ S]$. If $t \in a^2 \circ S$, then $a \in (a^2 \circ S] \subseteq (a \circ (a^2 \circ S] \circ S] \subseteq ((a] \circ (a^2 \circ S] \circ (S]] = (a \circ (a^2 \circ S) \circ S] \subseteq (S \circ a^2 \circ S]$. If $t \in S \circ a^2 \circ S$, then $a \in (S \circ a^2 \circ S]$. Therefore, S is intra-regular.

Theorem 3.13. *Let S be an ordered semihypergroup. Then the hyperideals of S are prime if and only if they form a chain under inclusion and S is intra-regular.*

Proof. Let all hyperideals of S be prime. Then they are weakly prime. By Theorem 3.8, they form a chain with respect to the inclusion relation. By hypothesis, the hyperideals of S are also semiprime. Thus S is intra-regular by Lemma 3.12.

Conversely, suppose that S is an intra-regular ordered semihypergroup and the hyperideals of S form a chain. We prove that the hyperideals of S are prime. In fact, let T be a hyperideal of S and $a, b \in S$ such that $a \circ b \subseteq T$. By Lemma 3.12, the hyperideals of S are semiprime. Then, by Lemma 3.11(2), there exists $c \in a \circ b \subseteq T$ such that $I(a) \cap I(b) = I(c)$. By hypothesis, we have $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then $a \in I(a) = I(a) \cap I(b) = I(c) \subseteq T$, i.e., $a \in T$. If $I(b) \subseteq I(a)$, then $b \in I(b) = I(a) \cap I(b) = I(c) \subseteq T$, i.e., $b \in T$. We have thus shown that T is prime.

In the following we shall investigate the relationships among the prime hyperideals, weakly prime hyperideals and semiprime hyperideals in ordered semihypergroups.

Theorem 3.14. *Let S be an ordered semihypergroup and T a hyperideal of S. Then T is prime if and only if T is weakly prime and semiprime. In a commutative ordered semihypergroup the prime and weakly prime hyperideals coincide.*

Proof. Let S be an ordered semihypergroup and T a prime hyperideal of S. Clearly T is weakly prime and semiprime. Conversely, assume that T is weakly prime and semiprime and let $a, b \in S$ such that $a \circ b \subseteq T$. Then, we have

$$\begin{aligned}(b \circ S \circ a] \circ (b \circ S \circ a] &\subseteq (b \circ S \circ a \circ b \circ S \circ a] \subseteq (S \circ (a \circ b) \circ S] \\ &\subseteq (S \circ T \circ S] \subseteq (T] = T.\end{aligned}$$

Since T is semiprime, we have $(b \circ S \circ a] \subseteq T$. Thus, by Lemma 2.6, we have

$$\begin{aligned}(S \circ b \circ S] \circ (S \circ a \circ S] &\subseteq (S \circ b \circ S \circ S \circ a \circ S] \subseteq (S \circ (b \circ S \circ a) \circ S] \\ &\subseteq (S \circ (b \circ S \circ a] \circ S] \subseteq (S \circ T \circ S] \subseteq (T] = T.\end{aligned}$$

Since $(S \circ b \circ S], (S \circ a \circ S]$ are hyperideals of S, and T is weakly prime, we have $(S \circ b \circ S] \subseteq T$ or $(S \circ a \circ S] \subseteq T$. Similar to the proof of (1)$\Longrightarrow$(2) in Theorem 3.6, we have $a \in T$ or $b \in T$. Therefore, T is prime.

In particular, let S be a commutative ordered semihypergroup. Then every weakly prime hyperideal of S is prime. Indeed, let T be a weakly prime hyperideal of S and $a, b \in S$ such that $a \circ b \subseteq T$. Then, we have

$$\begin{aligned}I(a) \circ I(b) &\subseteq (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S] \circ (b \cup S \circ b \cup b \circ S \cup S \circ b \circ S] \\ &\subseteq ((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S) \circ (b \cup S \circ b \cup b \circ S \cup S \circ b \circ S)] \\ &= (a \circ b \cup S \circ a \circ b] \subseteq (T \cup S \circ T] = (T] = T.\end{aligned}$$

Since T is weakly prime, by Theorem 3.6 we have $a \in T$ or $b \in T$. Thus T is prime.

