Toward strictly singular fractional operator restricted by Fredholm-Volterra in Sobolev space

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Abstract. In this paper, a reliable numerical technique is proposed for solving a class of singular fractional differential equations involving Fredholm and Volterra operators subjected to suitable three-point boundary conditions. The solution methodology is presented based on reproducing-kernel method (RKM), which is used directly without employing linearization and perturbation. However, a favorable Hilbert spaces are construcred, and then the orthonormal function system is generated by using Gram-Schmidt orthogonalization process. Error analysis is given in Sobolev space. Numerical example is tested to multipoint singular fractional differential problems with Fredholm and Volterra operators to show the theoretical statements of the RKHS method. The results obtained indicate that the RKHS method is easy to implement, reliability and capability with a great potential of such singular problems.
Keywords: singular integral operator, fractional differential equation, reproducing-kernel method, Caputo fractional derivative, Gram-Schmidt process.

1. Introduction

The multipoint singular boundary value problems (BVPs) arise in a variety of differential applied mathematics and physics such as gas dynamics, nuclear physics, chemical reaction, studies of atomic structures, and atomic calculations. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint singular BVP as in [1]. Many problems in the theory of elastic stability can be handled using multipoint singular BVPs as in [2]. In optimal bridge design, large size bridges are sometimes contrived with multipoint supports, which corresponds to a mul-

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tipoint singular BVP as in [3]. Therefore, it appears to be very important to develop numerical or analytical methods for solving such problems.

Most scientific problems and phenomenons in different fields of sciences and engineering occur nonlinearly with a set of finite singularity. To be more precisely, most of them can not be handled analytically. So these nonlinear singular equations should be solved using numerical methods or other analytical methods. Anyhow, when we use multipoint singular BVPs, the obtained numerical solutions could be not that required accurate outcomes or may even fail to converge due to singularity problem [4, 5, 6]; whilst analytical methods commonly used to solve nonlinear singular differential equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors. Thus, it is expected to have some restrictions to handle these kind of problems; because of two difficulties presenting in both nonlinearity of equations and the singularity case of BVPs.

In this paper, we aim to use appropriate theory for building the Hilbert spaces to develop IRKM algorithm for handling second-order singular ordinary differential equations with three-point boundary conditions. In particular, we provide the analytical-numerical solutions for the following singular differential-operator equation:

$$\begin{aligned}
&D^{2\alpha}u\left(x\right)+P(x)D^{\alpha}\left(x\right)+Q(x)u\left(x\right)=F\left(x,Su\left(x\right),Tu\left(x\right)\right),x\in[0,1],\\
(1)\qquad &Su\left(x\right)=\lambda_1\textstyle\int_0^1 k_1\left(x,\xi\right)u\left(\xi\right)d\xi,\\
&Tu\left(x\right)=\lambda_2\textstyle\int_0^x k_2\left(x,\xi\right)u\left(\xi\right)d\xi,
\end{aligned}$$

with the boundary conditions

$$(2)\qquad \begin{aligned}
&u\left(0\right)=0,\\
&u\left(1\right)-\alpha u\left(\eta\right)=0,\ 0<\eta<1,\ \alpha>0,\ \alpha\eta<1,
\end{aligned}$$

where $0<\alpha\leq 1$, λ_1 and λ_2 are constant parameters, $D^{\alpha}\left(x\right)$ is indicated to fractional derivative in the Caputo sense, $k_1\left(x,t\right),k_2\left(x,t\right)$ are continuous arbitrary kernel functions over $0<\xi<x<1$, $F\left(x,w_1,w_2\right)$ is continuous terms in $W_2^1\left[0,1\right]$ as $w_i=w_i\left(x\right)\in W_2^3\left[0,1\right]$, $-\infty<w_i<\infty$, $i=1,2$, $P\left(x\right)$ and $Q\left(x\right)$ are continuous real-valued functions and may be equal to zero at some $\{x_i\}_{i=1}^m\in[0,1]$, $u(x)$ is an unknown analytical function in $W_2^3\left[0,1\right]$ to be determined, and $W_2^1\left[0,1\right],W_2^3\left[0,1\right]$ are reproducing-kernel spaces. Here, we assume that Eq. (1) with conditions (2) has a unique smooth solution. Further, the Caputo fractional derivative of order $m-1<\alpha\leq m, m\in N$, can be defined as follows

$$D^{\alpha}u\left(x\right)=\frac{1}{\Gamma(m-\alpha)}\int_0^x(x-t)^{m-\alpha-1}u^{(m)}(t)dt,\ 0<t<x,$$

