Classes of weighted tent function spaces and mixed norms with some applications

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Abstract. In this paper, some new definitions for weighted classes of analytic functions are introduced. Moreover, certain properties are presented for functions belonging to the defined classes in the unit disk. Besides, a class of weighted tent functions is also considered. Furthermore, some properties for identity operator are studied for the new tent function spaces.
Keywords: mixed norms, tent functions, Q_p-functions.

1. Introduction and preliminaries

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. Assume that $H(\Delta)$ is the class of all holomorphic functions f on Δ.

An interesting class of analytic functions, which called Bloch space (see [10, 12]) and it is defined by:

$$\mathcal{B} = \{f : f \text{ analytic in } \Delta \text{ and } \sup_{z\in\Delta}(1 - |z|^2)|f'(z)| < \infty\}.$$

For more discussions on Bloch-type classes in $\mathbb{C}$, we may refer to [10,12,20,27,28,29] and the cited references therein.

Analytic Q_p-spaces are introduced by Aulaskari and Lappan (see [12]) as follows:

$$Q_p = \left\{f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta}\int_\Delta \left|f'(z)\right|^2 g^p(z,a)dA(z) < \infty\right\},$$

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where $0 < p < \infty$ and $dA(z) = dxdy$. Also, the function $g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$ defines the Green's function in Δ. For more details on analytic Q_p spaces, we can refer to [22, 29].

In [30] Zhao defined $F(p,q,s)$ and $F_0(p,q,s)$ classes as follows:

Definition 1.1. *Let $f \in H(\Delta)$. Assume that $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If*

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a|\to 1} \int_\Delta |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) = 0,$$

then $f \in F_0(p,q,s)$.

The classes $F(p,q,s)$ were discussed by Zhao in [30] and Rättyä in [24]. From ([30], Theorem 2.10), it is clear that, for $p \ge 1$, the classes $F(p,q,s)$ are Banach spaces using the norm

$$\|f\| = \|f\|_{F(p,q,s)} + |f(0)|.$$

The important various studies on $F(p,q,s)$ classes can be found in [20, 21, 30].

In the present article, ω stands for a reasonable function, which means that an analytic function satisfying some natural conditions.

Definition 1.2 ([25, 26])**.** *Let $\omega : (0,1] \to (0,\infty)$ and let $0 < \alpha < \infty$. Suppose that $f \in H(\Delta)$, then f is said to belong to the α,ω-Bloch class $\mathcal{B}^\alpha_\omega$ if*

$$\mathcal{B}^\alpha_\omega(f) = \|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{z\in\Delta} \frac{(1-|z|)^\alpha}{\omega(1-|z|)} |f'(z)| < \infty.$$

Definition 1.3 ([25, 26])**.** *Let $\omega : (0,1] \to (0,\infty)$. Suppose that $0 < q < \infty$ and $0 < p < \infty$. Suppose that $f \in H(\Delta)$, then*

$$\begin{aligned} f \in B^{p,q}_\omega \iff B^{p,q}_\omega(f) &= \|f\|^q_{B^{p,q}_\omega} \\ &= \sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^q \left(1-|z|^2\right)^{q-p} \frac{\left(1-|\varphi_a(z)|^2\right)^p}{\omega^q(1-|z|)} dA(z) < \infty. \end{aligned}$$

In the next definition, we clear one of the motivations for the present article.

Definition 1.4. *Let $\omega : (0,1] \to (0,\infty)$. Suppose that $0 < q < \infty$ and $0 < p < \infty$. The function $f \in H(\Delta)$ is said to belong to the $B^q(p,\omega,\varphi)$-class if*

$$\|f\|^q_{B^q(p,\omega,\varphi)} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^q \frac{(1-|\varphi_a(z)|^2)^p}{\omega^q(1-|z|)} dA(z) < \infty.$$

For some results in this article, we will use the following:

$$(1)\qquad \big(1-|\varphi_a(z)|^2\big)=\frac{\big(1-|a|^2\big)\big(1-|z|^2\big)}{|1-\bar{a}z|^2},$$

and

$$(2)\qquad 1-|z|\leq|1-\bar{a}z|\leq 1+|z| \ \text{ and also } \ 1-|a|\leq|1-\bar{a}z|\leq 1+|a|.$$

Two equivalent quantities L_f and M_f, which are depending on $f\in H(\Delta)$, and we write $L_f\approx M_f$, is we get a constant $C>0$, such that

$$\frac{1}{C}M_f\leq L_f\leq C\,M_f.$$

The symbole $A\lesssim B$ can be used instead of the inequality $A\leq C_1B$, where C_1 is a constant and $C_1>0$.

