# Nonlinear left $*$-Lie triple mappings of standard operator algebras

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**Abstract.** Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ which is closed under the adjoint operation. For $A, B \in \mathcal{A}$, define by ${}_*[A, B] = AB - B^*A$ the left $*$-Lie product of $A$ and $B$. In this paper, we prove that a mapping $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies $\phi({}_*[A, {}_*[B, C]]) = {}_*[\phi(A), {}_*[B, C]] + {}_*[A, {}_*[\phi(B), C]] + {}_*[A, {}_*[B, \phi(C)]]$, for all $A, B, C \in \mathcal{A}$ is automatically linear. Moreover, $\phi$ is an inner $*$-derivation.

**Keywords:** left $*$-Lie triple product, derivation, standard operator algebras.

## 1. Introduction

Let $\mathcal{A}$ be an algebra. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *nonlinear Lie derivation* if $\phi([A, B]) = [\phi(A), B] + [A, \phi(B)]$ holds true for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$ is the usual Lie product. Furthermore, if $\mathcal{A}$ is an algebra with involution, a mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *nonlinear $*$-Lie derivation* if for any $A, B \in \mathcal{A}$, $\phi([A, B]_*) = [\phi(A), B]_* + [A, \phi(B)]_*$, where $[A, B]_* = AB - BA^*$ is the skew Lie product of $A$ and $B$. Note that for both cases no additivity is assumed on $\phi$. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if $\phi(AB) = \phi(A)B + A\phi(B)$, for all $A, B \in \mathcal{A}$. $\phi$ is a *$*$-derivation* provided that

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$\phi(A^*) = \phi(A)^*$, for all $A \in \mathcal{A}$. A derivation on $\mathcal{A}$ is inner if there exists $T \in \mathcal{A}$ such that $\phi(A) = AT - TA$. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *Jordan derivation* if $\phi(A^2) = \phi(A)A + A\phi(A)$, for all $A \in \mathcal{A}$. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a *Jordan left $*$-derivation* if $\phi(A^2) = \phi(A)A + A^*\phi(A)$ holds true for any $A \in \mathcal{A}$.

Concerning Lie product, Lu and Liu [6] proved that every Lie derivation on $\mathcal{B}(\mathcal{X})$ can be expressed as the sum of an additive derivation of $\mathcal{B}(\mathcal{X})$ into itself and a central mapping on $\mathcal{B}(\mathcal{X})$ vanishing on each commutator. This result was generalized to the case of Lie derivation on prime rings in [3]. The skew Lie product is found playing an important role in the problem of representing quadratic functionals with sesquilinear functionals (see, for example, [8, 9, 10]) and in the problem of characterizing ideals (see, for example, [1, 7]). In [13] Yu and Zhang showed that every nonlinear $*$-Lie derivation from a factor von Neumann algebra on an infinite dimensional complex Hilbert space into itself is an additive $*$-derivation. In [5], Li, Lu and Fang arrived the same conclusion on von Neumann algebra without central abelian projections. Recently, Jing [4] proved that every nonlinear $*$-Lie derivation of standard operator algebra on complex Hilbert space is an inner $*$-derivation.

In this paper, we define left $*$-Lie product by ${}_*[A, B] = AB - B^*A$, for all $A, B \in \mathcal{A}$, in fact, it have a close relationship to Jordan left $*$-derivation [11]. And we call a nonlinear mapping $\phi$ is a nonlinear left $*$-Lie triple mapping if it satisfying $\phi({}_*[A, {}_*[B, C]]) = {}_*[\phi(A), {}_*[B, C]] + {}_*[A, {}_*[\phi(B), C]] + {}_*[A, {}_*[B, \phi(C)]]$ for all $A, B, C \in \mathcal{A}$. We shall show every nonlinear left $*$-Lie triple mapping of standard operator algebras which are closed under adjoint operation on infinite dimensional complex Hilbert space is automatically linear. Moreover it is an inner $*$-derivation.

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field, $\mathcal{B}(\mathcal{H})$ will represent the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We will denote by $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ the subalgebra of all bounded finite rank operators. We call a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ a *standard operator algebra* if it contain $\mathcal{F}(\mathcal{H})$. Note that, different from von Neumann algebra which are always weakly closed, a standard operator algebra is not necessarily closed. Recall that an algebra $\mathcal{A}$ is *prime* if $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. An operator $P \in \mathcal{B}(\mathcal{H})$ is said to be a projection provided $P^* = P$ and $P^2 = P$. It is well known that every standard operator algebra is prime and its commutant is $\mathbb{C}I$.