Many authors have studied different kind of analysis about solvability of second-order, three point singular BVPs. In order to get more information about the properties and the applications have been studied, the reader can refer to [7, 8, 9, 10]. On the other hand, there is a few research papers about investigating second-order, three point singular BVPs restricted by Fredholm-Volterra operators numerically. Reproducing kernel theory has many applications in complex analysis, harmoni analysis, and quantum mechanics [11, 12, 13, 14, 15, 16]. Recently, a lot of research work has been carried out to apply of the RKHS algorithm for wide classes of stochastic and deterministic problems involving operator equations, differential equations, integral equations, and integro-differential equations. The RKHS algorithm was successfully used by many authors to investigate many scientific applications side by side with their theories. To understand the properties and the fundamentals of reproducing kernel Hilbert spaces, the reader can return to the references [17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Fractional DEs are a type of differential equations that involving an unknown function with fractional derivatives. These types of equations are used to formulate problems involving functions of single or several variables and to aid a solution of many physical phenomena in terms of fractional DEs. The Caputo's definition will be used, taking the feature of Caputo's approach that the initial conditions of fractional DEs with Caputo's derivatives take the traditional form. For more details, we refer to [27, 28, 29, 30, 31, 32].

The outline of the paper is as follows. In section 2, two appropriate inner product spaces are constructed to apply RKHS method to solve the presented BVP with Fredholm Volterra operator. In section 3, Gram-Schmidt orthogonalization process is used to obtain the orthonormal basis. Meanwhile, the efficiency of the method is proposed in section 3 by proving that the numerical solution converges to the analytical solution uniformly. After all, numerical algorithm with numerical example are presented to show how the process does work in section 4. Finally, we summarize up the process with some concluding remarks in section 5.

2. Toward to reproducing-kernel function

In this section, a method for constructing a reproducing kernel function that satisfying the two-point boundary conditions $v\left(0\right)=0$ and $v\left(1\right)=0$ is presented. By applying some good properties of the reproducing kernel space, a very simple numerical method is provided for obtaining approximation to the solution of Eqs. (1) and (2). Here, $L^2\left[0,1\right]=\{v \mid \int_0^1 v^2\left(x\right)dx<\infty\}$ and $l^2=\{A \mid \sum_{i=1}^{\infty}\left(A_i\right)^2<\infty\}$.

Definition 2.1 ([33]). *Let Π be a Hilbert space of function $\theta:\Omega\to\Pi$ on a set Ω. A function $\Gamma:\Omega\times\Omega\to\mathbb{C}$ is a reproducing kernel of Π if the following conditions are satisfied. Firstly, $\Gamma\left(\cdot,x\right)\in\Pi$ for each $x\in\Omega$. Secondly, $\langle\theta\left(\cdot\right),\Gamma\left(\cdot,x\right)\rangle=\theta\left(x\right)$ for each $\theta\in\Pi$ and each $x\in\Omega$.*

To solve Eqs. (1) and (2) using RKHS algorithm, we first define and construct a reproducing kernel space $W_2^3\left[0,1\right]$ in which every function satisfies the two-point boundary conditions $v(0) = 0$ and $v(1) = 0$. After that, we utilize a reproducing kernel space $W_2^1[0,1]$.

Definition 2.2. *The space* $W_2^3[0,1]$ *is defined as* $W_2^3[0,1] = \{v \mid v, v', v''$ *are absolutely continuous on* $[0,1]$, $v, v', v'', v''' \in L^2[0,1]$, *and* $v(0) = 0, v(1) = 0\}$. *On the other hand, the inner product and the norm in* $W_2^3[0,1]$ *are defined, respectively, by*

$$\langle v(x), w(x)\rangle_{W_2^3} = \sum_{i=0}^{2} v^{(i)}(0)\, w^{(i)}(0) + \int_0^1 v'''(x)w'''(x)dx, \tag{3}$$

and $||v||_{W_2^3} = \sqrt{\langle v(x), v(x)\rangle_{W_2^3}}$, *where* $v, w \in W_2^3[0,1]$.

