On some properties of certain subclasses of univalent functions

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Abstract. This study interested in two subclasses of analytic functions defined on the open unit disc of the complex plain, we discuss some neighborhood properties, integral means inequalities and some results concerning the partial sums of the functions belonging to these subclasses.
Keywords: analytic function, neighborhood, integral means, partial sums.

1. Introduction

Let T denoted to class of function of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0), \tag{1.1}$$

which are analytic function in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $T^*(\alpha)$ and $C(\alpha)$ the subclasses of starlike functions of order α, and convex functions of order α, respectively. Theses two subclases are defined by Silverman [11] as following:

$$T^*(\alpha) = \left\{ f \in T : \Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U) \right\}, \tag{1.2}$$

and

$$C(\alpha) = \left\{ f \in T : \Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; z \in U) \right\}. \tag{1.3}$$

For $\mu > 0$ and $a, c \in \mathbb{C}$, are such that $\Re\{c - a\} \geq 0$, Raina and Sharma [9] (see also [3], [4]) defined the integral operator $J_\mu^{a,c} : T \longrightarrow T$, as following:

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(i) for $\Re\{c-a\}>0$ and $\Re\{a\}>-\mu$ by

$$J_{\mu}^{a,c}f(z)=\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)\Gamma(c-a)}\int_0^1(1-t)^{c-a-1}t^{a-1}f(zt^{\mu})dt; \tag{1.4}$$

(ii) for $a=c$ by

$$J_{\mu}^{a,a}f(z)=f(z), \tag{1.5}$$

where Γ stands for Euler's Gamma function (which is valid for all complex numbers except the non-positive integers).

For $f(z)$ defined by (1.1), it is easily from (1.4) and (1.5) that:

$$J_{\mu}^{a,c}f(z)=z-\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{k=2}^{\infty}\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_kz^k \qquad (\mu>0,\Re\{c\}\geq\Re\{a\}>-\mu). \tag{1.6}$$

Let $M_{\mu}^{a,c}(\alpha;A,B)$ be the subclass of functions $f\in T$ for which:

$$\frac{z(J_{\mu}^{a,c}f(z))'}{J_{\mu}^{a,c}f(z)}\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha \qquad (-1\leq B<A\leq 1, 0\leq\alpha<1), \tag{1.7}$$

that is, that

$$M_{\mu}^{a,c}(\alpha;A,B)=\left\{f\in T:\left|\frac{\frac{z(J_{\mu}^{a,c}f(z))'}{J_{\mu}^{a,c}f(z)}-1}{B\frac{z(J_{\mu}^{a,c}f(z))'}{J_{\mu}^{a,c}f(z)}-[B+(A-B)(1-\alpha)]}\right|<1, z\in U\right\}. \tag{1.8}$$

Also, let $N_{\mu}^{a,c}(\alpha;A,B)$ be the subclass of functions $f\in T$ for which:

$$1+\frac{z(J_{\mu}^{a,c}f(z))''}{(J_{\mu}^{a,c}f(z))'}\prec(1-\alpha)\frac{1+Az}{1+Bz}+\alpha,$$

form (1.7) and (1.8), it is clear that

$$f(z)\in N_{\mu}^{a,c}(\alpha;A,B)\iff zf'(z)\in M_{\mu}^{a,c}(\alpha;A,B). \tag{1.9}$$

It is easily to see that:

(i) $M_{\mu}^{a,a}(\alpha;A,B)=T^*(A,B,\alpha)$ and $N_{\mu}^{a,c}(\alpha;A,B)=C(A,B,\alpha)$, see [2, with $p=1$];

(ii) $M_{\mu}^{a,a}(\alpha;\beta,-\beta)=T^*(\alpha,\beta)$ and $N_{\mu}^{a,a}(\alpha;\beta,-\beta)=C(\alpha,\beta)$ the subclasses of starlike and convex of order $0\leq\alpha<1$ and type $0<\beta\leq 1$, see [6];

(iii) $M_{\mu}^{a,a}(\alpha;1,-1)=T^*(\alpha)$ and $N_{\mu}^{a,a}(\alpha;1,-1)=C(\alpha)$ the subclasses of starlike and convex of order $0\leq\alpha<1$, see [11].