Theorem 3.15. *Let S be an ordered semihypergroup and $\{T_i \mid i \in I\}$ a family of prime hyperideals of S. Then $\bigcap_{i \in I} T_i$ is a semiprime hyperideal of S if $\bigcap_{i \in I} T_i \neq \emptyset$.*

Proof. Let T_i be a prime hyperideal of S for any $i \in I$. Assume that $\bigcap_{i \in I} T_i \neq \emptyset$. Then, by Lemma 2.7, $\bigcap_{i \in I} T_i$ is a hyperideal of S. Moreover, we can show that $\bigcap_{i \in I} T_i$ is semiprime. In fact, let $a \in S$ be such that $a \circ a \subseteq \bigcap_{i \in I} T_i$. Then $a \circ a \subseteq T_i$ for every $i \in I$. Hence, by hypothesis, $a \in T_i$ for every $i \in I$. It thus follows that $a \in \bigcap_{i \in I} T_i$. Therefore, $\bigcap_{i \in I} T_i$ is a semiprime hyperideal of S.

In the above theorem we have shown that every nonempty intersection of prime hyperideals of an ordered semihypergroup S is semiprime. But the nonempty intersection of prime hyperideals of S is not necessarily a prime hyperideal of S. We can illustrate it by the following example.

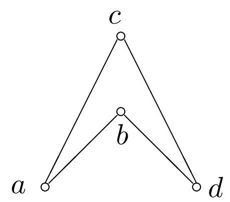
Example 3.16. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$
d	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$

$$\leq := \{(a,a),(a,b),(a,c),(b,b),(c,c),(d,b),(d,c),(d,d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a,b),(a,c),(d,b),(d,c)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup ([25]). We can easily verify that $T_1 = \{a, b, d\}$, $T_2 = \{a, c, d\}$ are prime hyperideals of S. But $T_1 \cap T_2 = \{a, d\}$ is not a prime hyperideal of S. In fact, since $b \circ c = \{a, d\} \subseteq \{a, d\}$, but $b \notin \{a, d\}$ and $c \notin \{a, d\}$.

Let S be an ordered semihypergroup. A hyperideal A of S is called *proper* if $A \neq S$. A proper hyperideal T of S is called *maximal* if A is a hyperideal of S such that $T \subset A$, we have $A = S$. Equivalently, if for any proper hyperideal A of S such that $T \subseteq A$, we have $A = T$.

Theorem 3.17. *If S is an ordered semihypergroup satisfying $S = (S^2]$, then every maximal hyperideal of S is weakly prime.*

Proof. Let M be a maximal hyperideal of S and A, B hyperideals of S such that $A \circ B \subseteq M$. Then $A \subseteq M$ or $B \subseteq M$. Indeed, suppose that $A \nsubseteq M$ and $B \nsubseteq M$. Then $M \subset M \cup A, M \subset M \cup B$. By Lemma 2.7, $M \cup A$ and $M \cup B$ are two hyperideals of S. Since M is maximal, we have $M \cup A = S$ and $M \cup B = S$. Then

$$\begin{aligned} S &= (S^2] = ((M \cup A) \circ (M \cup B)] \\ &= (M \circ M \cup M \circ B \cup A \circ M \cup A \circ B] \subseteq (M] = M, \end{aligned}$$

from which we deduce that $M = S$. It contradicts the fact that M is maximal. Thus M is weakly prime.

Corollary 3.18. *If S is an intra-regular ordered semihypergroup, then every maximal hyperideal of S is weakly prime.*

Proof. Suppose that S is an intra-regular ordered semihypergroup and $a \in S$. Then, we have $a \in (S \circ a \circ a \circ S] \subseteq (S^2]$. It implies that $S \subseteq (S^2]$, and thus $S = (S^2]$. Consequently, by Theorem 3.17, every maximal hyperideal of S is weakly prime.