It is easy to see that $\langle u(x), v(x)\rangle_{W_2^3}$ satisfies all the requirements of the inner product as follows; first, $\langle u(x), u(x)\rangle_{W_2^3} \geq 0$; second, $\langle u(x), v(x)\rangle_{W_2^3} = \langle v(x), u(x)\rangle_{W_2^3}$; third, $\langle \gamma u(x), v(x)\rangle_{W_2^3} = \gamma\langle u(x), v(x)\rangle_{W_2^3}$; fourth, $\langle u(x) + w(x), v(x)\rangle_{W_2^3} = \langle u(x), v(x)\rangle_{W_2^3} + \langle w(x), v(x)\rangle_{W_2^3}$, where $u, v, w \in W_2^3[0,1]$. Indeed, it is obvious that when $u(x) = 0$, then $\langle u(x), u(x)\rangle_{W_2^3} = 0$, while on the other aspect as well, if $\langle u(x), u(x)\rangle_{W_2^3} = 0$, then by Eq. (3); $\langle u(x), u(x)\rangle_{W_2^3} = \sum_{i=0}^{2}(u^{(i)}(0))^2 + \int_0^1 (u'''(x))^2 dx = 0$, therefore, $u(0) = u'(0) = u''(0) = 0$ and $u'''(x) = 0$. Thus, one can obtain $u(x) = 0$.

The Hilbert space $W_2^3[0,1]$ is called a reproducing kernel if for each fixed $x \in [0,1]$, there exist $R^{\{1\}}(x,y) \in W_2^3[0,1]$ (simply $R_x^{\{1\}}(y)$) such that $\langle v(y), R_x^{\{1\}}(y)\rangle_{W_2^3} = v(x)$ for any $v(y) \in W_2^3[0,1]$ and $y \in [0,1]$.

Theorem 2.1 ([34])**.** *The Hilbert space* $W_2^3[0,1]$ *is a complete reproducing kernel with reproducing kernel function*

$$R_x^{\{1\}}(y) = \begin{cases} a_1(x) + a_2(x)y + a_3(x)y^2 + a_4(x)y^3 \\ \qquad + a_5(x)y^4 + a_6(x)y^5, y \leq x, \\ b_1(x) + b_2(x)y + b_3(x)y^2 + b_4(x)y^3 \\ \qquad + b_5(x)y^4 + b_6(x)y^5, y > x, \end{cases} \tag{4}$$

where $a_i(x)$ *and* $b_i(x)$, $i = 1, 2, ..., 6$, *are unknown coefficients of* $R_x^{\{1\}}(y)$ *and are given as*

a_i*'s coefficients*	b_i*'s coefficients*
$a_1(x) = 0,$	$b_1(x) = \frac{1}{120}x^5,$
$a_2(x) = -\frac{1}{156}x(-36 + 30x + 10x^2 - 5x^3 + x^4),$	$b_2(x) = -\frac{1}{312}x(-72 + 60x + 20x^2 + 3x^3 + 2x^4),$
$a_3(x) = -\frac{1}{624}x(120 - 126x + 10x^2 - 5x^3 + x^4),$	$b_3(x) = -\frac{1}{624}x(120 - 126x - 42x^2 - 5x^3 + x^4),$
$a_4(x) = -\frac{1}{1872}x(120 - 126x + 10x^2 - 5x^3 + x^4),$	$b_4(x) = -\frac{1}{1872}x(120 + 30x + 10x^2 - 5x^3 + x^4),$
$a_5(x) = \frac{1}{3744}x(-36 + 30x + 10x^2 - 5x^3 + x^4),$	$b_5(x) = \frac{1}{3744}x(120 + 30x + 10x^2 - 5x^3 + x^4),$
$a_6(x) = \frac{1}{18720}(156 - 120x - 30x^2 - 10x^3 + 5x^4 - x^5),$	$b_6(x) = -\frac{1}{18720}x(120 + 30x + 10x^2 - 5x^3 + x^4).$

