A common fixed point theorem without continuity under weak compatible mappings in uniform convex Banach spaces

Deepti Thakur
Mathematics Section
College of Applied Sciences
Sohar
Oman
thakurdeepti@yahoo.com

Abstract. In this paper, we prove a common fixed point theorem for four mappings under the condition of weak compatibility on a closed subset of a uniformly convex Banach space without taking under consideration the continuity of mappings. We provide an example in support of our result.
Keywords: common fixed point, uniform convex Banach space, weakly compatible mappings, closed subset.

1. Introduction

Imdad et al. [3] obtained some results on common fixed points for three mappings defined on a closed subset of a uniformly convex Banach space. Their results extended and refined some results of Husain and Sehgal [2] and Khan and Imdad [10]. Rashwan [11] extended results of Imdad et al. [3] by employing four compatible mappings of type (A) instead of three weakly commuting mappings and by using one continuous mapping as opposed to two. In this paper, we improve the result of Rashwan[11] by removing the condition of continuity and using weak compatible mappings. For preliminaries and definitions we refer to ([1], [4], [5], [6],[7],[8],[9],[12]).

2. Preliminaries

Throughout the paper, X stands for a uniformly convex Banach space. Let R^+ denote the set of all non negative real numbers and F be the family of mappings f from $(R^+)^5$ into R^+ such that f is upper semicontinuous, non-decreasing in each coordinate variable. The modulus of convexity of X is a function δ from $(0,2]$ into $(0,1]$ defined by

$$\delta(\epsilon) = inf\{1 - \frac{1}{2}\|x-y\|, x, y \in X, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\}.$$

Moreover, if X is uniformly convex, then δ is strictly increasing, $\delta(\epsilon) \to 0$ as $\epsilon \to 0, \delta(2) = 1, \eta(t) < 2$ when $t < 1$ and η is the inverse of δ.

For our main theorem we need the following lemma:

Lemma 2.1 ([1]). *Let X be uniformly convex Banach space and B_r, the closed ball X centred at the origin with radius $r > 0$. If $x_1, x_2, x_3 \in B_r$ satisfy*

$$\|x_1 - x_2\| \geq \|x_2 - x_3\| \geq d > 0 \text{ and if } \|x_2\| \geq \left(1 - \frac{1}{2}\delta\left(\frac{d}{l}\right)\right)l,$$

then

$$\|x_1 - x_3\| \leq \eta\left(1 - \frac{1}{2}\delta\left(\frac{d}{l}\right)\right) \|x_1 - x_2\|.$$

Now, we shall give some definitions;

Definition 2.1 ([12]). Let S and T be self commuting on X. Then $\{S, T\}$ is called a weakly commuting pair on X if $\|STx - TSx\| \geq \|Sx - Tx\|$ for all $x \in X$.

Definition 2.2 ([4]). Let $S, T : X \to X$ be mappings. S and T are said to be compatible if $\lim_{n\to\infty} \|STx_n - TSx_n\| = 0$, whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X.$$

Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible. On the other hand, examples are given in [4], [5], [6], and [12] to show neither of the above implications are reversible.

Definition 2.3 ([8]). A pair of mappings S and T is called weakly compatible pair in fuzzy metric space if they commute at coincidence points; i.e. if $Tu = Su$ for some $u \in X$, then $TSu = STu$. It is easy to see that if S and T are compatible , then they are weakly compatible and the converse is not true in general.

Definition 2.4 ([9]). Let $S, T : X \to X$ be mappings. S and T are said to be compatible of type (A) if

$$\lim_{n\to\infty} \|STx_n - SSx_n\| = 0, \lim_{n\to\infty} \|STx_n - TTx_n\| = 0.$$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Imdad et al. [3] proved the following:

Theorem 2.1. *Let X be uniformly convex Banach space and K a non empty closed subset of X. Let A, S and T be three self mappings of K satisfying the following conditions:*

$$S \text{ and} T \text{ are continuous}, AK \subset SK \cap TK, \tag{2.1}$$

$$\{A, S\} \text{ and} \{A, T\} \text{ are weakly commuting pairs on } K, \tag{2.2}$$

there exists a function $f \in F$ *such that for every* $x, y \in K$*:*

$$\|Ax - Ay\| \le f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - Ay\|, \|Ty - Ax\|, \|Ty - Ay\|), \tag{2.3}$$

where f has the additional requirements:

$$\text{for}\, t > 0, f(t, t, 0, \alpha t, t) \le \beta t \;\text{and}\, f(t, t, \alpha t, 0, t) \le \beta t \;\text{being}\; \beta < 1 \;\text{for}\; \alpha < 2 \tag{2.4}$$

and $\beta = 1$ *for* $\alpha = 2, \alpha, \beta \in R^+$,

$$f(t, 0, t, t, 0) < t \;\text{for}\; t > 0; \tag{2.5}$$

Then, there exists a point u in K such that:

(i) *u is the common fixed point of* A, S *and* T.

(ii) *For any* $x_0 \in K$*, the sequence* $\{Ax_n\}$ *defined by*

$$Tx_{2n} = Ax_{2n-1}, Sx_{2n+1} = Ax_{2n}, n = 0, 1, 2, ...,$$

converges strongly to u.

Rashwan [11] proved a following common fixed point theorem for four compatible mappings of type (A) which extends and improves Theorem (2.1).

Theorem 2.2. *Let X and K be as in Theorem* (2.1). *Let A, B, S and T be four self mappings of K satisfying the following conditions:*

$$\text{One of}\; A, B, S \;\text{and}\; T \;\text{are continuous and}\; AK \subseteq TK \;\text{and}\; BK \subseteq SK, \tag{2.6}$$

$$\{A, S\} \;\text{and}\; \{B, T\} \;\text{are compatible of type}\; (A), \tag{2.7}$$

there exists a function $f \in F$ *such that for every* $x, y \in K$*:*

$$\|Ax - By\| \le f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - By\|, \|Ty - Ax\|, \|Ty - By\|), \tag{2.8}$$

where f satisfies the condition (2.4) *and* (2.5) *as in Theorem* (2.1).

Then there exists a point u in K such that:

(i) *u is the common fixed point of* $A, B, S,$ *and* T;

(ii) *For any* $x_0 \in K$*, the sequence* $\{y_n\}$ *defined by*

$$y_{2n} = Sx_{2n} = Bx_{2n-1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n}, n = 0, 1, 2, ...$$

converges strongly to u.

3. Main results

Theorem 3.1. *Let X be uniformly convex Banach space and K a non empty closed subset of X. Let $A, B, S,$ and T be four mappings of K satisfying the following conditions:*

$$AK \subset TK \, and \, BK \subset SK, \tag{3.1}$$

$$\{A, S\} \, and \, \{B, T\} \, are \, weakly \, compatible, \tag{3.2}$$

there exists a function $f \in F$ such that for every $x, y \in K$:

$$\begin{aligned} &\|Ax - By\| \leq f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - By\|, \\ &\|Ty - Ax\|, \|Ty - By\|), \end{aligned} \tag{3.3}$$

where f has the additional requirements:

$$\begin{aligned} &for \, t > 0, f(t, t, 0, \alpha t, t) \leq \beta t \, and \, f(t, t, \alpha t, 0, t) \\ &\leq \beta t \; being \; \beta < 1 \, for \, \alpha < 2 \end{aligned} \tag{3.4}$$

$and \, \beta = 1 \, for \, \alpha = 2, \alpha, \beta \in R^+,$

$$f(t, 0, t, t, 0) < t \, for \, t > 0, \tag{3.5}$$

Then there exists a point z in K such that:
(i) *z is the common fixed point of $A, B, S,$ and T.*
(ii) *For any $x_0 \in K$, the sequence $\{y_n\}$ defined by*

$$y_{2n} = Sx_{2n} = Bx_{2n-1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n}, n = 0, 1, 2, ...,$$

converges strongly to z.

Proof. Let $x_0 \in K$. Since $AK \subset TK$,$BK \subset SK$ we can always define a sequence $\{y_n\}$ as $y_{2n} = Sx_{2n} = Bx_{2n-1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n}, n = 0, 1, 2, ...$ converges strongly to z..