Corollary 3.19. *If S is an ordered semihypergroup with an identity, then every maximal hyperideal of S is weakly prime.*

Proof. Let S be an ordered semihypergroup containing an identity e. Then, we have $S = e \circ S \subseteq S \circ S = S^2 \subseteq (S^2] \subseteq S$, which implies that $S = (S^2]$. Therefore, every maximal hyperideal of S is weakly prime by Theorem 3.17.

The following is an immediate corollary of Theorem 3.14 and Corollary 3.19.

Corollary 3.20. *If S is a commutative ordered semihypergroup with an identity, then every maximal hyperideal of S is prime.*

4. Hyperideal extensions of ordered semihypergroups

In the current section we consider the extensions of hyperideals in commutative ordered semihypergroups. Moreover, we define n-prime hyperideals and n-semiprime hyperideals of ordered semihypergroups, and investigate the relationship between extensions of hyperideals and n-prime hyperideals.

Definition 4.1. Let I be a hyperideal of S, $x \in I$. The set

$$< x, I >:= \{a \in S \mid x \circ a \subseteq I\}$$

is called the *extension* of I by x.

Proposition 4.2. *Let I be a hyperideal of a commutative ordered semihypergroup S, $x \in S$. Then the following statements hold:*

(1) *$< x, I >$ is a hyperideal of S.*
(2) *$I \subseteq < x, I > \subseteq < y, I >$, for any $y \in x^2$.*
(3) *If $x \in I$, then $< x, I >= S$.*
(4) *I is prime if and only if $< x, I >= I$ for any $x \in S \backslash I$.*

Proof. (1) Let $a \in < x, I >, S \ni b \leq a$. Then $b \in < x, I >$. In fact, since $x \circ a \subseteq I, S \supseteq x \circ b \preceq x \circ a$, by Lemma 2.6(8) we have $x \circ b \subseteq I$, i.e., $b \in < x, I >$. Furthermore, let $a \in < x, I >$ and $b \in S$. Then $x \circ a \subseteq I$, and, for any $c \in a \circ b$, we have

$$x \circ c \subseteq x \circ (a \circ b) = (x \circ a) \circ b \subseteq I \circ S \subseteq I.$$

It implies that $c \in < x, I >$, and we have $a \circ b \subseteq < x, I >$. Hence $< x, I >$ is a right hyperideal of S. Since S is commutative, we obtain the requested result.

(2) If $a \in I$, then $x \circ a \subseteq S \circ I \subseteq I$, i.e., $a \in < x, I >$. It implies that $I \subseteq < x, I >$. Furthermore, let $a \in < x, I >$. Then $x \circ a \subseteq I$, and, for any $y \in x^2$, we have

$$y \circ a \subseteq x^2 \circ a = x \circ (x \circ a) \subseteq S \circ I \subseteq I.$$

Thus it can be shown that $< x, I > \subseteq < y, I >$ for any $y \in x^2$.

(3) Let $a \in S, x \in I$. Then $x \circ a \subseteq I \circ S \subseteq I$, and we have $a \in < x, I >$. It thus follows that $< x, I > = S$.

(4) Assume that I is a prime hyperideal of S. Let $x \in S \backslash I$ and $a \in < x, I >$. Then $x \circ a \subseteq I$, and, by hypothesis, we have $a \in I$. It implies that $< x, I > \subseteq I$. By (2), the inverse inclusion holds. Hence $< x, I > = I$ for any $x \in S \backslash I$.

Conversely, let $x, y \in S$ be such that $x \circ y \subseteq I$. Then $y \in < x, I >$. We claim that $x \in I$ or $y \in I$. If $x \notin I$, then, by hypothesis, we have $y \in < x, I > = I$. Thus I is prime.