Definition 2.3 ([17]). *The space* $W_2^1[0,1]$ *is defined as* $W_2^1[0,1] = \{v \mid v$ *is absolutely continuous on* $[0,1]$ *and* $v' \in L^2[0,1]\}$. *On the other hand, the inner product and the norm in* $W_2^1[0,1]$ *are defined, respectively, by*

$$\langle v(x), w(x)\rangle_{W_2^1} = v(0)\,w(0) + \int_0^1 v'(x)w'(x)dx,$$

and $||v||_{W_2^1} = \sqrt{\langle v(x), v(x)\rangle_{W_2^1}}$, *where* $v, w \in W_2^1[0,1]$.

Theorem 2.2 ([17]). *The Hilbert space* $W_2^1[0,1]$ *is a complete reproducing kernel with reproducing kernel function*

$$R_x^{\{2\}}(y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases}$$

3. Structure of the method

Here, the formulation of a differential linear operator is presented in $W_2^3[0,1]$. After that, we use the Gram-Schmidt orthogonalization process on the orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ and normalizing them on $W_2^3[0,1]$ to obtain the required orthogonalization coefficients in order to obtain the analytical-numerical solutions of Eqs. (1) and (2) using RKHS algorithm.

Let us consider the differential operator $L : W_2^3[0,1] \to W_2^1[0,1]$ such that $Lv(x) = D^{2\alpha}v(x) + P(x)D^{\alpha}v(x) + Q(x)v(x)$, $v(x) = u(x) - \phi(x)$, where $\phi(x)$ satisfies $\phi(0) = 0$ and $\phi(1) = \gamma$, that is, $\phi(x) = \gamma x$ and $v(x) = u(x) - \gamma x$. Thus, Eqs. (1) and (2) can be equivalently converted into the form:

$$Lv(x) = F(x, (v+\phi)(x), S(v+\phi)(x), T(v+\phi)(x)) - \left(\phi' P + \phi Q\right)(x), \tag{5}$$

with respect to the two-point boundary conditions

$$v(0) = 0,\ v(1) = 0. \tag{6}$$

Theorem 3.1. *The operator* $L : W_2^3[0,1] \to W_2^1[0,1]$ *is bounded and linear.*

Proof. Clearly, $\|Lv\|^2_{W_2^1} \le M\|v\|^2_{W_2^3}$, where $M > 0$. From the definition of $W_2^1[0,1]$, we have $\|Lv\|^2_{W_2^1} = \langle Lv(x), Lv(x)\rangle_{W_2^1} = [(Lv)(0)]^2 + \int_0^1[(Lv)'(x)]^2dx$. By the Schwarz inequality and reproducing properties $v(x) = \langle v(y), R_x^{\{1\}}(y)\rangle_{W_2^3}$, $(Lv)(x) = \langle v(y), (LR_x^{\{1\}})(y)\rangle_{W_2^3}$, and $(Lv)'(x) = \langle v(y), (LR_x^{\{1\}})'(y)\rangle_{W_2^3}$, we get

$$|(Lv)(x)| = \left|\left\langle v(x), (LR_x^{\{1\}})(x)\right\rangle_{W_2^3}\right| \le \left\|LR_x^{\{1\}}\right\|_{W_2^3}\|v\|_{W_2^3} = M_1\|v\|_{W_2^3},$$

$$\left|(Lv)'(x)\right| = \left|\left\langle v(x), (LR_x^{\{1\}})'(x)\right\rangle_{W_2^3}\right| \le \left\|(LR_x^{\{1\}})'\right\|_{W_2^3}\|v\|_{W_2^3} = M_2\|v\|_{W_2^3},$$

where $M_i > 0$, $i = 1, 2$. Thus, $\|Lv\|^2_{W_2^1} = [(Lv)(0)]^2 + \int_0^1[(Lv)'(x)]^2dx \le (M_1^2 + M_2^2)\|v\|^2_{W_2^3}$. The linearity part is obvious. □