## 2. The main result and its proof

The main result in this paper is as follows.

**Theorem 2.1.** *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator $I$. If $\mathcal{A}$ is* *closed under the adjoint operation and* $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *satisfies*

$$\phi({}_*[A,\ {}_*[B,C]]) = {}_*[\phi(A),\ {}_*[B,C]] + {}_*[A,\ {}_*[\phi(B),C]] + {}_*[A,\ {}_*[B,\phi(C)]],$$

*for all* $A, B, C \in \mathcal{A}$*, then* $\phi$ *is a linear* $*$*-derivation. Moreover, there exists an operator* $T \in \mathcal{B}(\mathcal{H})$ *satisfying* $T + T^* = 0$ *such that* $\phi(A) = AT - TA$*, for all* $A \in \mathcal{A}$*, that is,* $\phi$ *is inner.*

To complete the proof of the main theorem, we begin with the following lemmas.

**Lemma 2.1.** *Let* $\mathcal{A}$ *be a standard operator algebra containing identity* $I$ *on a complex Hilbert space which is closed under adjoint operation. If* $AB = B^*A$ *holds true for all* $A \in \mathcal{A}$*, then* $B \in \mathbb{R}I$*.*

**Proof.** In fact, take $A = I$, then $B = B^*$. Thus the condition becomes $AB = BA$. It follows that $B \in \mathbb{C}I$, the center of $\mathcal{A}$, and so $B \in \mathbb{R}I$. □

**Lemma 2.2.** $\phi(0) = 0$.

**Proof.** It follows from the following:

$$\phi(0) = \phi({}_*[0,\ {}_*[0,0]]) = {}_*[\phi(0),\ {}_*[0,0]] +_* [0,\ {}_*[\phi(0),0]] + {}_*[0,\ {}_*[0,\phi(0)]] = 0.$$

□

**Lemma 2.3.** $\phi(\mathbb{R}I) \subseteq \mathbb{R}I, \phi(\mathbb{C}I) \subseteq \mathbb{C}I$*. For any* $A \in \mathcal{A}$ *with* $A = A^*$*,* $\phi(A^*) = \phi(A)^*$*.*

**Proof.** For any $\lambda \in \mathbb{R}$, we consider

$$\begin{aligned}
0 &= \phi({}_*[I,\ {}_*[A,\lambda I]]) \\
&= {}_*[\phi(I),\ {}_*[A,\lambda I]] + {}_*[I,\ {}_*[\phi(A),\lambda I]] + {}_*[I,\ {}_*[A,\phi(\lambda I)]] \\
&= {}_*[I,\ {}_*[A,\phi(\lambda I)]] \\
&= (A + A^*)\phi(\lambda I) - \phi(\lambda I)^*(A + A^*).
\end{aligned}$$

This gives us $(A+A^*)\phi(\lambda I) = \phi(\lambda I)^*(A+A^*)$ holds true for all $A \in \mathcal{A}$. That is, $B\phi(\lambda I) = \phi(\lambda I)^*B$ holds true for all $B = B^* \in \mathcal{A}$. Since every element in $\mathcal{A}$ is a linear span of two self-adjoint operators, it follows that $B\phi(\lambda I) = \phi(\lambda I)^*B$ holds true for all $B \in \mathcal{A}$. By Lemma 2.1, we have $\phi(\lambda I) \in \mathbb{R}I$. Hence $\phi(\mathbb{R}I) \subseteq \mathbb{R}I$. Let $A = A^* \in \mathcal{A}$. Since $\phi(I) \in \mathbb{R}I$, we have that

$$\begin{aligned}
0 &= \phi({}_*[I,\ {}_*[I,A]]) \\
&= {}_*[\phi(I),\ {}_*[I,A]] + {}_*[I,\ {}_*[\phi(I),A]] + {}_*[I,\ {}_*[I,\phi(A)]] \\
&= {}_*[I,\ {}_*[I,\phi(A)]] \\
&= 2\phi(A) - 2\Phi(A)^*.
\end{aligned}$$