Proposition 4.3. *Let I be a hyperideal of a commutative ordered semihypergroup S and $\{I_\alpha \mid \alpha \in \mathcal{A}\}$ a family of prime hyperideal of S. If $I = \bigcap_{\alpha \in \mathcal{A}} I_\alpha$, then, for any $x \in S$, $< x, I >$ is a semiprime hyperideal of S whenever $I \neq \emptyset$.*

Proof. Let $x \in S$. We first show that

$$< x, I > = < x, \bigcap_{\alpha \in \mathcal{A}} I_\alpha > = \bigcap_{\alpha \in \mathcal{A}} < x, I_\alpha > .$$

In fact,

$$\begin{aligned} a \in < x, \bigcap_{\alpha \in \mathcal{A}} I_\alpha > &\iff x \circ a \subseteq \bigcap_{\alpha \in \mathcal{A}} I_\alpha \\ &\iff x \circ a \subseteq I_\alpha, \forall \alpha \in \mathcal{A} \\ &\iff a \in < x, I_\alpha >, \forall \alpha \in \mathcal{A} \\ &\iff a \in \bigcap_{\alpha \in \mathcal{A}} < x, I_\alpha > . \end{aligned}$$

Now we consider the following cases:

Case 1. If $x \in I_\alpha$ for any $\alpha \in \mathcal{A}$, then, by Proposition 4.2(3), we have $< x, I_\alpha > = S$. Then $< x, I > = \bigcap_{\alpha \in \mathcal{A}} < x, I_\alpha > = S$, and $< x, I >$ is a semiprime hyperideal of S.

Case 2. Let $x \in S \backslash I_\alpha$ for some $\alpha \in \mathcal{A}$. Then, by Proposition 4.2(4), $< x, I_\alpha > = I_\alpha$. Let $\mathcal{B} := \{\alpha \in \mathcal{A} \mid x \notin I_\alpha\}$. Then $\mathcal{B} \neq \emptyset$ and we have

$$< x, I > = \bigcap_{\alpha \in \mathcal{B}} I_\alpha.$$

It thus follows from Theorem 3.15 that $< x, I >$ is semiprime.

Proposition 4.4. *Let S be a commutative ordered semihypergroup containing an identity e, and $x, y \in S$. Then $I(x) \subseteq I(y)$ if and only if, for any hyperideal J of S, we have $< x, J > \supseteq < y, J >$.*

Proof. Let J be a hyperideal of S and $a \in < y, J >$. Then $y \circ a \subseteq J$. Since S is a commutative ordered semihypergroup containing an identity e and $I(x) \subseteq I(y)$, we have $x \in I(y) = (y \cup S \circ y] \subseteq (e \circ y \cup S \circ y] = (S \circ y]$, that is, $x \preceq z \circ y$ for some $z \in S$. Then, we have

$$x \circ a \preceq (z \circ y) \circ a = z \circ (y \circ a) \subseteq S \circ J \subseteq J.$$

It thus follows from Lemma 2.6(9) that $x \circ a \subseteq J$. Hence $a \in < x, J >$.

Conversely, since $I(y)$ is a hyperideal of S, by hypothesis, we have $< x, I(y) > \supseteq < y, I(y) >$. Since $y \in I(y)$, by Proposition 4.2(3), $< y, I(y) >= S$. Then we can deduce that $< x, I(y) >= S$. Hence $e \in < x, I(y) >$, and $x \in x \circ e \subseteq I(y)$. Thus it can be obtained that $I(x) \subseteq I(y)$.

In order to characterize the hyperideal extensions of ordered semihypergroups, we need introduce the concept of n-prime hyperideals of ordered semihypergroups.

Let n be any positive integer such that $n \geq 2$. For any $x_i \in S$ $(i = 1, 2, \cdots, n)$, and j being a positive integer such that $2 \leq j \leq n-1$, we define

$$\begin{aligned} I_{1,n} &:= x_2 \circ x_3 \circ \cdots \circ x_{n-1} \circ x_n, \\ I_{j,n} &:= x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n, \\ I_{n,n} &:= x_1 \circ x_2 \circ \cdots \circ x_{n-2} \circ x_{n-1}. \end{aligned}$$