To construct an orthogonal function system of $W_2^3[0,1]$; put $\varphi_i(x) = R_{x_i}^{\{2\}}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and L^* is the adjoint operator of L. In other words, $\langle v(x), \psi_i(x)\rangle_{W_2^3} = \langle v(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lv(x), \varphi_i(x)\rangle_{W_2^1} = Lv(x_i)$, $i = 1, 2, \ldots.$ The orthonormal function system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \tag{7}$$

where $\beta_{ij} = \frac{1}{\|\psi_1\|_{W_2^3}}$ for $i = j = 1$, $\beta_{ij} = \frac{1}{\sqrt{\|\psi_i\|_{W_2^3}^2 - \sum_{k=1}^{i-1}(\langle\psi_i(x),\overline{\psi}_k(x)\rangle_{W_2^3})^2}}$ for $i = j \neq 1$, and $\beta_{ij} = -\frac{1}{\sqrt{\|\psi_i\|_{W_2^3}^2 - \sum_{k=1}^{i-1}(c_{ik})^2}} \times \sum_{k=j}^{i-1}\langle\psi_i(x), \overline{\psi}_k(x)\rangle_{W_2^3}\beta_{kj}$ for $i > j$.

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete function system of the space $W_2^3[0,1]$.*

Proof. Clearly, $\psi_i(x)=L^*\varphi_i(x) = \langle L^*\varphi_i(x), R_x^{\{1\}}(y)\rangle_{W_2^3}=\langle\varphi_i(x), L_y R_x^{\{1\}}(y)\rangle_{W_2^1}$ $= L_y R_x^{\{1\}}(y)|_{y=x_i} \in W_2^3[0,1]$, so, $\psi_i(x) = .L_y R_x^{\{1\}}(y)|_{y=x_i}$. For each fixed $v \in W_2^3[0,1]$, let $\langle v(x), \psi_i(x)\rangle_{W_2^3} = 0$, so, $\langle v(x), \psi_i(x)\rangle_{W_2^3} = \langle v(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lv(x), \varphi_i(x)\rangle_{W_2^1} = Lv(x_i) = 0$. But since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, therefore $Lv(x) = 0$. It follows that $v(x) = 0$ from the existence of L^{-1}. $\square$

Theorem 3.3. *For each $v \in W_2^3[0,1]$, $\sum_{i=1}^{\infty}\langle v(x), \bar{\psi}_i(x)\rangle \bar{\psi}_i(x)$ is convergent in the sense of the norm of $W_2^3[0,1]$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the analytical solution of Eqs. (5) and (6) is*

$$\begin{aligned} v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[F(x_k, (v+\phi)(x_k), S(v+\phi)(x_k), T(v+\phi)(x_k)) \\ - (\phi' P + \phi Q)(x_k)]\bar{\psi}_i(x). \end{aligned} \tag{8}$$

Proof. Using Eq. (7), it easy to see that

$$\begin{aligned} v(x) &= L^{-1}F(x, (v+\phi)(x), S(v+\phi)(x_k), T(v+\phi)(x)) - (\phi' P + \phi Q)(x) \\ &= \sum_{i=1}^{\infty}\langle v(x), \bar{\psi}_i(x)\rangle_{W_2^3}\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle v(x), \psi_k(x)\rangle_{W_2^3}\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle v(x), L^*\varphi_k(x)\rangle_{W_2^3}\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lv(x), \varphi_k(x)\rangle_{W_2^1}\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle F(x, (v+\phi)(x), S(v+\phi)(x), T(v+\phi)(x)) \\ &\quad - (\phi' P + \phi Q)(x), \varphi_k(x)\rangle_{W_2^1}\bar{\psi}_i(x) \end{aligned}$$

$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[F(x_k,(v+\phi)(x_k),S(v+\phi)(x_k),T(v+\phi)(x_k)) - (\phi' P+\phi Q)(x_k)]\bar{\psi}_i(x).$$