Hence $\phi(A) = \phi(A)^*$. For any $\lambda \in \mathbb{C}$ and $A \in \mathcal{A}$ with $A = A^* \in \mathcal{A}$, applying above results, we see that

$$\begin{aligned} 0 &= \phi({}_*[C,\ {}_*[\lambda I, A]]) \\ &= {}_*[\phi(C),\ {}_*[\lambda I, A]] + {}_*[C,\ {}_*[\phi(\lambda I), A]] + {}_*[C,\ {}_*[\lambda I, \phi(A)]] \\ &= {}_*[C,\ {}_*[\phi(\lambda I), A]] \end{aligned}$$

holds true for all $C \in \mathcal{A}$. It follows from Lemma 2.1 that ${}_*[\phi(\lambda I), A] \in \mathbb{R}I$. This yields that $[\phi(\lambda I), A] \in \mathbb{R}I$, for all $A \in \mathcal{A}$ with $A = A^*$. By the Kleinecke-Shirokov theorem (cf. [2, Problem 230]) , we get $[\phi(\lambda I), A] = 0$, that is, $\phi(\lambda I)A = A\phi(\lambda I)$, for all $A \in \mathcal{A}$ with $A = A^*$. It follows that $\phi(\lambda I)A = A\phi(\lambda I)$ for any $A \in \mathcal{A}$, and so $\phi(\lambda I) \in \mathbb{C}I$. Therefore, $\phi(\mathbb{C}I) \subseteq \mathbb{C}I$. □

**Lemma 2.4.** *$\phi(\frac{1}{2}I) = \phi(\frac{1}{2}iI) = 0$ and $\phi(iA) = i\phi(A)$, for all $A \in \mathcal{A}$, where $i$ is the imaginary unit.*

**Proof.** We compute

$$\begin{aligned} 0 &= \phi({}_*[-\frac{1}{2}I,\ {}_*[-\frac{1}{2}iI, -\frac{1}{2}iI]]) \\ &= {}_*[\phi(-\frac{1}{2}I),\ {}_*[-\frac{1}{2}iI, -\frac{1}{2}iI]] + {}_*[-\frac{1}{2}I,\ {}_*[\phi(-\frac{1}{2}iI), -\frac{1}{2}iI]] \\ &+ {}_*[-\frac{1}{2}I,\ {}_*[-\frac{1}{2}iI, \phi(-\frac{1}{2}iI)]] \\ &= {}_*[\phi(-\frac{1}{2}I), -\frac{1}{2}I] + {}_*[-\frac{1}{2}I, -i\phi(-\frac{1}{2}iI)] + {}_*[-\frac{1}{2}I, -\frac{1}{2}i(\phi(-\frac{1}{2}iI) - \phi(-\frac{1}{2}iI)^*)] \\ &= i\phi(-\frac{1}{2}iI) - i\phi(-\frac{1}{2}iI)^*. \end{aligned}$$

It follows that $\phi(-\frac{1}{2}iI) = -\phi(-\frac{1}{2}iI)^*$. Similarly, by the equality $0 = {}_*[\frac{1}{2}I,\ {}_*[\frac{1}{2}iI, \frac{1}{2}iI]]$, we can get $\phi(\frac{1}{2}iI) = -\phi(\frac{1}{2}iI)^*$. We may also compute

$$\begin{aligned} \phi(-\frac{1}{2}iI) &= \phi({}_*[-\frac{1}{2}I,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]) \\ &= {}_*[\phi(-\frac{1}{2}I), -\frac{1}{2}iI] + {}_*[-\frac{1}{2}I, -i\phi(\frac{1}{2}iI)] + {}_*[-\frac{1}{2}I, -\phi(-\frac{1}{2}iI)] \\ &= 2i\phi(-\frac{1}{2}I) + \phi(-\frac{1}{2}iI). \end{aligned}$$

It follows that $\phi(-\frac{1}{2}I) = 0$. The equality $-\frac{1}{2}I = {}_*[\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]$ implies

$$\begin{aligned} 0 = \phi(-\frac{1}{2}I) &= \phi({}_*[\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]) \\ &= {}_*[\phi(\frac{1}{2}iI),\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]] + 0 + {}_*[\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, \phi(-\frac{1}{2}iI)]] \\ &= i\phi(\frac{1}{2}iI) - i\phi(-\frac{1}{2}iI). \end{aligned}$$