The object of the present paper is to determine the neighborhood properties for each of the subclasses $M_{\mu}^{a,c}(\alpha;A,B)$ and $N_{\mu}^{a,c}(\alpha;A,B)$. Moreover, investigate integral means inequalities, and some results concerning partial sums for functions belonging to the subclass $M_{\mu}^{a,c}(\alpha; A, B)$. We will make use of the following lemmas, also otherwise mentioned, we assume in the reminder of this paper that, $0 \leq \alpha < 1, -1 \leq B < A \leq 1, \mu > 0$, $a, c \in \mathbb{R}, c > a > -\mu$ and $z \in U$.

Lemma 1 ([8]). *Let the function $f(z)$ be given by (1.1). Then $f \in M_{\mu}^{a,c}(\alpha; A, B)$, if and only if*

$$\sum_{k=2}^{\infty} [(1-B)(k-1) + (A-B)(1-\alpha)] \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} a_k \leq (A-B)(1-\alpha)\frac{\Gamma(a+\mu)}{\Gamma(c+\mu)}. \tag{1.10}$$

Lemma 2 ([8]). *Let the function $f(z)$ be given by (1.1). Then $f \in N_{\mu}^{a,c}(\alpha; A, B)$, if and only if*

$$\sum_{k=2}^{\infty} [(1-B)(k-1) + (A-B)(1-\alpha)] \frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)} k a_k \leq (A-B)(1-\alpha)\frac{\Gamma(a+\mu)}{\Gamma(c+\mu)}. \tag{1.11}$$

2. Neighborhood results

Following the earlier investigations of Goodman [5] and Ruscheweyh [10], the $\delta-$ neighborhood is defined as following:

$$N_{\delta}(f) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \sum_{k=2}^{\infty} k\,|a_k - b_k| \leq \delta \right\}. \tag{2.1}$$

For the identity function $e(z) = z$, we immediately have

$$N_{\delta}(e) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \sum_{k=2}^{\infty} k\,|b_k| \leq \delta \right\}, \tag{2.2}$$

where the function f is given by (1.1).

Theorem 1. *If the function $f(z)$ defined by* (1.1) *is in the subclass $M_{\mu}^{a,c}(\alpha; A, B)$.*

Then $M_{\mu}^{a,c}(\alpha; A, B) \subset N_{\delta}(e)$, where

$$\delta = \frac{2\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)\left[(1-B)+(A-B)(1-\alpha)\right]\Gamma(a+2\mu)}. \tag{2.3}$$

Proof. Since $f \in M_{\mu}^{a,c}(\alpha; A, B)$, by using Lemma 1 and from (1.10), we find

$$\frac{\Gamma(c+\mu)\left[(1-B)+(A-B)(1-\alpha)\right]\Gamma(a+2\mu)}{2\Gamma(a+\mu)\Gamma(c+2\mu)} \sum_{k=2}^{\infty} k a_k \leq$$

$$\sum_{k=2}^{\infty}[(1-B)(k-1)+(A-B)(1-\alpha)]\frac{\Gamma(a+k\mu)}{\Gamma(c+k\mu)}a_k \leq (A-B)(1-\alpha).$$

It is clear

$$\sum_{k=2}^{\infty} ka_k \leq \frac{2\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)\left[(1-B)+(A-B)(1-\alpha)\right]\Gamma(a+2\mu)} = \delta.$$

Corollary 1. *If $f \in T$ is in the class $T^*(A,B,\alpha)$. Then*

$$T^*(A,B,\alpha) \subset N(e),$$

where

$$\delta = \frac{2(A-B)(1-\alpha)}{(1-B)+(A-B)(1-\alpha)}.$$

Corollary 2. *If $f \in T$ is in the class $T^*(\alpha,\beta)$. Then*

$$T^*(\alpha,\beta) \subset N(e),$$

where

$$\delta = \frac{4\beta(1-\alpha)}{1+2\beta(2-\alpha)}.$$

Corollary 3. *If $f \in T$ is in the class $T^*(\alpha)$. Then*

$$T^*(\alpha) \subset N(e),$$

where

$$\delta = \frac{2(1-\alpha)}{2-\alpha}.$$

by similarly applying Lemma 2 instead of Lemma1, we can prove following.