2. Some weighted analytic functions

Certain important properties of analytic $B^{p,q}_\omega$ classes in Δ will be considered in this section. Essential relations connecting between mixed norms of weighted $B^{p,q}_\omega$-type classes and the norms of weighted $Q_{p,\omega}$-type classes are considered.

Proposition 2.1. *$f\in H(\Delta)$ and let $f\in\mathcal{B}_\omega$. Then for $0\leq p<\infty$ and $0<q<\infty$ with $0\leq\frac{q-2}{2}<\infty$ and $\frac{q}{2}-1<p$, we have that*

$$\int_\Delta\big|f'(z)\big|^q\big(1-|z|^2\big)^{q-p}\frac{\big(1-|\varphi_a(z)|^2\big)^p}{\omega^q(1-|z|)}dA(z)\leq 4\pi\lambda\mathcal{B}^q_\omega(f).$$

Proof. Since (see [25]),

$$\frac{(1-|z|^2)}{\omega(1-|z|)}\big|f'(z)\big|\leq\mathcal{B}_\omega(f).$$

Then,

$$\begin{aligned}&\int_\Delta\big|f'(z)\big|^q\big(1-|z|^2\big)^{q-p}\frac{\big(1-|\varphi_a(z)|^2\big)^p}{\omega^q(1-|z|)}dA(z)\\ &\leq\mathcal{B}^{p,q}_\omega(f)\int_\Delta\big(1-|z|^2\big)^{-p}\big(1-|\varphi_a(z)|^2\big)^p dA(z)\\ &=\mathcal{B}^{p,q}_\omega(f)\int_\Delta\big(1-|\varphi_a(z)|^2\big)^{-p}\big(1-|z|^2\big)^p\frac{\big(1-|a|^2\big)^2}{|1-\bar{a}z|^4}dA(z),\end{aligned}$$

where, the Jacobian determinant is

$$(3)\qquad J_1=\frac{\big(1-|a|^2\big)^2}{|1-\bar{a}z|^4}$$

stands for Jacobian determinant. For $0 \le p < 2$, applying equality (1), we obtain that,

$$\int_{\Delta} |f'(z)|^q \big(1-|z|^2\big)^{q-p} \frac{\big(1-|\varphi_a(z)|^2\big)^p}{\omega^q(1-|z|)} dA(z)$$
$$\le \mathcal{B}^q_\omega(f) \int_{\Delta} \frac{\big(1-|a|^2\big)^{(2-p)}}{|1-\bar{a}ry|^{2(2-p)}} d\Gamma = 4\pi\lambda \mathcal{B}^q_\omega(f).$$

When $2 \le p < \infty$, we can prove the Proposition 2.1, in view of the inequality

$$1-|a| \le |1-\bar{a}r| \le 1+|a|.$$

Hence, Proposition 2.1 is completely proved.

Now, in view of Proposition 2.1, we can give the following corollary:

Corollary 2.1. *For $0 \le p < \infty$ and $0 < q < \infty$, we have*

$$\mathcal{B}_\omega \subset B^{p,q}_\omega.$$

Relationships between weighted $Q_{p_1,\omega}$ classes and $B^{p,q}_\omega$ classes are described in the following result.

