On uniformly primary hyperideals and uniformly 2-absorbing primary hyperideals

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Abstract. Let R be a multiplicative hyperring. In this paper, we introduce the concepts of uniformly primary hyperideal and uniformly 2- absorbing primary hyperideal of R, which impose a certain boundedness condition on the usual notions of primary hyperideal and 2- absorbing primary hyperideal, respectively. We will show some properties of them.
Keywords: 2-absorbing primary hyperideal, uniformly primary hyperideal, uniformly 2-absorbing primary hyperideal, Noether strongly 2-absorbing primary hyperideal, special 2-absorbing primary hyperideal.

1. Introduction

The theory of algebraic hyperstructures was introduced in 1934 by Marty [8] during the 8^{th} Congress of Scandinavian Mathematicians. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non commutative groups. Later on, many researchers have observed that the theory of hyperstructures also have many applications in both pure and applied sciences. A comprehensive review of this theory can be found in [3],[6],[9], [4] and [11]. The notion of multiplicative hyperring was introduced by R. Rota [10] in 1982. For example, applications of hyperstructures in chemistry and physics can be studied in Chapter 8, [6].

A triple $(R, +, \circ)$ is called a multiplicative hyperring if
(1)$(R, +)$ is an abelian group;
(2)$(R, \circ)$ is semihypergroup;
(3) for all $a, b, c \in R$, we have $a\circ(b+c) \subseteq a\circ b+a\circ c$ and $(b+c)\circ a \subseteq b\circ a+c\circ a$;
(4) for all $a, b \in R$, we have $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

For any two nonempty subsets A and B of R and $x \in R$, we define

$$A \circ B = \bigcup_{a\in A,\ b\in B} a \circ b, \qquad A \circ x = A \circ \{x\}$$

A non empty subset I of a multiplicative hyperring R is a hyperideal if

(1) If $a, b \in I$, then $a - b \in I$;

(2) If $x \in I$ and $r \in R$, then $r \circ x \subseteq I$.

The concept of 2-absorbing hyperideal was introduced in [7]. Really, it is a generalization of prime hyperideal. Precisely, a nonzero proper hyperideal I of a multiplicative hyperring R is called to be 2-absorbing if $x \circ y \circ z \subseteq I$ where $x, y, z \in R$, then $x \circ y \subseteq I$ or $y \circ z \subseteq I$ or $x \circ z \subseteq I$.

In this paper, we introduce the concepts of uniformly primary hyperideal and uniformly 2-absorbing primary hyperideal of R, which impose a certain boundedness condition on the usual notions of primary hyperideal and 2- absorbing primary hyperideal, respectively.

Among many results in this paper, it is shown (Theorem 3.6) that hyperideal Q of R is a uniformly P-primary hyperideal if and only if:

(1) Q is a P-primary hyperideal of R, and

(2) there exists a positive integer n such that $P = \{x \in R \mid x^n \subseteq Q\}$.

Moreover, $ord_H(Q) = k$ if and only if k is the smallest positive integer for which condition (2) holds. It is shown (Theorem 4.9) that if R_1 and R_2 be multiplicative hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good homomorphism. Then the following statements hold:

(1) If Q_2 is a uniformly 2-absorbing primary hyperideal of R_2, then $\phi^{-1}(Q_2)$ is a uniformly 2-absorbing primary hyperideal of R_1 with $2\_ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq 2\_ord_{H_{R_2}}(Q_2)$.

(2) If ϕ is an epimorphism and Q_1 is a uniformly 2-absorbing primary hyperideal of R_1 containing $ker(\phi)$, then $\phi(Q_1)$ is a uniformly 2-absorbing primary hyperideal of R_2 with $2\_ord_{H_{R_2}}(\phi(Q_1)) \leq 2\_ord_{H_{R_1}}(Q_1)$.

It is shown (Theorem 5.5) that if Q is a Noether strongly 2-absorbing primary hyperideal of R, then Q is a uniformly 2-absorbing primary hyperideal of R and $2\_ord_H(Q) \leq 2\_\mathfrak{e}_H(Q)$.

2. Preliminaries

Definition 2.1 ([5]). A nonzero proper hyperideal P of R is called a *prime hyperideal* if $x \circ y \subseteq P$ for $x, y \in R$ implies that $x \in P$ or $y \in P$. The intersection of all prime hyperideals of R containing I is called the prime radical of I, being denoted by $r(I)$. If the multiplicative hyperring R does not have any prime hyperideal containing I, we define $r(I) = R$.

Definition 2.2 ([5]). Let $\mathbf{C}$ be the class of all finite products of elements of R i.e. $\mathbf{C} = \{r_1 \circ r_2 \circ ... \circ r_n \ : \ r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A hyperideal I of R is said to be a $\mathbf{C}$-*hyperideal* of R if for any $A \in \mathbf{C}, A \cap I \neq \varnothing \Rightarrow A \subseteq I$.