Theorem 2. *If the function $f(z)$ defined by* (1.1) *is in the subclass $N_{\mu}^{a,c}(\alpha;A,B)$. Then $N_{\mu}^{a,c}(\alpha;A,B) \subset N_{\delta}(e)$,* where

$$\delta = \frac{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)\left[(1-B)+(A-B)(1-\alpha)\right]\Gamma(a+2\mu)}. \tag{2.4}$$

Corollary 4. *If $f \in T$ is in the class $C(A,B,\alpha)$. Then*

$$C(A,B,\alpha) \subset N(e),$$

where

$$\delta = \frac{(A-B)(1-\alpha)}{(1-B)+(A-B)(1-\alpha)}.$$

Corollary 5. *If $f \in T$ is in the class $C(\alpha, \beta)$. Then*

$$C(\alpha, \beta) \subset N(e),$$

where

$$\delta = \frac{2\beta(1-\alpha)}{(1+\beta)+2\beta(1-\alpha)}.$$

Corollary 6. *If $f \in T$ is in the class $C(\alpha)$. Then*

$$C(\alpha) \subset N(e),$$

where

$$\delta = \frac{1-\alpha}{2-\alpha}.$$

We will determine the neighborhood properties for each of the following (slightly modified) function subclass $M_{\mu}^{a,c,\rho}(\alpha; A, B)$.

A functions $f \in T$ is said to be in the class $M_{\mu}^{a,c,\rho}(\alpha; A, B)$ if there exists a function $g \in M_{\mu}^{a,c}(\alpha; A, B)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \rho \qquad (z \in U; 0 \leq \rho < 1). \tag{2.5}$$

The proofs of the following results involving the neighborhood properties for the subclass $M_{\mu}^{a,c,\rho}(\alpha; A, B)$, is similar to those given in [1].

Theorem 3. *If $g \in M_{\mu}^{a,c}(\alpha; A, B)$. Suppose also that*

$$\rho = 1 - \frac{\delta\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)}{2[\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)-\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)]}, \tag{2.6}$$

then

$$N_{\delta}(g) \subset M_{\mu}^{a,c,\rho}(\alpha; A, B).$$

Proof. let $f(z)$ be in $N_{\delta}(g)$. We then find from the definition (2.1) that

$$\sum_{k=2}^{\infty} k\,|a_k - b_k| \leq \delta, \tag{2.7}$$

since $g \in M_{\mu}^{a,c}(\alpha; A, B)$, we have

$$\sum_{k=2}^{\infty} b_k \leq \frac{\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)\left[(1-B)+(A-B)(1-\alpha)\right]\Gamma(a+2\mu)}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=2}^{\infty}|a_k - b_k|}{1-\sum_{k=2}^{\infty} b_k} \\ &\leq \frac{\delta}{2}\frac{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)-\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)} \\ &= 1-\rho,\end{aligned}$$

provided that ρ is given precisely by (2.6). Thus, by definition, $f \in M_{\mu}^{a,c,\rho}(\alpha; A, B)$ for ρ given by (2.6). This evidently completes our proof of Theorem 3.