Theorem 2.1. *Let $0 < q < 2$ and $2 < p_1 < 4-q$. Then,*

$$\cup_{p_1} Q_{p_1,\omega} \subset \cap_{p,q} B^{p,q}_\omega.$$

Proof. Let $f \in Q_{p_1,\omega}$ for $2 < p_1 < 4-q$ and $0 < q < 2$. By Hölder's inequality, we deduce

$$\int_{\Delta} |f'(z)|^q \big(1-|z|^2\big)^{q-p} \frac{\big(1-|\varphi_a(z)|^2\big)^p}{\omega^q(1-|z|)} dA(z)$$
$$\le \left\{ \int_{\Delta} \left[\frac{|f'(z)|^q}{\omega^q(1-|z|)} \big(1-|\varphi_a(z)|^2\big)^{\frac{qp_1}{2}} \right]^{\frac{2}{q}} dA(z) \right\}^{\frac{q}{2}}$$
$$\times \left\{ \int_{\Delta} \left[\big(1-|z|^2\big)^{q-p} \big(1-|\varphi_a(z)|^2\big)^{p-\frac{qp_1}{2}} \right]^{\frac{2}{2-q}} dA(z) \right\}^{\frac{2-q}{2}}$$
$$= \left\{ \int_{\Delta} |f'(z)|^2 \frac{\big(1-|\varphi_a(z)|^2\big)^{p_1}}{\omega^2(1-|z|)} dA(z) \right\}^{\frac{q}{2}}$$
$$\times \left\{ \int_{\Delta} \big(1-|z|^2\big)^{\frac{2(q-p)}{2-q}} \big(1-|\varphi_a(z)|^2\big)^{\frac{2p-qp_1}{2-q}} dA(z) \right\}^{\frac{2-q}{2}} \tag{4}$$

Because,

$$f \in Q_{p_1,\omega} \Longleftrightarrow \sup_{a\in\Delta} \int_{\Delta} |f'(z)|^2 \frac{\big(1-|\varphi_a(z)|^2\big)^{p_1}}{\omega^2(1-|z|)} dA(z) < \infty \quad \text{(see [25]).}$$

Then, by equality (1) and change z by $\varphi_a(z)$, we get

$$\int_{\Delta} |f'(z)|^q (1-|z|^2)^{q-p} \frac{(1-|\varphi_a(z)|^2)^p}{\omega^q(1-|z|)} dA(z)$$

$$\leq k\|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}} \left\{ \int_{\Delta} \frac{(1-|a|^2)^{2(\frac{q-p}{2-q}+1)}(1-|z|^2)^{2(\frac{q-p}{2-q})}}{|1-\bar{a}z|^{4+4\left(\frac{q-p}{2-q}\right)}} dA(z) \right\}^{\frac{2-q}{2}}$$

$$\leq k\|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}} \left\{ (1-|a|^2)^{(\frac{4-2p}{2-q})} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{4-2p}{2-q})}{n!\,\Gamma(\frac{4-2p}{2-q})} |a|^{2n} \right\}^{\frac{2-q}{2}}$$

$$= k\|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}} \left\{ \frac{2-q}{q-2p+1} \right\}^{\frac{2-q}{2}}, \tag{5}$$

where the constant $k > 0$. Hence,

$$\|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}} = \int_{\Delta} |f'(z)|^2 \frac{(1-|\varphi_a(z)|^2)^{p_1}}{\omega^2(1-|z|)} dA(z).$$

$$\Longrightarrow \int_{\Delta} |f'(z)|^q (1-|z|^2)^{q-p} \frac{(1-|\varphi_a(z)|^2)^p}{\omega^q(1-|z|)} dA(z) \leq k_1 \|f\|^{\frac{q}{2}}_{Q_{p_1,\omega}},$$

where the the constant $k_1 > 0$. Then,

$$\| f \|_{B^{p,q}_{\omega}} \leq \| f \|_{Q_{p_1,\omega}} < \infty.$$

Therefore $f \in B^{p,q}_{\omega}$, where $0 < q < 2$ and $2 < p_1 < 4-q$. The proof of the theorem is therefore established.