Hence

$$\phi(\frac{1}{2}iI) = \phi(-\frac{1}{2}iI). \tag{1}$$

Since the equality $\frac{1}{2}iI = {}_*[\frac{1}{2}I,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]$ hold true, we have

$$\begin{aligned}\phi(\frac{1}{2}iI) &= \phi({}_*[\frac{1}{2}I,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]) \\ &= {}_*[\phi(\frac{1}{2}I),\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]] + 0 + {}_*[\frac{1}{2}I,\ {}_*[-\frac{1}{2}I, \phi(-\frac{1}{2}iI)]] \\ &= {}_*[\phi(\frac{1}{2}I), \frac{1}{2}iI] + {}_*[\frac{1}{2}I, -\phi(-\frac{1}{2}iI)] \\ &= i\phi(\frac{1}{2}I) - \phi(-\frac{1}{2}iI).\end{aligned}$$

It follows that

$$\phi(\frac{1}{2}iI) + \phi(-\frac{1}{2}iI) = i\phi(\frac{1}{2}I). \tag{2}$$

Finally, by the equality $\frac{1}{2}I = {}_*[-\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, -\frac{1}{2}iI]]$, we can get

$$\begin{aligned}\phi(\frac{1}{2}I) &= \phi({}_*[-\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, \frac{1}{2}iI]]) \\ &= {}_*[\phi(-\frac{1}{2}iI),\ {}_*[-\frac{1}{2}I, \frac{1}{2}iI]] + 0 + {}_*[-\frac{1}{2}iI,\ {}_*[-\frac{1}{2}I, \phi(\frac{1}{2}iI)]] \\ &= {}_*[\phi(-\frac{1}{2}iI), \frac{1}{2}I] + {}_*[-\frac{1}{2}iI, -\phi(-\frac{1}{2}iI)] \\ &= i\phi(-\frac{1}{2}iI) + i\phi(-\frac{1}{2}iI) = 2i\phi(-\frac{1}{2}iI).\end{aligned}$$

It follows that

$$2\phi(-\frac{1}{2}iI) = -i\phi(\frac{1}{2}I). \tag{3}$$

Hence by Eq. (1), Eq. (2) and Eq. (3), we have $\phi(\frac{1}{2}iI) = \phi(-\frac{1}{2}iI) = 0$. For every $A \in \mathcal{A}$, it follows from $iA = {}_*[A,\ {}_*[\frac{1}{2}I, \frac{1}{2}iI]]$ that

$$\phi(iA) = \phi({}_*[A,\ {}_*[\tfrac{1}{2}I, \tfrac{1}{2}iI,]]) = {}_*[\phi(A),\ {}_*[\tfrac{1}{2}I, \tfrac{1}{2}iI]] = i\phi(A). \qquad \square$$

We now choose a nontrivial projection $P_1 \in \mathcal{A}$ and let $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j, i, j = 1, 2,$. Then we have the Peirce decomposition of $\mathcal{A}$ as $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. Note that any operator $A \in \mathcal{A}$ can be expressed as $A = A_{11} + A_{12} + A_{21} + A_{22}$, and $A_{ij}^* \in \mathcal{A}_{ji}$ for any $A_{ij} \in \mathcal{A}_{ij}$.

**Lemma 2.5.** *For any $A \in \mathcal{A}$,*
(1) ${}_*[A,\ {}_*[I, i(P_2 - P_1)]] = 0$ *implies* $A_{11} = A_{22} = 0$,
(2) ${}_*[I,\ {}_*[P_1, A]] = 0$ *implies* $A_{12} = 0$,
(3) ${}_*[I,\ {}_*[P_2, A]] = 0$ *implies* $A_{21} = 0$,
(4) ${}_*[A,\ {}_*[I, iP_1]] = 0$ *implies* $A_{11} = A_{12} = A_{21} = 0$,
(5) ${}_*[A,\ {}_*[I, iP_2]] = 0$ *implies* $A_{22} = A_{12} = A_{21} = 0$.

**Proof.** We only show (1). The proofs of (2), (3), (4) and (5) go similarly. We compute

$$\begin{aligned} 0 &= {}_*[A,\ {}_*[I, i(P_2-P_1)]] = {}_*[A, 2i(P_2-P_1)] \\ &= 2i(A(P_2-P_1)+(P_2-P_1)A) \\ &= 4i(A_{22}-A_{11}), \end{aligned}$$

which leads to $A_{22}=A_{11}=0$. □

**Lemma 2.6.** *For any $A_{12}\in\mathcal{A}_{12}$ and $B_{21}\in\mathcal{A}_{21}$, we have*

$$\phi(A_{12}+B_{21})=\phi(A_{12})+\Phi(B_{21}).$$

**Proof.** Let $M=\phi(A_{12}+B_{21})-\phi(A_{12})-\phi(B_{21})$. We now show that $M=0$. On one hand, since ${}_*[A_{12},\ {}_*[I,i(P_2-P_1)]]={}_*[B_{21},\ {}_*[I,i(P_2-P_1)]]=0$, we have