Theorem 2.3 ([5], Proposition 3.2). *Let I be a hyperideal of R. Then, $D \subseteq r(I)$ where $D = \{r \in R : r^n \subseteq I \ for \ some \ n \in \mathbb{N}\}$. The equality holds when I is a $\mathbf{C}$-hyperideal of R.*

In this paper, we assume that all hyperideals are $\mathbf{C}$-hyperideal.

Definition 2.4 ([5])**.** A nonzero proper hyperideal Q of R is called a *primary hyperideal* if for any $x, y \in R$, $x \circ y \subseteq Q$ and $x \notin Q$, then $y^n \subseteq Q$ for some $n \in \mathbb{N}$.

Since $r(Q) = P$ is a prime hyperideal of R by Propodition 3.6 in [5], Q is referred to as a P-primary hyperideal of R.

Definition 2.5 ([1])**.** A nonzero proper hyperideal I of R is called *2-absorbing primary hyperideal* of R if $x \circ y \circ z \subseteq I$ for some $x, y, z \in R$, then $x \circ y \subseteq I$ or $x \circ z \subseteq r(I)$ or $y \circ z \subseteq r(I)$.

Theorem 2.6 ([1], Theorem 4.2)**.** *Let I be a 2-absorbing primary hyperideal of R. Then $P = r(I)$ is a 2-absorbing hyperideal. We say that I is a P-2-absorbing primary hyperideal of R.*

Definition 2.7. Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be multiplicative hyperrings. A mapping from R_1 into R_2 is said to be a *good homomorphism* if for all $x, y \in R_1$, $\phi(x +_1 y) = \phi(x) +_2 \phi(y)$ and $\phi(x \circ_1 y) = \phi(x) \circ_2 \phi(y)$.

Definition 2.8. For $x \in R$, we define $(I :_R x) = \{r \in R \mid r \circ x \subseteq I\}$.

3. Uniformly primary hyperideals

Definition 3.1. Let Q be a proper hyperideal of R. Q is a *uniformly primary hyperideal* of R if there exists a positive integer n such that whenever $x, y \in R$ satisfy $x \circ y \subseteq Q$ and $x \notin Q$ then $y^n \subseteq Q$. If k is the smallest positive integer for which the above property holds ,then it is denoted by $ord_{H_R}(Q) = k$, or simply $ord_H(Q) = k$.

Definition 3.2. P-primary hyperideal Q of R is said to be a *Noether strongly primary hyperideal* if $P^n \subseteq Q$ for some positive integer n. If k is the smallest positive integer for which the above property holds ,then it is denoted by $\mathfrak{e}_{H_R}(Q) = k$, or simply $\mathfrak{e}_H(Q) = k$.

Example 3.3. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers. We define the hyperoperation $a \circ b = \{2ab, 4ab\}$, for all $a, b \in \mathbb{Z}$. The hyperideal $3\mathbb{Z} = \{3n \mid n \in \mathbb{Z}\}$ of the multiplicative hyperring $(\mathbb{Z}, +, \cdot)$ is a Noether strongly primary hyperideal.

Theorem 3.4. *If Q is a Noether strongly P-primary hyperideal of R then Q is a uniformly P-primary hyperideal of R. Also, $ord_H(Q) \leq \mathfrak{e}_H(Q)$.*

Proof. Assume that Q is a Noether strongly P-primary hyperideal of R. Let $x \circ y \subseteq Q$ for some $x, y \in R$ such that $x \notin Q$. Thus $y \in P$ and so $y^{\mathfrak{e}_H(Q)} \subseteq P^{\mathfrak{e}_H(Q)} \subseteq Q$. Hence, Q is a uniformly P-primary hyperideal of R such that $ord_H(Q) \leq \mathfrak{e}_H(Q)$. □

Example 3.5. In Example 3.3, the hyperideal $3\mathbb{Z} = \{3n \mid n \in \mathbb{Z}\}$ of the multiplicative hyperring $(\mathbb{Z}, +, \cdot)$ is a uniformly primary hyperideal.

Theorem 3.6. *Hyperideal Q of R is a uniformly P-primary hyperideal if and only if:*

(1) *Q is a P-primary hyperideal of R.*

(2) *there exists a positive integer n such that $P = \{x \in R \mid x^n \subseteq Q\}$.*

Moreover, $ord_H(Q) = k$ if and only if k is the smallest positive integer for which condition (2) holds.

Proof. $\Longrightarrow$ Let Q be a uniformly P-primary hyperideal of R with $ord_H(Q) = k$. Thus we have condition (1) clearly. Suppose that $x \in P$. So there exists some positive integer t with $x^{t-1} \circ x = x^t \subseteq Q$ such that $x^{t-1} \nsubseteq Q$. Since $ord_H(Q) = k$, we have $x^k \subseteq Q$. Thus the proof is completed.

$\Longleftarrow$ Assume that $x \circ y \subseteq Q$ for some $x, y \in R$ such that $x \notin Q$. Then we have $y \in P$. On the other hand, by (2), there exists a positive integer n with $y^n \subseteq Q$ such that n is independent of y. Thus, Q is a uniformly primary hyperideal of R.