3. Mixed norms

The mixed norm space $H_{p,q,\gamma}(\Delta)$, $0 < p, q < \infty$ and $-1 < \gamma < \infty$, consists of all $f \in H(\Delta)$ such that (see [18, 21])

$$\|f\|^q_{p,q,\gamma} = \int_0^1 M_p^q(f,r)(1-r)^{\gamma} dr < \infty,$$

where

$$M_p(f,r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

Now, let $p, q, \alpha > 0$, $f \in H(\Delta)$, then

$$f \in H(p,q,\alpha) \Longleftrightarrow \|f\|^q_{p,q,\alpha} = \frac{1}{2\pi} \int_0^1 (1-r)^{\alpha q - 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{q}{p}} dr < \infty$$

The above definition appeared in (see [19]). $H(p,q,\alpha)$ class was studied and discussed by Flett (see [16, 17]). It should be mentioned that the class $H(p,q,\alpha)$

an interesting class of functions, which contains some known classes such analytic Hardy and analytic Bergman type classes.

One of our main aims in the present article is to introduce the following definition for the weighted mixed norm spaces, then we study some of its important properties.

Definition 3.1. *Suppose that* $\omega : (0,1] \to (0,\infty)$ *and* $0 < p, q, \alpha < \infty$, *and* $f \in H(\Delta)$, *then*

$$f \in H(p,q,\alpha,\omega) \Leftrightarrow \|f\|^q_{p,q,\alpha,\omega} = \frac{1}{2\pi}\int_0^1 \frac{(1-r)^{\alpha q-1}}{\omega(1-r)}\left(\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{\frac{q}{p}} dr < \infty.$$

Theorem 3.1. *Let* $0 < q < \infty$ *and* $0 \le p < \infty$, $f \in H(\Delta)$. *Suppose that* $f'(\varphi_a(w))$ *is a nondecreasing function. Let* $\omega : (0,1] \to (0,\infty)$, *then*

$$f \in B^{p,q}_{\omega} \Leftrightarrow \sup_{a\in\Delta}\int_\Delta |f'(z)|^q(1-|z|^2)^{q-p}\frac{\big(g(z,a)\big)^p}{\omega^q(1-|z|)}dA(z) < \infty.$$

Proof. Let us consider the equivalence

$$\begin{aligned}&\int_\Delta |f'(z)|^q(1-|z|^2)^{q-p}\frac{(1-|\varphi_a(z)|^2)^p}{\omega^q(1-|z|)}dA(z)dA(z)\\ &\approx \int_\Delta |f'(z)|^q(1-|z|^2)^{q-p}\frac{\big(g(z,a)\big)^p}{\omega^q(1-|z|)}dA(z),\end{aligned}$$

The change of variables $w = \varphi_a(z)$, resulting that

$$\begin{aligned}&\int_\Delta |f'(\varphi_a(w))|^q(1-|\varphi_a(w)|^2)^{q-p}\frac{(1-|w|^2)^p}{\omega^q(1-|\varphi_a(w)|)}\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^2 dA(w)\\ &\approx \int_\Delta |f'(\varphi_a(w))|^q(1-|\varphi_a(w)|^2)^{q-p}\frac{\log\left(\frac{1}{|w|}\right)}{\omega^q(1-|\varphi_a(w)|)}\left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^2 dA(w).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\Delta |f'(\varphi_a(w))|^q\frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\ &\approx \int_\Delta |f'(\varphi_a(w))|^q\frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w),\end{aligned}$$

we aim to evaluate constants $C_1(p)$ and $C_2(p)$ with

$$\begin{aligned}&C_1(p)\int_\Delta |f'(\varphi_a(w))|^q\frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\ &\le \int_\Delta |f'(\varphi_a(w))|^q\frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\ &\le C_2(p)\int_\Delta |f'(\varphi_a(w))|^q\frac{\log\left(\frac{1}{|w|}\right)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w).\end{aligned}$$