$$\begin{aligned} 0 &= \phi({}_*[A_{12}+B_{21},\ {}_*[I,i(P_2-P_1)]]) \\ &= {}_*[\phi(A_{12}+B_{21}),\ {}_*[I,i(P_2-P_1]]+{}_*[A_{12}+B_{21},\ {}_*[\phi(I),i(P_2-P_1)]] \\ &\quad + {}_*[A_{12}+B_{21},\ {}_*[I,\phi(i(P_2-P_1))]]. \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &= \phi({}_*[A_{12},\ {}_*[I,i(P_2-P_1)]])+\phi({}_*[B_{21},\ {}_*[I,i(P_2-P_1)]]) \\ &= {}_*[\phi(A_{12})+\phi(B_{21}),\ {}_*[I,i(i(P_2-P_1))]]+{}_*[A_{12}+B_{21},\ {}_*[\phi(I),i(P_2-P_1)]] \\ &\quad + {}_*[A_{12}+B_{21},\ {}_*[I,\phi(i(P_2-P_1))]]. \end{aligned}$$

Comparing the above two equalities, we arrive at ${}_*[M,\ {}_*[I,i(P_2-P_1)]]=0$. It follows from Lemma 2.5 (1), that $M_{11}=M_{22}=0$.

Since ${}_*[I,\ {}_*[P_1,B_{21}]]=0$, we have that

$$\begin{aligned} &{}_*[\phi(I),\ {}_*[P_1,A_{12}+B_{21}]]+{}_*[I,\ {}_*[\phi(P_1),A_{12}+B_{21}]]+{}_*[I,\ {}_*[P_1,\phi(A_{12}+B_{21})]] \\ &= \phi({}_*[I,\ {}_*[P_1,A_{12}+B_{21}]]) \\ &= \phi({}_*[I,\ {}_*[P_1,A_{12}]])+\phi({}_*[I,\ {}_*[P_1,B_{21}]]) \\ &= {}_*[\phi(I),\ {}_*[P_1,A_{12}+B_{21}]]+{}_*[I,\ {}_*[\phi(P_1),A_{12}+B_{21}]]+{}_*[I,\ {}_*[P_1,\phi(A_{12}+B_{21})]]. \end{aligned}$$

Hence ${}_*[I,\ {}_*[P_1,M]]=0$. By Lemma 2.5 (2), we get that $M_{12}=0$. Similarly, by using the fact ${}_*[I,\ {}_*[P_2,A_{12}]]=0$, one can show $M_{21}=0$. □

**Lemma 2.7.** *For any $A_{11}\in\mathcal{A}_{11}, B_{12}\in\mathcal{A}_{12}, C_{21}\in\mathcal{A}_{21}$, and $D_{22}\in\mathcal{A}_{22}$,*

(1) $\phi(A_{11}+B_{12}+C_{21})=\phi(A_{11})+\phi(B_{12})+\phi(C_{21})$.

(2) $\phi(B_{12}+C_{21}+D_{22})=\phi(B_{12})+\phi(C_{21})+\phi(D_{22})$.

**Proof.** (1) Since ${}_*[A_{11},\ {}_*[I, iP_2]] = 0$, by Lemma 2.6, we obtain

$$\begin{aligned}
&{}_*[\phi(A_{11} + B_{12} + C_{21}),\ {}_*[I, iP_2]] + {}_*[A_{11} + B_{12} + C_{21},\ {}_*[\phi(I), iP_2]] \\
&+ {}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, \phi(iP_2)]] \\
&= \phi({}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, iP_2)]]) \\
&= \phi({}_*[A_{11},\ {}_*[I, iP_2]]) + \phi({}_*[B_{12} + C_{21},\ {}_*[I, P_2]]) \\
&= {}_*[\phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}),\ {}_*[I, iP_2]] + {}_*[A_{11} + B_{12} + C_{21},\ {}_*[\phi(I), iP_2]] \\
&+ {}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, i\phi(P_2)]].
\end{aligned}$$

Letting $M = \phi(A_{11}+B_{12}+C_{21})-\phi(A_{11})-\phi(B_{12})-\phi(C_{21})$, we get ${}_*[M,\ {}_*[I, iP_2]] = 0$. It follows from Lemma 2.5 (5) that $M_{12} = M_{21} = M_{22} = 0$.