The "moreover" statement follows from the definition of $ord_H(Q)$. $\square$

Theorem 3.7. *Let $Q_1 \subseteq Q_2$ be uniformly P-primary hyperideals of R. Then $ord_H(Q_1) \geq ord_H(Q_2)$.*

Proof. Put $k_1 = ord_(Q_1)$ and $k_2 = ord_H(Q_2)$. Then there exist elements $x, y \in R$ with $x \circ y \in Q_2$ such that $x \notin Q_2$, $y^n \subseteq Q_2$, and $y^{n-1} \nsubseteq Q_2$. Thus, we have $y \in P = r(Q_1)$ and so $y^{k_1} \subseteq Q_1 \subseteq Q_2$. Hence, $k_1 > k_2 - 1$, and then $k_1 \geq k_2$. $\square$

Theorem 3.8. *Let $\{Q_i\}_{i \in I}$ be a collection of uniformly P-primary hyperideals of R such that $max_{i \in I}\{ord_H(Q_i)\} = n$, where n is a positive integer. Then $Q = \bigcap_{i \in I} Q_i$ is a uniformly P-primary hyperideal of R with $ord_H(Q) = n$.*

Proof. By Proposition 3.3 in [5], we have $r(Q) = r(\bigcap_{i \in I} Q_i) = \bigcap_{i \in I} r(Q_i) = P$. Assume that $x \circ y \subseteq Q$ for some $x, y \in R$ such that $x \notin Q$. Thus there exists some $j \in I$ such that $x \circ y \subseteq Qj$ and $x \notin Q_j$. It means $y \in P$ and hence $y^n \subseteq Q$. Thus Q is a uniformly P-primary hyperideal of R with $ord_H(Q_j) \leq n$. Assume that $Q_t \in \{Q_i\}_{i \in I}$ be a uniformly P-primary hyperideal of R with $ord_H(Q) = n$. Hence, by Theorem 3.6, n is the smallest positive integer with $P = \{x \in R \mid x^n \subseteq Q_t\}$. Hence, there exists $x \in P$ but $x^{n-1} \nsubseteq Q_t$, and so $x^{n-1} \nsubseteq Q$. Consequently $ord_H(Q) = n$. $\square$

Theorem 3.9. *Let R_1 and R_2 be multiplicative hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good homomorphism. If Q_2 is a uniformly P-primary hyperideal of R_2, then $\phi^{-1}(Q_2)$ is a uniformly $\phi^{-1}(P)$-primary hyperideal of R_1 with $ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq ord_{H_{R_2}}(Q_2)$.*

Proof. The proof of the first statement is easy. Let Q_2 be a uniformly P-primary hyperideal of R_2 with $k = ord_{H_{R_2}}(Q_2)$. By Theorem 3.6, we have $P = \{y \in R_2 \mid y^k \subseteq Q_2\}$. Hence, $\phi^{-1}(P) = \{x \in R_1 \mid x^k \subseteq \phi^{-1}(Q_2)\}$.Therefore, we conclude that $\phi^{-1}(Q_2)$ is a uniformly $\phi^{-1}(P)$-primary hyperideal of R such that $ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq ord_{H_{R_2}}(Q_2)$. □

Corollary 3.10. Let I and Q be hyperideals of R such that $I \subseteq Q$. Then Q is a uniformly P-primary hyperideal of R with $ord_{H_R}(Q) = k$ if and only if Q/I is a uniformly P/I -primary hyperideal of R/I with $ord_{H_{R/I}}(Q/I) = k$.

Proof. $\Longrightarrow$ It is straightforward.

$\Longleftarrow$ Assume that Q/I is a uniformly P/I -primary hyperideal of R/I with $ord_{H_{R/I}}(Q/I) = k$. By Theorem 3.9, Q is a uniformly P-primary hyperideal of R with $ord_{H_R}(Q) \leq k$. By Theorem 3.6, there exists $x + I \in P/I$ but $(x + I)^{k-1} \nsubseteq Q/I$. Hence, $x \in P$ and $x^{k-1} \nsubseteq Q$. Thus $ord_{H_R}(Q) = k$. □

4. Uniformly 2-absorbing primary hyperideals

Definition 4.1. Let Q be a proper hyperideal of R. Q is a *uniformly 2-absorbing primary hyperidea*l of R if there exists a positive integer n such that whenever $x, y, z \in R$ satisfy $x \circ y \circ z \subseteq Q$, $x \circ y \nsubseteq Q$ and $x \circ z \nsubseteq r(Q)$, then $(y \circ z)^n \subseteq Q$. If k is the smallest positive integer for which the above property holds ,then it is denoted by $2\_ord_{H_R}(Q) = k$, or simply $2\_ord_H(Q) = k$.