Case 1. Let $C_2(p)=2^p$. Using,

$$1-|a|\le|1-\bar{a}w|\le 1+|a| \quad \text{and} \quad 1-|w|\le|1-\bar{a}w|\le 1+|w|, \tag{6}$$

we obtain,

$$\begin{aligned}
&\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\
&-2^p\int_\Delta |f'(\varphi_a(w))|^q \frac{\log\big(\frac{1}{|w|}\big)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\
&=(1-|a|^2)^{2-p}\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^q\ L(\log,|w|,p,q)}{|1-\bar{a}w|^{2q}\omega^q(1-|\varphi_a(w)|)}\,dA(w),
\end{aligned}$$

where

$$L(\log,|w|,p,q)=\frac{|1-\bar{a}w|^{2q}(1-|w|^2)^{-p}}{|1-\bar{a}w|^{q-p+4}}\Big[(1-|w|^2)^{-p}-2^p\log\big(\frac{1}{|w|}\big)\Big].$$

hence, after some simple computation, we deduce

$$\begin{aligned}
&\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\
&-2^p\int_\Delta |f'(\varphi_a(w))|^q \frac{\log\big(\frac{1}{|w|}\big)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\
&\le 2^{5+q-p}\pi\|f\|^q_{\mathcal{B}_\omega}(1-|a|^2)^{2-p}\int_0^1(1-r)^{-(p+q)}\big[(1-r)^{2-p}+2^p\log r\big]rdr
\end{aligned}$$

since, the last integral exists for all $0\le\frac{q-2}{2}<\infty$ and $\frac{q}{2}-1<p$.

Also, $f\in\mathcal{B}_\omega$, then

$$\begin{aligned}
&\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\
&\le C_1(p)\ \int_\Delta |f'(\varphi_a(w))|^q \frac{\log\big(\frac{1}{|w|}\big)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w).
\end{aligned}$$

Case 2. Let $C_1(p)=\big(\frac{11}{100}\big)^p$. Then,

$$\begin{aligned}
I_2=&\int_\Delta |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w)\\
&-C_2(p)\int_\Delta |f'(\varphi_a(w))|^q \frac{\log\big(\frac{1}{|w|}\big)(1-|w|^2)^{q-p}(1-|a|^2)^{q-p+2}}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)}dA(w),
\end{aligned}$$

which implies that

$$\begin{aligned} I_2 &= \int_{\Delta} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}G(\log,|w|,p)dA(w)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} \\ &= \int_{\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}G(\log,|w|,p)dA(w)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} \\ &+ \int_{\Delta\setminus\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}G(\log,|w|,p)dA(w)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} \\ &= J_1 + J_2. \end{aligned}$$

where

$$G(\log,|w|,p) = \left\{(1-|w|^2)^{-p} + \left(\frac{11}{100}\right)^p \left(\frac{\log|w|}{(1-|w|^2)}\right)^p\right\}.$$

Since $G(\log,|w|,p) \leq 0$; $\forall |w| \in [0, \frac{1}{10}]$, then using (2), we obtain that

$$\begin{aligned} J_1 &= \int_{\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q(1-|a|^2)^{q-p+2}G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\ &\geq k_2\ (1-|a|^2)^{q-p+4} \int_{\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)\ dA(w)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} \end{aligned}$$

and

$$\begin{aligned} J_2 &= (1-|a|^2)^{q-p+2} \int_{\Delta\setminus\Delta_{\frac{1}{10}}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \\ &\geq k_3(1-|a|^2)^{q-p+4} \int_{\frac{1}{10}}^1 |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w), \end{aligned}$$

where the constants k_2 and k_3 are positive.

Since, $\big|f'(\varphi_a(w))|^q \geq 0$; and $G(\log,|w|,p) \leq 0$; $\forall |w| = r \in [0, \frac{1}{10}]$.

Now, we want to compare the integral

$$k_2(1-|a|^2)^{q-p+2} \int_0^{\frac{1}{10}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w),$$

and the integral $k_3(1-|a|^2)^{q-p+4} \int_{\frac{5}{10}}^{\frac{6}{10}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w)$.

After simple calculation, we can obtain that

$$\begin{aligned} &k_2(1-|a|^2)^{q-p+2} \int_0^{\frac{1}{10}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w), \\ &< k_3(1-|a|^2)^{q-p+4} \int_{\frac{5}{10}}^{\frac{6}{10}} |f'(\varphi_a(w))|^q \frac{(1-|w|^2)^q G(\log,|w|,p)}{|1-\bar{a}w|^{q-p+4}\omega^q(1-|\varphi_a(w)|)} dA(w) \end{aligned}$$

Since from the assumptions, we have that, $f'(\varphi_a(w))$ is a nondecreasing function, $\forall 0 \leq |w| < 1$. Thus,

$$I_2 = J_1 + J_2 \geq 0.$$

The proof of Theorem 3.1 is finished.