We now show that $M_{11} = 0$. By noting ${}_*[B_{12}, [I,\ {}_*i(P_2-P_1)]] = {}_*[C_{21},\ {}_*[I, i(P_2-P_1)]] = 0$, we have

$$\begin{aligned}
&\phi({}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, i(P_2 - P_1)]]) \\
&= \phi({}_*[A_{11},\ {}_*[I, i(P_2 - P_1), I]]) + \phi({}_*[B_{12},\ {}_*[I, i(P_2 - P_1)]]) \\
&+ \phi({}_*[C_{21},\ {}_*[I, i(P_2 - P_1)]]).
\end{aligned}$$

By using the similar argument, we can get ${}_*[M,\ {}_*[I, i(P_2-P_1)]] = 0$. Therefore, $M_{11} = 0$ by Lemma 2.5 (3).

(2) Considering ${}_*[\phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}),\ {}_*[I, iP_1]]$ and $\phi({}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, i(P_2 - P_1)]])$, with the same argument as in (1), one can get $\phi(B_{12} + C_{21} + D_{22}) = \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22})$. □

**Lemma 2.8.** *For any* $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}$, *and* $D_{22} \in \mathcal{A}_{22}$,

$$\phi(A_{11} + B_{12} + C_{21} + D_{22}) = \phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22}).$$

**Proof.** Let $M = \phi(A_{11}+B_{12}+C_{21}+D_{22}) - \phi(A_{11} - \phi(B_{12}) - \phi(C_{21}) - \phi(D_{22})$. Noticing that ${}_*[D_{22},\ {}_*[I, iP_1]] = 0$ and applying (1) in Lemma 2.7, we have

$$\begin{aligned}
&{}_*[\phi(A_{11} + B_{12} + C_{21} + D_{22}),\ {}_*[I, iP_1]] + {}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[\phi(I), iP_1]] \\
&+ {}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[I, \phi(iP_1)]] \\
&= \phi({}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[I, iP_1]]) \\
&= \phi({}_*[A_{11} + B_{12} + C_{21},\ {}_*[I, iP_1]]) + \phi({}_*[D_{22},\ {}_*[I, iP_1]]) \\
&= {}_*[\phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22}),\ {}_*[I, iP_1]] \\
&+ {}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[\phi(I), iP_1]] \\
&+ {}_*[A_{11} + B_{12} + C_{21} + D_{22},\ {}_*[I, \phi(iP_1)]].
\end{aligned}$$

It follows that ${}_*[M,\ {}_*[I, iP_1]] = 0$, so $M_{11} = M_{12} = M_{21} = 0$ by Lemma 2.5. Using the fact that ${}_*[A_{11},\ {}_*[I, iP_2]] = 0$ and the similar argument above, we can get ${}_*[M,\ {}_*[I, iP_2]] = 0$ which leads $M_{22} = 0$, completing the proof. □

**Lemma 2.9.** *For any $A_{jk}, B_{jk} \in \mathcal{A}_{jk}$, where $1 \leq j \neq k \leq 2$, we have*

$$\phi(A_{jk} + B_{jk}) = \phi(A_{jk}) + \phi(B_{jk}).$$

**Proof.** On one hand, by Lemma 2.7,

$$\phi(iA_{jk} + iB_{jk} + iA_{jk}^* + iA_{jk}^*B_{jk}) = \phi(iA_{jk} + iB_{jk}) + \phi(i(A_{jk}^*) + \phi(i(A_{jk}^*B_{jk}).$$

On the other hand, since

$${}_*[P_j + B_{jk},\ {}_*[P_k + A_{jk}, \frac{i}{2}I]] = i(A_{jk} + B_{jk}) + i(A_{jk}^*) + i(A_{jk}^*B_{jk}),$$