Theorem 4.2. *If Q is a 2-absorbing hyperideal of R, then Q is a uniformly 2-absorbing primary hyperideal with $2\_ord_H(Q) = 1$.*

Proof. It is obvious. □

Example 4.3. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers. Corresponding to every subset $A \in P^*(\mathbb{Z}) = P(R)\backslash\{\varnothing\}$ $(|A| \geq 2)$, there exists a multiplicative hyperring $(\mathbb{Z}_A, +, \circ)$ where $\mathbb{Z}_{\mathbb{A}} = \mathbb{Z}$ and for any $x, y \in \mathbb{Z}_{\mathbb{A}}, x \circ y = \{x \cdot a \cdot y \mid a \in A\}$ [5]. In the multiplicative hyperring of integers Z_A with $A = \{5, 7\}$, the principal hyperideals $< 2 >$ and $< 3 >$ are prime hyperideals by Proposition 4.3 in [5]. Hence, hyperideal $< 2 > \cap < 3 >$ is a 2-absorbing hyperideal and so $< 2 > \cap < 3 >$ is a uniformly 2-absorbing primary hyperideal.

Theorem 4.4. *If Q is a uniformly 2-absorbing primary hyperideal of R, then Q is a 2-absorbing primary hyperideal of R with $2\_ord_H(Q) = 1$.*

Proof. It is clear. □

Theorem 4.5. *Let Q be a proper hyperideal of R. If Q is a uniformly 2-absorbing primary hyperideal of R, then one of the following conditions must hold:*

(1) $r(Q) = P$ is a prime hyperideal.

(2) $r(Q) = P_1 \cap P_2$, where P_1 and P_2 are the only distinct prime hyperideals of R that are minimal over Q.

Proof. Apply Theorem 4.5 in [1]. □

Theorem 4.6. *Let Q be a proper hyperideal of R. Then the following conditions are equivalent:*

(1) *Q is uniformly 2-absorbing primary hyperideal.*

(2) *There exists a positive integer n such that for every $x, y \in R$ either $(xoy)^n \subseteq Q$ or $(Q :_R x \circ y) \subseteq (Q :_R x) \cup (r(Q) :_R y)$.*

(3) *There exists a positive integer n such that for every $x, y \in R$ either $(x \circ y)^n \subseteq Q$ or $(Q :_R x \circ y) = (Q :_R x) or (Q :_R x \circ y) \subseteq (r(Q) :_R y)$.*

(4) *There exists a positive integer n such that for every $x, y \in R$ and every hyperideal I of R, $x \circ y \circ I \subseteq Q$ implies that either $x \circ I \subseteq Q$ or $y \circ I \subseteq r(Q)$ or $(x \circ y)^n \subseteq Q$.*

Proof. (1) $\Rightarrow$ (2)Let Q be a uniformly 2-absorbing primary hyperideal of R such that $2\_ord_H(Q) = n$. Suppose that $(x \circ y)^n \nsubseteq Q$ for some $x, y \in R$. Let $a \in (Q :_R x \circ y)$. Hence $aoy \subseteq Q$. Therefore we have $a \circ x \subseteq Q$ or $a \circ y \subseteq r(Q)$. It means $a \in (Q :_R x)$ or $a \in (r(Q) :_R y)$. This implies that $(Q :_R x \circ y) \subseteq (Q :_R x) \cup (r(Q) :_R y)$.

(2)$\Rightarrow$ (3)If an hyperideal is a subset of the union of two hyperideals, then it is a subset of one of them.

(3)$\Rightarrow$ (4) Assume that n is a positive number such that for every $x, y \in R$ either $(x \circ y)^n \subseteq Q$ or $(Q :_R x \circ y) = (Q :_R x) or (Q :_R x \circ y) \subseteq (r(Q) :_R y)$. Assume that I is a hyperideal of R with $x \circ y \circ I \subseteq Q$ for some $x, y \in R$ such that $(x \circ y)^n \nsubseteq Q$. Therefore $I \subseteq (Q :_R x \circ y)$. Hence we have $I \subseteq (Q :_R x)$ or $I \subseteq (r(Q) :_R y)$. Thus $xoI \subseteq Q$ or $y \circ I \subseteq r(Q)$.

(4)$\Rightarrow$ (1) Straightforward. □

Theorem 4.7. *Let Q_1 be a uniformly P-primary hyperideal of R and Q_2 be a uniformly P-2-absorbing primary hyperideal of R such that $Q_1 \subseteq Q_2$. Then $2\_ord_H(Q_2) \leq ord_H(Q_1)$.*

Proof. Assume that $ord_H(Q_1) = k_1$ and $2\_ord_H(Q_2) = k_2$. Thus there are $x, y, z \in R$ with $x \circ y \circ z \subseteq Q_2$ such that $x \circ y \nsubseteq Q_2$, $x \circ z \nsubseteq r(Q_2)$, $(y \circ z)^{k_2} \subseteq Q_2$ and $(y \circ z)^{k_2 - 1} \nsubseteq Q_2$. Hence $y \circ z \subseteq r(Q_2) = r(Q_1)$. Thus $(y \circ z)^{k_1} \subseteq Q_1 \subseteq Q_2$ by Theorem 3.6. Thus $k_2 > k_1 - 1$. Then $k_2 \geq k_1$. □