Theorem 3.2. *Let $f \in H(\Delta)$. Suppose that $\omega : (0,1] \to (0,\infty)$. Let $0 < p \leq \infty$, and $0 < \alpha, q < \infty$. Then, the function*

$$f(z) = \frac{\omega(1-|z|)}{(1-|z|)^{\gamma}}$$

belongs to the weighted mixed norm space

$$H(p,q,\alpha,\omega) \Longleftrightarrow \gamma < 1 + \alpha$$

and

$$f \in H(p,\infty,\alpha,\omega) \Longleftrightarrow \gamma \leq \frac{1}{p} + \alpha.$$

Proof. Using similar steps to the corresponding result in [11], with some simple modifications, we can easily establish the proof of Theorem 3.2.

4. Logarithmic tent spaces

In this section, we introduce weighted $(p,q;\ln,\omega)$-Carleson measures on the unit disk Δ. Then we study $(p,q;\ln,\omega)$-Carleson measures for the weighted classes of Bloch-type and $B^q(p,\ln,\omega,\varphi)$ type-spaces. Moreover, we define the conformally invariant Bloch space and the modified Möbius-invariant seminorms. Boundedness (resp., compactness) for the $B^q(p,\ln,\omega,\varphi)$ classes, which contained in the weighted tent-type space $\mathcal{T}_p^{\infty}(\mu_{\ln,\omega,p,q},q)$ are also discussed.

For more studies about tent spaces, we refer to [14,15,23].

Definition 4.1. *For $0 \leq p < \infty$, $0 < q < \infty$. Let $\omega : (0,1] \to (0,\infty)$, we will call that the weighted positive measure μ defined on Δ is a bounded weighted $(p,q;\ln,\omega)$-Carleson measure provided,*

$$\mu_{\ln,\omega,p,q}(S(I,\omega)) = O\left(\frac{|\, I \,|^p \ln(1+|I|)}{\omega^q(|I|)}\right), \tag{7}$$

for all subarcs I of $\partial\Delta$ where, $|\, I \,|$ denotes the arc length of $I \subset \partial\Delta$ and $S(I)$ defines the Carleson box, which is based on I, that is,

$$S(I) = \{z \in \Delta : z/|z| \in I,\ 1 - |z| \leq |I|/2\pi\}.$$

Remark 4.1. When $p = 1, \omega \equiv 1$ and $|I| = e - 1$, then the usual standard definition of Carleson measure is obtained.

If $\omega \equiv 1$ and $|I| = e-1$, then the p-Carleson measure concept is also deduced.

Definition 4.2. *In Definition 4.1, if the right side of (7) becomes* $o(\frac{|I|^p \ln(1+|I|)}{\omega^q(|I|)})$ *as* $|I| \to 0$, *then we get the definition of weighted compact* $(p, q; \ln, \omega)$*-Carleson measure.*

Let $B^q(p, \ln, \omega, \varphi)$ be a class of all analytic functions f on Δ satisfying

$$\|f\|^q_{B^q(p,\ln,\omega,\varphi)} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^q \frac{(1-|\varphi_a(z)|^2)^p \, \ln(2-|z|)}{\omega^q(1-|z|)} \, dA(z) < \infty,$$

where $0 < p < \infty$ and $0 < q < \infty$. Meanwhile, $\mathcal{T}_p^\infty(q, \mu_{\ln,\omega,p,q})$ denotes the weighted tent-type class of all $\mu_{\ln,\omega,p,q}$-measurable functions f on Δ satisfying

$$\|f\|^q_{\mathcal{T}_p^\infty(\mu_{\ln,\omega,p,q})} = \sup_{S(I)\subseteq\Delta} |I|^{-1} \int_{S(I)} |f|^q \, d\mu_{\ln,\omega,p,q} < \infty\,;$$

where

$$|I| = (2\pi)^{-1} \int_I |d\xi| \quad \text{and} \quad S(I) = \{r\xi \in \Delta : 1-|I| \le r < 1\,, \xi \in I\}$$

are the normalized length of the subarc I of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and the Carleson square in Δ respectively. The norm of $f \in B^q(p, \ln, \omega, \varphi)$ is given by:

$$\sup_{S(I)\subseteq\Delta} |I|^{-p} \int_{S(I)} |f'(z)|^q \frac{(1-|z|^2)^p \, \ln(2-|z|)}{\omega^q(1-|z|)} \, dA(z).$$