using Lemma2.8 again,

$$\begin{aligned}
&\phi(iA_{jk} + iB_{jk} + iA_{jk}^* + iA_{jk}^*B_{jk})\\
&= \phi({}_*[P_j + B_{jk},\ {}_*[P_k + A_{jk}, \frac{i}{2}I]])\\
&= {}_*[\phi(P_j + B_{jk}),\ {}_*[P_k + A_{jk}, \frac{i}{2}I]] + {}_*[P_j + B_{jk},\ {}_*[\phi(P_k + A_{jk}), \frac{i}{2}I]]\\
&+ {}_*[P_j + B_{jk},\ {}_*[P_k + A_{jk}, \phi(\frac{i}{2}I)]]\\
&= {}_*[\phi(P_j) + \phi(B_{jk}),\ {}_*[P_k + A_{jk}, \frac{i}{2}I]] + {}_*[P_j + B_{jk},\ {}_*[\phi(P_k) + \phi(A_{jk}), \frac{i}{2}I]]\\
&+ {}_*[P_j + B_{jk},\ {}_*[P_k + A_{jk}, \phi(\frac{i}{2}I)]]\\
&= \phi({}_*[P_j,\ {}_*[P_k, \frac{i}{2}I]]) + \phi({}_*[B_{jk},\ {}_*[P_j, \frac{i}{2}I]]) + \phi({}_*[P_j,\ {}_*[A_{jk}, \frac{i}{2}I]])\\
&+ \phi({}_*[B_{jk},\ {}_*[A_{jk}, \frac{i}{2}I]])\\
&= \phi(iB_{jk}) + \phi(iA_{jk} + iA_{jk}^*) + \phi(iA_{jk}^*B_{jk})\\
&= \phi(iB_{jk}) + \phi(iA_{jk}) + \phi(iA_{jk}^*) + \phi(iA_{jk}^*B_{jk}).
\end{aligned}$$

Note that in the last identity above, we are using Lemma 2.6. We now can conclude that $\phi(A_{jk} + B_{jk}) = \phi(A_{jk}) + \phi(B_{jk})$ by Lemma 2.4. □

**Lemma 2.10.** *For any $A_{jj}, B_{jj} \in \mathcal{A}_{jj}$, where $1 \leq j \leq 2$, we have*

$$\phi(A_{jj} + B_{jj}) = \phi(A_{jj}) + \phi(B_{jj}).$$

**Proof.** Let $k \in \{1, 2\}$, with $k \neq j$. We compute

$$\begin{aligned}
&{}_*[\phi(A_{jj} + B_{jj}),\ {}_*[I, iP_k]] + {}_*[A_{jj} + B_{jj},\ {}_*[\phi(I), iP_k]] + {}_*[A_{jj} + B_{jj},\ {}_*[I, \phi(iP_k)]]\\
&= \phi({}_*[A_{jj} + B_{jj},\ {}_*[I, iP_k]]) = 0\\
&= \phi({}_*[A_{jj},\ {}_*[I, \phi(iP_k)]]) + \phi({}_*[B_{jj}, [I,\ {}_*\phi(iP_k)]])\\
&= {}_*[\phi(A_{jj}) + \phi(B_{jj}),\ {}_*[I, iP_k]] + {}_*[A_{jj} + B_{jj},\ {}_*[\phi(I), iP_k]]\\
&+ {}_*[A_{jj} + B_{jj},\ {}_*[I, \phi(iP_k)]].
\end{aligned}$$

Write $M = \phi(A_{jj}+B_{jj}) - \phi(A_{jj}) - \phi(B_{jj})$. The above computation yields that ${}_*[M,\ {}_*[I, iP_k]] = 0$. By Lemma 2.4, we have $M_{jk} = M_{kj} = M_{kk} = 0$. We now show that $M_{jj} = 0$. For any $C_{jk} \in \mathcal{A}_{jk}$, by Lemma 2.7,

$$\begin{aligned}
&{}_*[\phi(C_{jk}),\ {}_*[A_{jj}+B_{jj}, \frac{1}{2}iP_j)]] + {}_*[C_{jk},\ {}_*[\phi(A_{jj}+B_{jj}), \frac{1}{2}iP_j]]\\
&+ {}_*[C_{jk},\ {}_*[A_{jj}+B_{jj}, \phi(\frac{1}{2}iP_j)]] = \phi({}_*[C_{jk},\ {}_*[A_{jj}+B_{jj}, \frac{1}{2}iP_j]])\\
&= \phi({}_*[C_{jk},\ {}_*[A_{jj}, \frac{1}{2}iP_j]]) + \phi({}_*[C_{jk},\ {}_*[B_{jj}, \frac{1}{2}iP_j]])\\
&= {}_*[\phi(C_{jk}),\ {}_*[A_{jj}+B_{jj}, \frac{1}{2}iP_j)]] + {}_*[C_{jk},\ {}_*[\phi(A_{jj}+B_{jj}), \frac{1}{2}iP_j]]\\
&+ {}_*[C_{jk},\ {}_*[A_{jj}+B_{jj}, \phi(\frac{1}{2}iP_j)]].
\end{aligned}$$