Theorem 4.8. *Let $\{Q_i\}_{i \in I}$ be a chain of uniformly P-2-absorbing primary hyperideals of R such that $max_{i \in I}\{2\_ord_H(Q_i)\} = n$, where n is a positive integer. Then $Q = \bigcap_{i \in I} Q_i$ is a uniformly P-2-absorbing primary hyperideal of R with $2\_ord_H(Q) \leq n$.*

Proof. By Proposition 3.3 in [5], we have $r(Q) = r(\bigcap_{i \in I} Q_i) = \bigcap_{i \in I} r(Q_i) = P$. Assume that $x \circ y \circ z \subseteq Q$ for some $x, y, z \in R$ such that $x \circ y \nsubseteq Q$ and $(y \circ z)^n \nsubseteq Q$. Since $\{Q_i\}_{i \in I}$ is a chain, there exists some $j \in I$ such that $xoy \nsubseteq Q_j$ and $(y \circ z)^n \nsubseteq Q_j$. Since Q_j is a uniformly 2-absorbing primary hyperideal of R with $2\_ord(Q_j) \leq n$, then $xoz \subseteq r(Q_i) = r(Q)$. Thus Q is a uniformly 2-absorbing primary hyperideal of R with $2\_ord_H(Q) \leq n$. □

Theorem 4.9. *Let R_1 and R_2 be multiplicative hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good homomorphism. Then the following statements hold:*

(1) *If Q_2 is a uniformly 2-absorbing primary hyperideal of R_2, then $\phi^{-1}(Q_2)$ is a uniformly 2-absorbing primary hyperideal of R_1 with $2\_ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq 2\_ord_{H_{R_2}}(Q_2)$.*

(2) *If ϕ is an epimorphism and Q_1 is a uniformly 2-absorbing primary hyperideal of R_1 containing $ker(\phi)$, then $\phi(Q_1)$ is a uniformly 2-absorbing primary hyperideal of R_2 with $2\_ord_{H_{R_2}}(\phi(Q_1)) \leq 2\_ord_{H_{R_1}}(Q_1)$.*

Proof. (1) Let $k = 2\_ord_{H_{R_2}}(Q_2)$ and $x \circ y \circ z \subseteq \phi^{-1}(Q_2)$ for some $x, y, z \in R_1$ such that $x \circ y \nsubseteq \phi^{-1}(Q_2)$ and $x \circ z \nsubseteq r(\phi^{-1}(Q_2))$. This implies that $\phi(x \circ y \circ z) = \phi(z) \circ \phi(y) \circ \phi(z) \subseteq Q_2$ such that $\phi(x \circ y) = \phi(x) \circ \phi(y) \nsubseteq Q_2$ and $\phi(x \circ z) = \phi(x) \circ \phi(z) \nsubseteq r(Q_2)$. Since Q_2 is a uniformly 2-absorbing primary hyperideal of R_2, we have $\phi^k(y \circ z) \subseteq Q_2$. Thus $\phi((y \circ z)^k) \subseteq Q_2$. It means $(y \circ z)^k \subseteq \phi^{-1}(Q_2)$. Hence $\phi^{-1}(Q_2)$ is a uniformly 2-absorbing primary hyperideal of R_1 such that $2\_ord_{H_{R_1}}(\phi^{-1}(Q_2)) \leq k = 2\_ord_{H_{R_2}}(Q_2)$.

(2) Let $k = 2\_ord_{H_{R_1}}(Q_1)$ and $x \circ y \circ z \subseteq \phi(Q_1)$ for some $x, y, z \in R_2$ such that $x \circ y \nsubseteq \phi(Q_1)$ and $x \circ z \nsubseteq r(\phi(Q_1))$. Since ϕ is an epimorphism, then there exist $a, b, c \in R_1$ with $\phi(a) = x$, $\phi(b) = y$ and $\phi(c) = z$. Thus $\phi(a \circ b \circ c) = x \circ y \circ z \subseteq \phi(Q_1)$ such that $\phi(a \circ b) = x \circ y \nsubseteq \phi(Q_1)$ and $\phi(a \circ c) = x \circ z \nsubseteq r(\phi(Q_1))$. Now take any $u \in a \circ b \circ c$. Then we get $\phi(u) \in \phi(a \circ b \circ c) \subseteq \phi(Q_1)$ and so $\phi(u) = \phi(w)$ for some $w \in Q_1$. This implies that $\phi(u - w) = 0 \in (0)$, that is, $u - w \in ker(\phi) \subseteq Q_1$ and so $u \in Q_1$. Since Q_1 is a **C**-hyperideal of R_1, then we conclude that $a \circ b \circ c \subseteq Q_1$. Since $\phi(r(Q_1)) \subseteq r(\phi(Q_1))$, then $a \circ b \nsubseteq Q_1$, and $a \circ c \nsubseteq r(Q_1)$, . Since Q_1 is a uniformly 2-absorbing primary hyperideal of R_1, then we have $(b \circ c)^k \subseteq Q_1$. Thus $\phi((b \circ c)^k) = (\phi(b) \circ \phi(c))^k = (y \circ z)^k \subseteq \phi(Q_1)$. Hence $\phi(Q_1)$ is a uniformly 2-absorbing primary hyperideal of R_2. Also, $2\_ord_{H_{R_2}}(\phi(Q_1)) \leq k = 2\_ord_{H_{R_1}}(Q_1)$. □

Corollary 4.10. Let Q be a hyperideal of R.