Definition 4.5. Let I be a hyperideal of an ordered semihypergroup S. I is called *n-prime* if for any $x_i \in S$ $(i = 1, 2, \cdots, n)$, $x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ x_n \subseteq I$ implies there exists a positive integer i $(1 \leq i \leq n)$ such that

$$I_{1,n}, I_{2,n}, \cdots, I_{i-1,n}, I_{i+1,n}, \cdots, I_{n,n} \subseteq I.$$

Definition 4.6. Let I be a hyperideal of an ordered semihypergroup S. I is called *n-semiprime* if for any $x_1, x_2, \cdots, x_n \in S$ with $x_1 = x_2 = \cdots = x_n$, $x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ x_n \subseteq I$ implies $I_{n,n} \subseteq I$.

Theorem 4.7. *Let S be an ordered semihypergroup. Then the following statements are true:*

(1) *Every n-prime hyperideal of S is n-semiprime.*
(2) *The prime hyperideals and 2-prime hyperideals of S coincide.*
(3) *The semiprime hyperideals and 2-semiprime hyperideals of S coincide.*

Proof. Straightforward.

Theorem 4.8. *Let S be an ordered semihypergroup. Then every $(n-1)$-prime hyperideal of S is an n-prime hyperideal of S for all positive integers $n \geq 3$.*

Proof. Suppose that I is an $(n-1)$-prime hyperideal of S. Let $x_1, x_2, \cdots, x_n \in S$ be such that $x_1 \circ x_2 \circ x_3 \circ x_4 \circ \cdots \circ x_{n-3} \circ x_{n-2} \circ x_{n-1} \circ x_n \subseteq I$. Then, for any $z \in x_{n-1} \circ x_n$, $x_1 \circ x_2 \circ x_3 \circ x_4 \circ \cdots \circ x_{n-3} \circ x_{n-2} \circ z \subseteq I$. We define:

$$\begin{aligned}
J_{1,n-1} &:= x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ z, \\
J_{2,n-1} &:= x_1 \circ x_3 \circ \cdots \circ x_{n-2} \circ z, \\
&\vdots \\
J_{n-2,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ z, \\
J_{n-1,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ x_{n-2}.
\end{aligned}$$

By hypothesis, there exists a positive integer i $(1 \leq i \leq n-1)$ such that

$$J_{1,n-1}, J_{2,n-1}, \cdots, J_{i-1,n-1}, J_{i+1,n-1}, \cdots, J_{n-1,n-1} \subseteq I.$$

We consider the following two cases:

Case 1. Let $J_{n-1,n-1} \nsubseteq I$. Then $J_{1,n-1}, J_{2,n-1}, \cdots, J_{n-2,n-1} \subseteq I$. Thus, by the arbitrariness of z, $I_{1,n}, I_{2,n}, \cdots, I_{n-2,n} \subseteq I$. By $I_{1,n} \subseteq I$, we have $x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1} \circ x_n \subseteq I$. We define

$$\begin{aligned}
K_{1,n-1} &:= x_3 \circ x_4 \circ \cdots \circ x_{n-1} \circ x_n, \\
K_{2,n-1} &:= x_2 \circ x_4 \circ \cdots \circ x_{n-1} \circ x_n, \\
&\vdots \\
K_{n-1,n-1} &:= x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1}.
\end{aligned}$$

By hypothesis, there exists a positive integer j $(1 \leq j \leq n-1)$ such that

$$K_{1,n-1}, K_{2,n-1}, \cdots, K_{j-1,n-1}, K_{j+1,n-1}, \cdots, K_{n-1,n-1} \subseteq I.$$

Then, we have

$$K_{n-2,n-1} = x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_n \subseteq I$$

or

$$K_{n-1,n-1} = x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1} \subseteq I.$$

Thus, since I is a hyperideal of S, we have

$$I_{n-1,n} = x_1 \circ K_{n-2,n-1} \subseteq I$$

or

$$I_{n,n} = x_1 \circ K_{n-1,n-1} \subseteq I.$$

Hence $I_{1,n}, I_{2,n}, \cdots, I_{n-2,n}, I_{n-1,n} \subseteq I$ or $I_{1,n}, I_{2,n}, \cdots, I_{n-2,n}, I_{n,n} \subseteq I$.