Therefore, ${}_*[C_{jk},\ {}_*[M, \frac{1}{2}iP_j]] = 0$ which leads to $M_{jj}^* C_{jk} = 0$, for all $C_{jk} \in \mathcal{A}_{jk}$. Since $\mathcal{A}$ is prime, we see that $M_{jj} = 0$. □

**Lemma 2.11.** *$\phi$ is an additive derivation with $\phi(A^*) = \phi(A)^*$, for all $A \in \mathcal{A}$.*

**Proof.** We first show that $\phi$ is additive. For arbitrary $A, B \in \mathcal{A}$, we write $A = \sum_{i,j=1}^{2} A_{ij}$ and $B = \sum_{i,j=1}^{2} B_{ij}$. By Lemma 2.8, Lemma 2.9 and Lemma 2.10, we obtain

$$\begin{aligned}
\phi(A+B) &= \phi(\sum_{i,j=1}^{2} A_{ij} + \sum_{i,j=1}^{2} B_{ij}) = \sum_{i,j=1}^{2} \Phi(A_{ij}+B_{ij})\\
&= \sum_{i,j=1}^{2} \phi(A_{ij}) + \sum_{i,j=1}^{2} \phi(B_{ij}) = \phi(\sum_{i,j=1}^{2} A_{ij}) + \phi(\sum_{i,j=1}^{2} B_{ij})\\
&= \phi(A) + \phi(B).
\end{aligned}$$

We now show $\phi(A^*) = \phi(A)^*$. For every $A \in \mathcal{A}$, we write $A = A_1 + iA_2$, where $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2i}$ are self-adjoint elements. By Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\phi(A^*) &= \phi(A_1 - iA_2) = \phi(A_1) - \phi(iA_2)\\
&= \phi(A_1) - i\phi(A_2) = \phi(A_1)^* - i\phi(A_2)^*\\
&= \phi(A_1)^* + (i\phi(A_2))^* = \phi(A_1 + iA_2)^* = \phi(A)^*.
\end{aligned}$$

To complete the proof, we need to show that $\phi$ is a derivation. By the additivity of $\phi$ and Lemma 2.5, we have $\phi(iI) = 2\phi(\frac{1}{2}iI) = 0$. Note that ${}_*[A,\ {}_*[B, iI]] = 2i(AB + B^*A)$. We compute

$$\begin{aligned}
2i\phi(AB + B^*A) &= \phi(2i(AB + B^*A))\\
&= \phi({}_*[A,\ {}_*[B, iI]])\\
&= {}_*[\phi(A),\ {}_*[B, iI]] + {}_*[A, {}_*[\phi(B), iI]] + {}_*[A,\ {}_*[B, \phi(iI)]]\\
&= 2i(\phi(A)B + B^*\phi(A)) + A\phi(B) + \phi(B)^*A).
\end{aligned}$$

It follows that

$$\phi(AB + B^*A) = \phi(A)B + B^*\phi(A) + A\phi(B) + \phi(B)^*A.$$

Replacing $B$ by $iB$ in the above equality, we get

$$\phi(AB - B^*A) = \phi(A)B - B^*\phi(A) + A\phi(B) - \phi(B)^*A.$$

Thus $\phi(AB) = \phi(A)B + A\phi(B)$, it is a derivation. □

**The proof of the main theorem.** By Lemma 2.11, we see that $\phi$ is an additive derivation with $\phi(A^*) = \phi(A)^*$. It follows from [12, Theorem 2.3] that $\phi$ is an linear inner derivation, that is, there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = AS - SA$, for all $A \in \mathcal{A}$. Since $\phi(A^*) = \phi(A)^*$, we have

$$A^*S - SA^* = \phi(A^*) = \phi(A)^* = S^*A^* - A^*S^*$$

for any $A \in \mathcal{A}$. This leads to $A^*(S + S^*) = (S + S^*)A^*$. Hence, $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Letting $T = S - \frac{1}{2}\lambda I$, one can check that $T + T^* = 0$ and $\phi(A) = AT - TA$, for all $A \in \mathcal{A}$.

**Corollary 2.1.** *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is nonlinear left ∗-Lie triple mapping, then $\phi$ is an inner ∗-derivation, that is, there exists an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\phi(A) = AT - TA$, for all $A \in \mathcal{A}$.*

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