Hence, Eq. (8) is the analytical solution of Eqs. (5) and (6). □

Let $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ be the normal orthogonal system derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$, then according to Eq. (8), the analytical solution of Eqs. (5) and (6) can be denoted by

$$v(x) = \sum_{i=1}^{\infty} B_i\bar{\psi}_i(x), \tag{9}$$

where $B_i = \sum_{k=1}^{i}\beta_{ik}[F(x_k,(v_{k-1}+\phi)(x_k),S(v_{k-1}+\phi)(x_k),T(v_{k-1}+\phi)(x_k)) - (\phi' P+\phi Q)(x_k)]$. In fact, B_i in Eq. (9) are unknown, we will approximate B_i using known A_i. For a numerical computations, we define the initial function $v_0(x_1) = 0$, put $v_0(x_1) = v(x_1)$, and define the n-term approximation $v_n(x)$ to $v(x)$ as

$$v_n(x) = \sum_{i=1}^{n} A_i\bar{\psi}_i(x), \tag{10}$$

$$A_i = \sum_{k=1}^{i}\beta_{ik}\big[F(x_k,(v_{k-1}+\phi)(x_k),S(v_{k-1}+\phi)(x_k),T(v_{k-1}+\phi)(x_k)) - (\phi' P+\phi Q)(x_k)\big]. \tag{11}$$

Theorem 3.4. *If $||v_n||_{W_2^3}$ is bounded and $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the n-term numerical solution $v_n(x)$ in the iterative formula of Eq. (10) converges to the analytical solution $v(x)$ of Eqs. (5) and (6) in the space $W_2^3[0,1]$ and $v(x) = \sum_{i=1}^{\infty} A_i\bar{\psi}_i(x)$, where A_i is given by Eq. (11).*

Proof. The proof is straightforward. □

If $\delta_n = ||v - v_n||_{W_2^3}$, where $v(x)$ and $v_n(x)$ are given by Eqs. (9) and (10), respectively, then $\delta_n^2 = \|\sum_{i=n+1}^{\infty} A_i\bar{\psi}_i\|_{W_2^3}^2 = \sum_{i=n+1}^{\infty}(A_i)^2$ and $\delta_{n-1}^2 = \|\sum_{i=n}^{\infty} A_i\bar{\psi}_i\|_{W_2^3}^2 = \sum_{i=n}^{\infty}(A_i)^2$. Thus, $\delta_{n-1} \geq \delta_n$, and consequently $\{\delta_n\}$ are monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$. By Theorem 3.3, $\sum_{i=1}^{\infty} A_i\bar{\psi}_i(x)$ is convergent, so, $\delta_n^2 = \sum_{i=n+1}^{\infty}(A_i)^2 \to 0$ or $\delta_n \to 0$ as $n \to \infty$.

4. Numerical example

In order to solve multipoint singular BVPs restricted by Fredholm-Volterra operators numerically and to show behavior, properties, efficiency, and applicability of the present RKHS algorithm, four multipoint singular BVPs restricted by

Fredholm-Volterra operators will be solved numerically in this section. Here, all the symbolic and numerical computations were performed by using MAPLE 13 software package.

Using RKHS algorithm, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$, applying $R_x^{\{1\}}(y)$ and $R_x^{\{2\}}(y)$ on $[0, 1]$. Some tabulate data are presented and discussed quantitatively at some selected grid points on $[0, 1]$ to illustrate the numerical solutions for the following multipoint singular BVPs restricted by the given Fredholm-Volterra operators.

Example 1. Consider the singularities at two endpoint of $[0, 1]$:

$$\begin{aligned} &D^{2\alpha}u(x) + \frac{1}{\sin(x)} D^{\alpha}u(x) - \frac{1}{x(x-1)} u(x) = [Tu](x) + f(x), \\ &[Tu](x) = \int_0^1 x^2 t u(t) dt + \int_0^x (x+1) t u(t) dt, \end{aligned}$$

subject to the three-point boundary conditions

$$\begin{aligned} &u(0) = 0, \\ &u(1) - 4u\left(\tfrac{1}{9}\right) = 0, \end{aligned}$$

where $0 < t < x < 1$. The analytical solution at $\alpha = 1$ is $u(x) = x(x-1)(x-\frac{1}{9})\cos(x)$.