(1) If S is a subhyperring of R and Q is a uniformly 2-absorbing primary hyperideal of R, then $Q \cap S$ is a uniformly 2-absorbing primary hyperideal of S with $2\_ord_S(Q \cap S) \leq 2\_ord_R(Q)$.

(2) Let I be a hyperideal of R such that $I \subseteq Q$. Then Q is a uniformly 2-absorbing primary hyperideal of R if and only if Q/I is a uniformly 2-absorbing primary hyperideal of R/I.

Proof. It follows from Theorem 4.9. □

5. Noether strongly 2-absorbing primary hyperideals

Definition 5.1. P-2-absorbing primary hyperideal Q of R is said to be a *Noether strongly 2-absorbing primary hyperideal* if $P^n \subseteq Q$ for some positive integer n. If k be the smallest positive integer for which the above property holds ,then it is denoted by $2\_\mathfrak{e}_{H_R}(Q) = k$, or simply $2\_\mathfrak{e}_H(Q) = k$.

Example 5.2. Consider the ring $(\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\} = \mathbb{Z}_6, \oplus, \odot)$ that for all $\bar{x},\bar{y} \in \mathbb{Z}_6$, $\bar{x} \oplus \bar{y}$ and $\bar{x} \odot \bar{y}$ are the remainder of $\frac{x+y}{6}$ and $\frac{x \cdot y}{6}$, respectively, which $+$ and $\cdot$ are ordinary addition and multiplication. We define the hyperoperation $x * y = \{\overline{xy}, \overline{2xy}, \overline{3xy}, \overline{4xy}, \overline{5xy}\}$, for all $\bar{x},\bar{y} \in \mathbb{Z}_6$. The hyperideal $\{\bar{0}\}$ of commutative multiplicative hyperring $(\mathbb{Z}_6, \oplus, \odot)$ is a Noether strongly 2-absorbing primary hyperideal.

Theorem 5.3. *Let Q_1 and Q_2 be Noether strongly primary hyperideals of R. Then, $Q_1 \cap Q_2$ and $Q_1 \circ Q_2$ are Noether strongly 2-absorbing primary hyperideals of R.*

Proof. Assume that Q_1 and Q_2 be primary hyperideals of R. By Theorem 4.6 in [1], $Q_1 \cap Q_2$ and $Q_1 \circ Q_2$ are 2-absorbing primary ideals of R, . □

Theorem 5.4. *If Q is a 2-absorbing hyperideal of R, then Q is a Noether strongly 2-absorbing primary hyperideal with $2\_\mathfrak{e}_H(Q) \leq 2$.*

Proof. Since Q is a 2-absorbing hyperideal, we conclude that it is a 2-absorbing primary hyperideal and $r(Q)^2 \subseteq Q$ by Theorem 4 in [7]. □

Theorem 5.5. *If Q is a Noether strongly 2-absorbing primary hyperideal of R, then Q is a uniformly 2-absorbing primary hyperideal of R and $2\_ord_H(Q) \leq 2\_\mathfrak{e}_H(Q)$.*

Proof. Assume that Q be a Noether strongly 2-absorbing primary hyperideal of R. Let $x \circ y \circ z \subseteq Q$ for some $x, y, z \in R$ such that $x \circ y \nsubseteq Q$ and $x \circ z \nsubseteq r(Q)$. Since Q is a 2-absorbing primary hyperideal of R, we have $y \circ z \subseteq r(Q)$. Hence $(y \circ z)^{2\_\mathfrak{e}_H(Q)} \subseteq (r(Q))^{2\_\mathfrak{e}_H(Q)} \subseteq Q$. Thus Q is a uniformly 2-absorbing primary hyperideal of R and $2\_ord_H(Q) \leq 2\_\mathfrak{e}_H(Q)$. □

Theorem 5.6. *Let Q be a proper hyperideal of R. Then the following conditions are equivalent:*

(1) *$r(Q)$ is a 2-absorbing hyperideal of R.*

(2) *For every $x, y, z \in R$, $x \circ y \circ x \subseteq Q$ implies that $xoy \subseteq r(Q)$ or $x \circ z \subseteq r(Q)$ or $y \circ z \subseteq r(Q)$.*

(3) *$r(Q)$ is a 2-absorbing primary hyperideal of R.*

(4) *$r(Q)$ is a Noether 2-absorbing primary hyperideal of R with $2\_\mathfrak{e}_H(r(Q)) =$ 1.*