Case 2. Let $J_{n-1,n-1} \subseteq I$. Then there exists a positive integer k $(1 \leq k \leq n-2)$ such that

$$J_{1,n-1}, J_{2,n-1}, \cdots, J_{k-1,n-1}, J_{k+1,n-1}, \cdots, J_{n-2,n-1} \subseteq I.$$

Thus, since z is an arbitrary element of $x_{n-1} \circ x_n$, we have

$$I_{1,n}, I_{2,n}, \cdots, I_{k-1,n}, I_{k+1,n}, \cdots, I_{n-2,n} \subseteq I.$$

Since I is a hyperideal of S and $J_{n-1,n-1} \subseteq I$, we have

$$I_{n-1,n} = J_{n-1,n-1} \circ x_n \subseteq I$$

and

$$I_{n,n} = J_{n-1,n-1} \circ x_{n-1} \subseteq I.$$

Hence, in this case, $I_{1,n}, I_{2,n}, \cdots, I_{j-1,n}, I_{j+1,n}, \cdots, I_{n-1,n}, I_{n,n} \subseteq I$ for some positive integer k $(1 \leq k \leq n-2)$.

Therefore, I is an n-prime hyperideal of S.

By the above theorem, we immediately obtain the following corollary:

Corollary 4.9. *Let S be an ordered semihypergroup. Then every prime hyperideal of S is an n-prime hyperideal of S for all positive integers $n \geq 2$.*

The converse of Theorem 4.8 is not true in general. We can illustrate it by the following example:

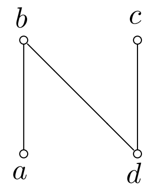
Example 4.10. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{b, d\}$	$\{b, d\}$	$\{d\}$	$\{d\}$
b	$\{b, d\}$	$\{b\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, b), (d, c), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec := \{(a, b), (d, b), (d, c)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. One can easily show that $I = \{d\}$ is a 3-prime hyperideal of S, but it is not 2-prime. Indeed, since $b \circ c = \{d\} \subseteq I$, but $b \notin I$ and $c \notin I$.

Now we give a relationship between extensions of hyperideals and n-prime hyperideals in commutative ordered semihypergroups.

Theorem 4.11. *Let S be a commutative ordered semihypergroup and I a hyperideal of S. Then I is n-prime if and only if any extension of I is $(n-1)$-prime for all positive integers $n \geq 3$.*

Proof. Let I be an n-prime hyperideal of S. For any $x \in S$, let $x_1, x_2, \cdots, x_{n-2}, x_{n-1} \in S$ be such that $x_1 \circ x_2 \circ \cdots \circ x_{n-2} \circ x_{n-1} \subseteq < x, I >$. Then $x \circ x_1 \circ x_2 \circ \cdots \circ x_{n-2} \circ x_{n-1} \subseteq I$. We define

$$\begin{aligned} J_{1,n} &:= x_1 \circ x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1}, \\ J_{2,n} &:= x \circ x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1}, \\ &\vdots \\ J_{n-1,n} &:= x \circ x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ x_{n-1}, \\ J_{n,n} &:= x \circ x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ x_{n-2}. \end{aligned}$$

Since I is an n-prime hyperideal of S, there exists a positive integer i $(1 \leq i \leq n)$ such that

$$J_{1,n}, J_{2,n}, \cdots, J_{i-1,n}, J_{i+1,n}, \cdots, J_{n,n} \subseteq I.$$

Thus, there exists a positive integer j $(2 \leq j \leq n)$ such that

$$J_{2,n}, J_{3,n}, \cdots, J_{j-1,n}, J_{j+1,n}, \cdots, J_{n,n} \subseteq I.$$

It implies that there exists a positive integer $k = j - 1$ $(1 \leq k \leq n-1)$ such that

$$x \circ I_{1,n-1}, x \circ I_{2,n-1}, \cdots, x \circ I_{k-1,n-1}, x \circ I_{k+1,n-1}, \cdots, x \circ I_{n-1,n-1} \subseteq I,$$

where

$$\begin{aligned} I_{1,n-1} &:= x_2 \circ x_3 \circ \cdots \circ x_{n-2} \circ x_{n-1}, \\ I_{l,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{l-1} \circ x_{l+1} \circ \cdots \circ x_{n-2} \circ x_{n-1} \ (2 \leq l \leq n-2), \\ I_{n-1,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{n-3} \circ x_{n-2}. \end{aligned}$$