Let $d_n = \|y_n - y_{n+1}\|, n = 0, 1, 2, ...$ $\lim_{n\to\infty} d_n = 0$. Now, for any even integer n, we have

$$\begin{aligned} d_n &= \|y_n - y_{n+1}\| = \|AX_n - BX_{n-1}\| \\ &\leq f\big(\|Sx_n - Tx_{n-1}\|, \|Sx_n - Ax_n\|, \|Sx_n - Bx_{n-1}\|, \\ &\|Tx_{n-1} - Ax_n\|, \|Tx_{n-1} - Bx_{n-1}\|\big), \end{aligned} \tag{3.6}$$

which implies $d_n = f(d_{n-1}, d_n, 0, d_{n-1} + d_n, d_{n-1})$.

Similarly for an odd n, we obtain

$$\begin{aligned} d_n &= \|y_n - y_{n+1}\| = \|AX_n - BX_{n-1}\| \\ &\leq f\Big[\|Sx_n - Tx_{n-1}\|, \|Sx_n - Ax_n\|, \|Sx_n - Bx_{n-1}\|, \\ &\|Tx_{n-1} - Ax_n\|, \|Tx_{n-1} - Bx_{n-1}\|\Big], \end{aligned} \tag{3.7}$$

which gives $d_n = f(d_{n-1}, d_{n-1}, 0, d_{n-1} + d_n, d_n)$. If $d_n > d_{n-1}$ for some $n \geq 1$, then $d_{n-1} + d_n = \alpha d_n$ with $\alpha < 2, \alpha \in R$.

Since f is non decreasing in each coordinate variable.

$$d_n = \begin{cases} f(d_n, d_n, 0, \alpha d_n, d_n), \text{ if } n \text{ is even}, \\ f(d_n, d_n, \alpha d_n, 0, d_n), \text{ if } n \text{ is odd}. \end{cases}$$

In both cases , by (3.4), we get $d_n \leq \beta d_n < d_n$, for some $\beta < 1, \beta \in R^+$, a contradiction. Thus $d_n \leq d_{n-1}$ for $n = 1, 2, 3, ...$ Suppose $d > 0$. Without loss of generality, we can pastulate that the origin of $X \in K$,

$$\lim_{n\to\infty} \sup \|y_n\| = l' > 0.$$

Let $l \in R^+$ be chosen in such a way that $l' < 1$ and $\eta[1 - (\frac{1}{2})\delta(\frac{d}{l})] < l'$, then there exists a sequence $\{n(k)\}, k = 0, 1, 2, ..., n(0) > 1$ of positive integers such that $\|y_{n(k)}\| \geq [(1 - \frac{1}{2})\delta(\frac{d}{l}))]$, where as it is $\|y_n\| \leq l$ for any $n \geq n(0)$. Since $d_{n(k)-1} \geq d_{n(k)} \geq d > 0, k = 0, 1, 2, ...$. From Lemma (2.1) it follows that

$$\|y_{n(k)-1} - y_{n(k)+1}\| \leq \eta(\frac{l'}{l})d_{n(k)-1}. \tag{3.8}$$

where $\eta(\frac{l'}{l}) < 2$ being $(\frac{l'}{l}) < 1$. Then by (3.6),(3.7) and (3.8), we have

$$d_{n(k)} = \begin{cases} f(d_{n(k)-1}, d_{n(k)-1}, 0, \eta(\frac{l'}{l})d_{n(k)-1}, d_{n(k)-1}), \text{ if } n \text{ is even}, \\ f(d_{n(k)-1}, d_{n(k)-1}, 0, \eta(\frac{l'}{l})d_{n(k)-1}, d_{n(k)-1}), \text{ if } n \text{ is odd}, \end{cases}$$

In both cases, (3.4) implies $d_{n(k)} \leq \beta d_{n(k)-1}$, for some $\beta < 1$. Observing that β does not depend on K, the foregoing inequality gives as $n \to \infty$ that $d \leq \beta d < d$, a contradiction. This means that $d = 0$. Now, we'll prove that $\{y_n\}$ is a cauchy sequence. Since $\lim_{n\to\infty} d_n = 0$, it is sufficient to show that the sequence $\{y_{2n}\}$ is a cauchy sequence. If not, then there is an $\epsilon > 0$ such that for every even integer 2k, k=0,1,2,..., there exists two sequences $\{2n(k)\}, \{2m(k)\}$ with $2k \leq 2n(k) \leq 2m(k)$ for which