(5) *$r(Q)$ is a uniformly 2-absorbing primary hyperideal of R.*

Proof. (1)$\Longrightarrow$(2) It is evident. (2)$\Longleftarrow$(1) Assume that $a \circ b \circ c \subseteq r(Q)$ for some $a, b, c \in R$. Thus, there exists a positive integer n such that $(a \circ b \circ c)^n = a^n \circ b^n \circ c^n \subseteq Q$. Then, we have $a^n \circ b^n \subseteq r(Q)$ or $a^n \circ c^n \subseteq r(Q)$ or $b^n \circ c^n \subseteq r(Q)$, by the hypothesis in (2). Thus $a \circ b \subseteq r(Q)$ or $a \circ c \subseteq r(Q)$ or $b \circ c \subseteq r(Q)$. Therefore $r(Q)$ is a 2-absorbing hyperideal. (3)$\Longleftrightarrow$(4) and (1)$\Longleftrightarrow$(3) are obvious. (5)$\Longrightarrow$(3) Is straightforward. (4)$\Longrightarrow$(5) It follows by Theorem 5.5. □

6. Special 2-absorbing primary hyperideals

Definition 6.1. Hyperideal Q of R is said to be a *special 2-absorbing primary hyperideal* if it is uniformly 2-absorbing primary hyperideal with $2\_ord_H(Q) = 1$.

Example 6.2. In Example 5.2, the hyperideal $\{\bar{0}\}$ of commutative multiplicative hyperring $(\mathbb{Z}_6, \oplus, \odot)$ is a a special 2-absorbing primary hyperideal.

Theorem 6.3. *Assume that Q is a proper hyperideal of R. Then the following conditions are equivalent:*

(1) *Q is special 2-absorbing primary hyperideal.*

(2) *For every $x, y \in R$ either $x \circ y \subseteq Q$ or $(Q :_R x \circ y) = (Q :_R x)$ or $(Q :_R x \circ y) \subseteq (r(Q) :_R y)$.*

(3) *For every $x, y \in R$ and every hyperideal I of R, $x \circ y \circ I \subseteq Q$ implies that either $x \circ y \subseteq Q$ or $x \circ I \subseteq Q$ or $y \circ I \subseteq r(Q)$.*

(4) *For every $x \in R$ and every hyperideal I of R either $x \circ I \subseteq Q$ or $(Q :_R x \circ I) \subseteq (Q :_R x) \cup (r(Q) :_R I)$.*

(5) *For every $a \in R$ and every hyperideal I of R either $x \circ I \subseteq Q$ or $(Q :_R x \circ I) = (Q :_R x)$ or $(Q :_R x \circ I) \subseteq (r(Q) :_R I)$.*

(6) *For every $x \in R$ and every hyperideals I, J of R, $x \circ I \circ J \subseteq Q$ implies that either $x \circ I \subseteq Q$ or $I \circ J \subseteq r(Q)$ or $x \circ J \subseteq Q$.*

(7) *For every hyperideals I, J of R either $I \circ J \subseteq r(Q)$ or $(Q :_R I \circ J) \subseteq (Q :_R I) \cup (Q :_R J)$.*

(8) *For every hyperideals I, J of R either $I \circ J \subseteq r(Q)$ or $(Q :_R I \circ J) = (Q :_R I)$ or $(Q :_R I \circ J) = (Q :_R J)$.*

(9) *For every hyperideals I, J, K of R, $I \circ J \circ K \subseteq Q$ implies that either $I \circ J \subseteq r(Q)$ or $I \circ K \subseteq Q$ or $J \circ K \subseteq Q$.*

Proof. (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3) This follows by Theorem 4.6.

(3)$\Longleftrightarrow$(4) Suppose that I be a hyperideal of R and $x \in R$ such that $x \circ I \nsubseteq Q$. Assume that $a \in (Q :_R x \circ I)$. we have $x \circ a \circ I \subseteq Q$, and therefore $a \in (Q :_R x)$ or $a \in (r(Q) :_R I)$. Hence $(Q :_R x \circ I) \subseteq (Q :_R x) \cup (r(Q) :_R I)$.

The proof of other cases are straightforward. □

Theorem 6.4. *Let Q be a special 2-absorbing primary hyperideal of R and $a \in R \setminus r(Q)$. The following conditions hold:*

(1) *$(Q :_R a) = (Q :_R a^n)$ for every $n \geq 2$;*

(2) *$(r(Q) :_R a) = r(Q :_R a)$.*

(3) *$(Q :_R a)$ is a special 2-absorbing primary hyperideal of R.*

Proof. (1) It is clear that $(Q :_R a) \subseteq (Q :_R a^n)$ for all $n \geq 2$. By induction on n, we show $(Q :_R a^n) \subseteq (Q :_R a)$. First, let $n = 2$ and $s \in (Q :_R a^2)$. We have $s \circ a^2 \subseteq Q$, and then either $s \circ a \subseteq Q$ or $a^2 \subseteq r(Q)$. The second case implies that $a \in r(Q)$ which is a contradiction. Hence $s \circ a \subseteq Q$ which means $s \in (Q :_R a)$. Thus $(Q :_R a) = (Q :_R a^2)$. Now, let $n > 2$. Assume that $(Q :_R a) = (Q :_R a^{n-1})$. Take $s \in (Q :_R a^n)$. We have $s \circ a^n \subseteq Q$. Since

$a \notin r(Q)$, we conclude that either $s \circ a^{n-1} \subseteq Q$ or $s \circ a \subseteq Q$. Both of them implies that $s \in (Q :_R a)$. Hence $(Q :_R a) = (Q :_R a^n)$.