Now, we give the following results:

Theorem 4.1. *Let* $\mu_{\ln,\omega,p,q}$ *be a nonnegative Borel measure on* Δ. *Then the identity operator* $I : B^q(p, \ln, \omega, \varphi) \longrightarrow \mathcal{T}_p^\infty(q, \mu_{\ln,\omega,p,q})$ *is bounded* $\Longleftrightarrow$

$$\|\mu_{\ln,\omega,p,q}\|^q_{\mathcal{LCM}_p} = \sup_{S(I)\subseteq\Delta} \frac{\mu_{\ln,\omega,p,q}(S(I))}{|I|^p (\log \frac{2}{|I|})^{-q}} < \infty.$$

Proof. Necessity. Given a subarc I of $\mathbb{T}$. If $f_{a,\ln,\omega}(z) = \frac{(1-\bar{a}z)\ln(2-|z|)}{\omega^q(1-|z|)}$ where $a = (1-|I|)\xi$ and ξ is the center of I, then

$$|f_{a,ln,\omega}(z)| \approx \frac{\ln(2|I|^{-p})\ln(1+|I|}{\omega^q(|I|^{-p})}, \qquad z \in S(I)$$

and

$$|I|^{-p} \int_{S(I)} |f_{a,\ln,\omega}|^q \, d\mu_{\ln,\omega,p,q} \le \|f_{a,\ln,\omega}\|^q_{B^q(p,\ln,\omega,\varphi)} \lesssim 1.$$

Accordingly, $\|\mu_{\ln,\omega,p,q}\|_{\mathcal{LCM}_p} \lesssim 1.$

Sufficiency. Assume that a nonnegative Borel measure $\mu_{\ln,\omega,p,q}$ on Δ is said to be weighted $(p,q;\ln,\omega)$-Carleson measures on the unit disk Δ for the weighted class $B^q(p,\ln,\omega,\varphi)$ of all $f \in \Delta$ which satisfying

$$\|f\|^q_{B^q(p,\ln,\omega,\varphi)} = \int_\Delta |f'(z)|^q \frac{(1-|z|^2)^p \ \ln(2-|z|)}{\omega^q(1-|z|)} \, dA(z) < \infty,$$

provided

$$\int_\Delta |f|^q \, d\mu_{\ln,\omega,p,q} \lesssim \|f\|^q_{B^q(p,\ln,\omega,\varphi)}.$$

This completes the proof.

Corollary 4.1. *Let $\mu_{\ln,\omega,p,q}$ be a nonnegative Borel measure on Δ. Then the identity operator $I_1 : B^q(p,\ln,\omega,\varphi) \longrightarrow \mathcal{T}^\infty_p(q,\mu_{\ln,\omega,p,q}) \Longleftrightarrow$*

$$\lim_{|I|\longrightarrow 0} \frac{\mu_{\ln,\omega,p,q}(S(I))}{|I|^p(\log\frac{2}{|I|})^{-q}} = 0.$$

Remark 4.2. An interesting and important question can be stated as follows:

Is the concept of analytic Tent function can be generalized using quaternion-functions?

For more information on several studied and various discussions using Clifford analysis, we can refer to the citations [1, 2, 3, 4, 5, 6, 7, 8, 9, 13] and others.

5. Conclusion

This paper starts with a concise overview of weighted function spaces in the sense of analytic functions.

New tools are used in studying some new weighted function spaces in Δ. Properties of weighted mixed normed spaces, which are generalizations of the so-called mixed normed spaces are introduced. Several inclusion/comparison results among these spaces are presented. Also we dealt with some properties of certain Carleson measures (which have introduced) and their relation with the Bloch-type spaces.

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