Hence, it can be easily shown that $I_{1,n-1}, I_{2,n-1}, \cdots, I_{k-1,n-1}, I_{k+1,n-1}, \cdots, I_{n-1,n-1} \subseteq < x, I >$. In other words, $< x, I >$ is indeed an $(n-1)$-prime hyperideal of S.

Conversely, suppose that any extension of I is an $(n-1)$-prime hyperideal of S. Let $x_1, x_2, \cdots, x_n \in S$ be such that $x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ x_n \subseteq I$. Then it can be easily shown that $x_1 \circ x_2 \circ \cdots \circ x_{n-2} \circ x_{n-1} \subseteq < x_n, I >$. Thus, by hypothesis, there exists a positive integer i $(1 \leq i \leq n-1)$ such that

$$(*) \qquad I_{1,n-1}, I_{2,n-1}, \cdots, I_{i-1,n-1}, I_{i+1,n-1}, \cdots, I_{n-1,n-1} \subseteq < x_n, I > .$$

To prove that I is n-prime, we consider the following two cases:

Case 1. If $I_{i,n-1} \subseteq < x_n, I >$, then we have

$$I_{1,n-1}, I_{2,n-1}, \cdots, I_{n-2,n-1}, I_{n-1,n-1} \subseteq < x_n, I > .$$

It thus follows that $I_{1,n}, I_{2,n}, \cdots, I_{n-2,n}, I_{n-1,n} \subseteq I$. Consequently, I is n-prime.

Case 2. Let $I_{i,n-1} \not\subseteq < x_n, I >$. Then, by $(*)$, we have

$$I_{1,n}, I_{2,n}, \cdots, I_{i-1,n}, I_{i+1,n}, \cdots, I_{n-1,n} \subseteq I.$$

We can prove that $I_{n,n} \subseteq I$. In fact, take a positive integer j such that $1 \leq j \leq n-1$ and $j \neq i$. Since $x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ x_n \subseteq I$, we have

$$x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n \subseteq < x_j, I > .$$

Now, we define

$$\begin{aligned}
K_{1,n-1} &:= x_2 \circ x_3 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n, \\
K_{2,n-1} &:= x_1 \circ x_3 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n, \\
&\vdots \\
K_{j-1,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{j-2} \circ x_{j+1} \circ \cdots \circ x_{n-1} \circ x_n, \\
K_{j,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+2} \circ \cdots \circ x_{n-1} \circ x_n, \\
&\vdots \\
K_{n-2,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-2} \circ x_n, \\
K_{n-1,n-1} &:= x_1 \circ x_2 \circ \cdots \circ x_{j-1} \circ x_{j+1} \circ \cdots \circ x_{n-2} \circ x_{n-1}.
\end{aligned}$$

By hypothesis, $< x_j, I >$ is an $(n-1)$-prime hyperideal of S, and there exists a positive integer k $(1 \leq k \leq n-1)$ such that

$$K_{1,n-1}, K_{2,n-1}, \cdots, K_{k-1,n-1}, K_{k+1,n-1}, \cdots, K_{n-1,n-1} \subseteq < x_j, I > .$$

It thus follows that there exists a positive integer l $(1 \leq l \leq n)$ and $l \neq j$ (assume $l < j$) such that

$$I_{1,n}, I_{2,n}, \cdots, I_{l-1,n}, I_{l+1,n}, \cdots, I_{j-1,n}, I_{j+1,n}, \cdots, I_{n,n} \subseteq I.$$

Since $j \neq i$ and $j \neq n$, we have $I_{i,n} \subseteq I$ or $I_{n,n} \subseteq I$. Again since $I_{i,n-1} \not\subseteq < x_n, I >$, we have $I_{i,n} \not\subseteq I$, and we deduce that $I_{n,n} \subseteq I$. Thus, in this case, we have

$$I_{1,n}, I_{2,n}, \cdots, I_{i-1,n}, I_{i+1,n}, \cdots, I_{n-1,n}, I_{n,n} \subseteq I,$$

and I is an n-prime hyperideal of S. This completes the proof.