$$\|y_{n(k)} - y_{m(k)}\| > \epsilon, \tag{3.9}$$

for each even integer 2k, let 2m(k) be the least even integer exceeding n(k) and satisfying (3.9) . Then $\|y_{2n(k)} - y_{2m(k)-2}\| \leq \epsilon$ and $\|y_{2n(k)} - y_{2m(k)}\| > \epsilon$, for each k=0,1,2,..., we have

$$\begin{aligned} \epsilon &\leq \|y_{2n(k)} - y_{2m(k)}\| \\ &\leq \epsilon \|y_{2n(k)} - y_{2m(k)-2}\| + \|y_{2m(k-2)} - y_{2m(k)-1}\| + \|y_{2m(k)-1} - y_{2m(k)}\| \\ &\leq \epsilon + d_{2m(k)-2} + d_{2m(k)-1}, \end{aligned}$$

which implies

$$\lim_{k\to\infty} \|y_{2n(k)} - y_{2m(k)}\| = \epsilon \tag{3.10}$$

Further, from triangular inequality , it follows that

$$\left| \left\| y_{2n(k)} - y_{2m(k)-1} \right\| - \left\| y_{2n(k)} - y_{2m(k)} \right\| \right| \leq d_{2m(k)-1}$$

and

$$\left| \left\| y_{2n(k)+1} - y_{2m(k)-1} \right\| - \left\| y_{2n(k)} - y_{2m(k)} \right\| \right| \leq d_{2m(k)-1} + d_{2n(k)}.$$

Hence for $k \to \infty$, we find by (3.10) that

$$\left\| y_{2n(k)} - y_{2m(k)-1} \right\| \to \epsilon \, and \, \left\| y_{2n(k)} - y_{2m(k)-1} \right\| \to \epsilon. \tag{3.11}$$

On the other hand ,using (3.3) we deduce that

$$\begin{aligned} \left\| y_{2n(k)} - y_{2m(k)} \right\| &\leq d_{2n(k)} + \left\| y_{2n(k)+1} - y_{2m(k)} \right\| \\ &\leq d_{2n(k)} + f\big(\left\| y_{2m(k)-1} - y_{2n(k)} \right\|, d_{2n(k)}, \\ &\left\| y_{2m(k)-1} - y_{2n(k)+1} \right\|, \left\| y_{2n(k)} - y_{2m(k)} \right\|, d_{2n(k)} \big) \end{aligned} \tag{3.12}$$

by (3.10), (3.11), the upper-semicontinuity and non-decreasing properties of f, and condition (3.5), we have from (3.12) for $k \to \infty$, $\epsilon \leq f(\epsilon, 0, \epsilon, \epsilon, 0) \leq \epsilon$, which is a contradiction. Therefore $\{y_{2n}\}$ is a cauchy sequence in K and so is $\{y_n.\}$

But K is a closed subset of Banach space X, therefore $\{y_n\}$ converges to a point z in K. On the other hand , the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to z. Since $BK \subset SK$, there exists a point u in K such that $Su = z$.

By using (3.3) we write $\|Au - Bx_{2n+1}\| \leq f(\|Su - Tx_{2n+1}\|, \|Su - Au\|, \|Su - Bx_{2n+1}\|, \|Tx_{2n+1} - Au\|, \|Tx_{2n+1} - Bx_{2n+1}\|)$.

Taking n tends to ∞ , we get $\|Au - z\| \leq f(\|z - z\|, \|z - Au\|, \|z - z\|, \|z - Au\|, \|z - z\|)$.