(2) Clearly, $r(Q :_R x) \subseteq (r(Q) :_R a)$. Assume that $s \in (r(Q) :_R a)$. Thus there exists a positive integer t such that $(s \circ a)^t \subseteq Q$. Hence $s^t \subseteq (Q :_R a)$, by part (1). Therefore $s \in r(Q :_R a)$. Consequently $(r(Q) :_R a) = r(Q :_R a)$.

(3) Assume that $x \circ y \circ z \subseteq (Q :_R a)$ for some $x, y, z \in R$. Thus $x \circ a \circ (y \circ z) \subseteq Q$ and then $x \circ a \subseteq Q$ or $x \circ y \circ z \subseteq Q$ or $y \circ z \circ a \subseteq r(Q)$. If $x \circ a \subseteq Q$, then we have $x \circ y \subseteq (Q :_R a)$. If $x \circ y \circ z \subseteq Q$, then we have either $x \circ y \subseteq Q \subseteq (Q :_R a)$ or $x \circ z \subseteq Q \subseteq (Q :_R a)$ or $y \circ z \subseteq r(Q) \subseteq r(Q :_R a)$. If $y \circ z \circ a \subseteq r(Q)$, then by part (2) we get $y \circ z \subseteq (r(Q) :_R a) = r(Q :_R a)$. Hence $(Q :_R a)$ is a special 2-absorbing primary hyperideal of R. □

Theorem 6.5. *Let Q be a special 2-absorbing primary hyperideal of R and P, P_1, P_2 be distinct prime hyperideals of R and $E_a = (Q :_R a)$.*

(1) If $r(Q) = P$, then $\{E_a \mid a \in R \setminus P\}$ is a totally ordered set.

(2) If $r(Q) = P_1 \cup P_2$, then $\{E_a \mid a \in R \setminus P_1 \cup P_2\}$ is a totally ordered set.

Proof. (1) Assume that $a, b \in R \setminus P$. We have $a \circ b \subseteq R \setminus P$. Clearly, $E_a \cup E_b \subseteq E_{a \circ b}$. Let $s \in E_{a \circ b}$. Thus $s \circ a \circ b \subseteq Q$. Since $a \circ b \nsubseteq r(Q)$ we have $s \circ a \subseteq Q$ or $s \circ b \subseteq Q$. Hence $E_{a \circ b} = E_a \cup E_b$. Therefore, we have either $E_{a \circ b} = E_a$ or $E_{a \circ b} = E_b$. Then either $E_b \subseteq E_a$ or $E_a \subseteq E_b$.

(2) It follows by using an argument to that in the proof of (1). □

Theorem 6.6. *Let R_1 and R_2 be multiplicative hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good homomorphism. Then the following statements hold:*

(1) *If Q_2 is a special 2-absorbing primary hyperideal of R_2, then $\phi^{-1}(Q_2)$ is a special 2-absorbing primary hyperideal of R_1.*

(2) *If ϕ is an epimorphism and Q_1 is a special 2-absorbing primary hyperideal of R_1 containing $ker(\phi)$, then $\phi(Q_1)$ is a special 2-absorbing primary hyperideal of R_2.*

Proof. It is similar to the proof of Theorem 4.9. □

References

[1] M. Anbarloei, *On 2-absorbing and 2-absorbing primary hyperideals of a multiplicative hyperring,* Cogent Mathematics, 4 (2017), 1-8.

[2] A. Badawi, U. Tekir, E. Yetkin, *On 2-absorbing primary ideals in commutative rings,* Bull. Korean Math. Soc., 51 (2014), 1163-1173.

[3] S. Corsini, *Prolegomena of hypergroup theory,* Second edition, Aviani editor, Italy, 1993.

[4] S. Corsini, V. Leoreanu, *Applications of hyperstructure theory,* Advances in Mathematics, vol. 5, Kluwer Academic Publishers, 2003.

[5] U. Dasgupta, *On prime and primary hyperideals of a multiplicative hyperrings,* Bull. Austral. Math. Soc., 75 (2007), 417-429.

[6] B. Davvaz, V. Leoreanu-Fotea, *Hyperring theory and applications,* International Academic Press, Palm Harbor, USA, 2007.

[7] P. Ghiasvand, *On 2-absorbing hyperideals of multiplicative hyperrings,* Second Seminar on Algebra and its Applications, (2014), 58-59.

[8] F. Marty, *Sur une generalization de la notion de groupe,* 8^{th} Congress Math. Scandenaves, Stockholm, (1934), 45-49.

[9] S. Omidi, B. Davvaz, *Contribution to study special kinds of hyperideals in ordered semihyperrings,* J. Taibah Univ. Sci., 2016.

[10] R. Rota, *Sugli iperanelli moltiplicativi*, Rend. Di Math., Series VII, 4 (1982), 711-724.

[11] T. Vougiouklis, *Hyperstructures and their representations,* Hadronic Press Inc., Florida, 1994.

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