Theorem 4.12. *Let S be a commutative ordered semihypergroup containing an identity e. Then the n-prime hyperideals and the $(n-1)$-prime hyperideals of S coincide for all positive integers $n \geq 3$.*

Proof. Let I be an n-prime hyperideal of S. By Theorem 4.11, $< e, I >$ is an $(n-1)$-prime hyperideal of S. Let $a \in < e, I >$. Then $a \in e \circ a \subseteq I$. Thus $< e, I > \subseteq I$. By Proposition 4.2(2), $< e, I > = I$. Hence I is $(n-1)$-prime. Consequently, by Theorem 4.8, the proof is completed.

Lemma 4.13. *Let S be a commutative ordered semihypergroup and I a semiprime hyperideal of S. Then $I = \bigcap_{x \in S} < x, I >$.*

Proof. By Proposition 4.2(2), $I \subseteq < x, I >$ for any $x \in S$. Then $I \subseteq \bigcap_{x \in S} < x, I >$. To prove the inverse inclusion, let $a \in \bigcap_{x \in S} < x, I >$. Then $a \in < a, I >$, and we have $a \circ a \subseteq I$. Since I is a semiprime hyperideal of S, we have $a \in I$. Therefore, we obtain the requested result.

Theorem 4.14. *Let I be a semiprime and n-prime hyperideal of a commutative ordered semihypergroup S, $n \geq 3$. Let*

$$\mathcal{P} := \{T \mid T \textit{ is an } (n-1)\textit{-prime hyperideal of } S \textit{ and } I \subseteq T\}.$$

Then $I = \bigcap_{T \in \mathcal{P}} T$.

Proof. Obviously, $I \subseteq \bigcap_{T \in \mathcal{P}} T$. On the other hand, since I is a semiprime hyperideal of S, by Lemma 4.13, $I = \bigcap_{x \in S} < x, I >$. Furthermore, since I is also n-prime, by Proposition 4.2(2) and Theorem 4.11, $< x, I >$ is an $(n-1)$-prime hyperideal of S for any $x \in S$ and $I \subseteq < x, I >$. Thus, for any $x \in S$, $< x, I > \in \mathcal{P}$. Hence $\bigcap_{T \in \mathcal{P}} T \subseteq \bigcap_{x \in S} < x, I > = I$. Therefore, $I = \bigcap_{T \in \mathcal{P}} T$.

The hypothesis that I is semiprime cannot be removed in the above theorem. Otherwise, Theorem 4.14 does not hold in general. We can illustrate it by the following example.

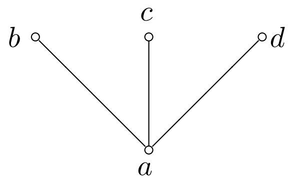
Example 4.15. We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, d\}$

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (c, c), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec := \{(a, b), (a, c), (a, d)\}.$$



Then $(S, \circ, \leq)$ is a commutative ordered semihypergroup and $\{a\}$ is a hyperideal of S. We can easily prove that $\{a\}$ is 3-prime, but it is not semiprime. In addition, with a small amount of effort one can verify that the sets $\{a, b, c\}$ and S are all prime hyperideals of S containing a, while $\{a, b, c\} \cap S = \{a, b, c\} \neq \{a\}$.

Corollary 4.16. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. If $(S, \circ)$ is a hypersemilattice, then every n-prime hyperideal $(n \geq 3)$ of S can be expressed as the intersection of all $(n-1)$-prime hyperideals of S containing it.*

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