This gives $\|Au - z\| \leq f(\|Au - z\|)$, which is a contradiction. Therefore we have $z = Au$. Thus $Au = Su = z$. Since $AK \subset TK$, there exists a point $v \in K$ such thst $Tv = z$. Then using (3.8), we have $\|Ax_{2n} - Bv\| \leq f(\|Sx_{2n} - Tv\|, \|Sx_{2n} - Ax_{2n}\|, \|Sx_{2n} - Bv\|, \|Tv - Ax_{2n}\|, \|Tv - Bv\|)$. Letting n tends to ∞, we get

$$\|z - Bv\| \leq f(\|z - Tv\|, \|z - z\|, \|z - Bv\|, \|Tv - z\|, \|Tv - Bv\|),$$
$$\|z - Bv\| \leq f(\|z - z\|, \|z - z\|, \|z - Bv\|, \|z - z\|, \|z - Bv\|).$$

This yields $\|z - Bv\| \leq f(\|z - Bv\|)$. which is a contradiction. Thus $z = Bv$. Therefore $z = Bv = Tv$. Hence $Au = Su = Bv = Tv = z$. Since A and S are weakly compatible, therefore A and S commute at their coincidence points i.e. $ASu = SAu$ or Az =Sz. Similarly $BTv = TBv$ or $Bz = Tz$.

Now we prove $Az = z$ by using (3.8), we have

$$\begin{aligned} \|Az - Bx_{2n+1}\| \leq f(&\|Sz - Tx_{2n+1}\|, \|Sz - Az\|, \|Sz - Bx_{2n+1}\|, \\ &\|Tx_{2n+1} - Az\|, \|Tx_{2n+1} - Bx_{2n+1}\|). \end{aligned}$$

Letting n tends to ∞, we have $\|Az - z\| \leq f(\|Sz - z\|, \|Sz - Az\|, \|Sz - z\|,$ $\|z - Az\|, \|z - z\|)$, $\|Az - z\| \leq f(\|Az - z\|, \|Az - Az\|, \|Az - z\|, \|z - Az\|,$ $\|z - z\|)$. This yields $\|Az - z\| \leq f(\|Az - z\|)$, which is a contradiction. Thus $Az = z$. Therefore $Az = Sz = z$. Similarly, we can show that $Bz = Tz = z$. This means that z is a fixed point of A, B, S and T.

For uniqueness of common fixed point , let $w \neq z$ be another common fixed point of A, B, S, and T. Then by (3.8), we have $\|Az - Bw\| \leq f(\|Sz - Tw\|,$ $\|Sz - Az\|, \|Sz - Bw\|, \|Tw - Az\|, \|Tw - Bw\|)$, $\|z - w\| \leq f(\|z-w\|, \|z-z\|,$ $\|z - w\|, \|w - z\|, \|w - w\|)$.

This gives $\|z - w\| \leq f(\|z - w\|)$, which is a contradiction. This $z = w$.

This completes the proof of the theorem.

Example 3.1. Let $X = K = [0, 2]$ with the Euclidean norm $\|.\|$. Define $A, B, S, T : K \to K$ by

$$Ax = \begin{cases} 0, & \text{if}\, x = 0, \\ 0.15, & \text{if}\ x > 0. \end{cases}$$

$$Bx = \begin{cases} 0, & \text{if}\, x = 0, \\ 0.35, & \text{if}\, x > 0. \end{cases}$$

$$Sx = \begin{cases} 0, \text{if}\, x = 0, \\ 0.35, & \text{if}\, 0 < x < 0.5, \\ x - 0.35, & \text{if}\, x \geq 0.5. \end{cases}$$

$$Tx = \begin{cases} 0, & \text{if}\, x = 0, \\ 0.15, & \text{if}\, 0 < x < 0.5, \\ x - 0.15, & \text{if}\, x \geq 0.5. \end{cases}$$

We see that A, B, S and T satisfy all the conditions of Theorem (3.1) and have a unique common fixed point of $0 \in X$. It may be noted in this example that the mappings A and S commute at coincidence point $0 \in X$. So A and S are weakly compatible maps. Similarly B and T are weakly compatible maps. To see the pairs $\{A, S\}$ and $\{B, T\}$ are non compatible, let us consider a decreasing sequence $\{x_n\}$ such that $x_n \to 0.5$. Then $\{Ax_n\} \to 0.15$, $\{Sx_n\} \to 0.15$, but $\lim_{n\to\infty} \|ASx_n - SAx_n\| \neq 0$. So the pair $\{A, S\}$ is noncompatible. Also $Bx_n \to 0.35, Tx_n \to 0.35$, but $\lim_{n\to\infty} \|BTx_n - TBx_n\| \neq 0$. So the pair {B,T} is non compatible. All the mappings involved in this example are discontinuous at the common fixed point.

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