Example 2. Consider the singularities at two endpoint of $[0, 1]$:

$$\begin{aligned} &D^{2\alpha}u(x) - \frac{1}{x^2(1-x)^2} D^{\alpha}u(x) + \frac{1}{\sinh(x)} u(x) \\ &= u^2(x) + \sinh^{-1}(u(x)) + [Tu](x) + f(x), \\ &[Tu](x) = \int_0^1 x t u^3(t)\, dt + \int_0^x (x-t) u^2(t)\, dt, \end{aligned}$$

subject to the three-point boundary conditions

$$\begin{aligned} &u(0) = 0, \\ &u(1) - u\left(\tfrac{1}{2}\right) = 0, \end{aligned}$$

where $0 < t < x < 1$. The analytical solution at $\alpha = 1$ is $u(x) = (x - \frac{1}{2})^2 (x-1)^2 \sinh(x)$.

Our next goal is to illustrate some numerical results of the RKHS solutions of the aforementioned examples in numeric values. In fact, results from numerical analysis are an approximation, in general, which can be made as accurate as desired. Because a computer has a finite word length, only a fixed number of digits are stored and used during computations. Next, the agreement between the analytical-numerical solutions is investigated for Examples 1 and 2 at $\alpha = 1$

and various x in $[0, 1]$ by computing the absolute errors and the relative errors of numerically approximating their analytical solutions for the corresponding equivalent equations as shown in Table 1, and Table 2, respectively. Anyhow, it is clear from the tables that, the numerical solutions are in close agreement with the analytical solutions for all examples, while the accuracy is in advanced by using only few tens of the RKHS iterations. Indeed, we can conclude that higher accuracy can be achieved by computing further RKHS iterations.

Table 1. The analytical-numerical solutions and errors at $\alpha = 1$ for Example 1.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.16	−0.00648674140330012	−0.00648836153953347	1.62014×10^{-6}	2.49761×10^{-4}
0.32	−0.04314675763472329	−0.04314611857352801	6.39061×10^{-7}	1.48113×10^{-5}
0.48	−0.08166976184992833	−0.08166610925390749	3.65260×10^{-6}	4.47240×10^{-5}
0.64	−0.09774018067274835	−0.09773862645916864	1.55421×10^{-6}	1.59015×10^{-5}
0.80	−0.07679256174137644	−0.07679374066742663	1.17893×10^{-6}	1.53521×10^{-5}
0.96	−0.01869522215933257	−0.01869699234994678	1.77019×10^{-6}	9.46868×10^{-5}

Table 2. The analytical-numerical solutions and errors at $\alpha = 1$ for Example 2.

x	Exact solution	Numerical solution	Absolute error	Relative error
0.16	0.01310653223586577	0.01310629969713650	2.32539×10^{-7}	1.77422×10^{-5}
0.32	0.00487640352847436	0.00487621848205233	1.85046×10^{-7}	3.79473×10^{-5}
0.48	0.00005393349784172	0.00005386640494187	6.70929×10^{-8}	1.24399×10^{-3}
0.64	0.00173897887325904	0.00173902449076913	4.56175×10^{-8}	2.62324×10^{-5}
0.80	0.00319718153587544	0.00319727450018004	9.29643×10^{-8}	2.90770×10^{-5}
0.96	0.00037729187128320	0.00037738952445887	9.76532×10^{-8}	2.58827×10^{-4}

5. Concluding remarks

In this work, we have used the reproducing kernel algorithm for solving linear and nonlinear second-order, three-point singular BVPs restricted by Fredholm-Volterra operators. In the meantime, we employed our algorithm and its conjugate operator to construct the complete orthonormal basis in the reproducing kernel space $W_2^3[0, 1]$. By separating the multipoint boundary conditions and adding the initial and boundary conditions to the reproducing kernel space that satisfying these points, we obtain the analytical-numerical solutions of the problem. The algorithm is applied in a direct way without using linearization, perturbation, or any restrictive assumptions. It may be concluded that RKHS algorithm is very powerful and efficient in finding the analytical-numerical solutions for a wide class of multipoint singular BVPs. It is worth mentioning here that the algorithm is capable of reducing the volume of the computational work and complexity while still maintaining the high accuracy of the numerical results.

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