A function $f \in T$ is said to be in the class $H_{\mu}^{a,c}(\alpha, \phi; A, B)$ if it satisfies the following non-homogeneous Cauchy-Euler differential equation:

$$z^2 \frac{d^2 f}{dz^2} + 2(\phi+1) z \frac{df(z)}{dz} + \phi(\phi+1) f(z) = (1+\phi)(2+\phi) g(z) \tag{2.8}$$

$$\big(g \in M_{\mu}^{a,c}(\alpha; A, B); \phi > -1\big)$$

Theorem 4. *If $f \in T$ is in the class $H_{\mu}^{a,c}(\alpha, \phi; A, B)$ then*

$$H_{\mu}^{a,c}(\alpha, \phi; A, B) \subset N_{\delta}(g), \tag{2.9}$$

where

$$\delta = \frac{4\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)} \left(\frac{2+\phi}{3+\phi}\right) \tag{2.10}$$

Proof. Suppose that $f \in H_{\mu}^{a,c}(\alpha, \phi; A, B)$ and f is given by (1.1). From (2.8)

$$a_k = \frac{(1+\phi)(2+\phi)}{(k+\phi)(k+\phi+1)} b_k \quad (k \geq 2),$$

$$\sum_{k=2}^{\infty} k\,|b_k - a_k| \leq \sum_{k=2}^{\infty} k b_k + \sum_{k=2}^{\infty} k a_k \quad (a_k \geq 0, b_k \geq 0),$$

we obtain

$$\sum_{k=2}^{\infty} k\,|b_k - a_k| \leq \sum_{k=2}^{\infty} k b_k + \sum_{k=2}^{\infty} \frac{(1+\phi)(2+\phi)}{(k+\phi)(k+\phi+1)} k b_k. \tag{2.11}$$

Next, since $g \in M_{\mu}^{a,c}(\alpha; A, B)$, from (1.10) of the Lemma 1 yields

$$\sum_{k=2}^{\infty} k b_k \leq \frac{2\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)}. \tag{2.12}$$

Finally, by making use of (2.11) on the right-hand side of (2.12), we find that

$$\begin{aligned}\sum_{k=2}^{\infty} k\,|b_k - a_k| &\leq \frac{2\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)} \left(1 + \frac{(1+\phi)}{(3+\phi)}\right) \\ &= \frac{2\Gamma(a+\mu)(A-B)(1-\alpha)\Gamma(c+2\mu)}{\Gamma(c+\mu)\,[(1-B)+(A-B)(1-\alpha)]\,\Gamma(a+2\mu)} \left(\frac{2(2+\phi)}{3+\phi}\right) = \delta.\end{aligned}$$

Thus, by definition (2.1) with $g(z)$ interchanged by $f(z)$, $f \in N_{\delta}(g)$.This, evidently, completes the proof of Theorem 4.

3. Integral means inequalities

We shall need the concept of subordination theorem of Littlewood [7] in our investigation.

Lemma 3. (Littlewood's theory [7]). *If the functions $f(z)$ and $g(z)$ are analytic in U with $g(z) \prec f(z)$ then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\tau d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \qquad (\tau > 0; 0 < r < 1). \tag{3.1}$$

Theorem 5. *Let $f \in M_\mu^{a,c}(\alpha; A, B)$ and suppose that*

$$f_2(z) = z - \frac{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)} z^2, \tag{3.2}$$

then for $\tau > 0, z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \le \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\tau d\theta \tag{3.3}$$

Proof. From (3.1), it would suffice to show that

$$1 - \sum_{k=2}^\infty a_k z^{k-1} \prec 1 - \frac{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)} z.$$

By setting

$$1 - \sum_{k=2}^\infty a_k z^{k-1} = 1 - \frac{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)}{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)} w(z),$$

we find that

$$\begin{aligned}
|w(z)| &= \left|\sum_{k=2}^\infty \frac{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)} a_k z^{k-1}\right| \\
&\le |z| \sum_{k=2}^\infty \frac{\Gamma(c+\mu)[(1-B)+(A-B)(1-\alpha)]\Gamma(a+2\mu)}{\Gamma(a+\mu)\Gamma(c+2\mu)(A-B)(1-\alpha)} a_k \\
&\le |z| \le 1,
\end{aligned} \tag{3.4}$$

by using (1.10). Hence $f(z) \prec g(z)$ which readily yields the integral means inequality (3.3).

4. Partial sums

In this section we will study the ratio of a function of the form (1.1) to its sequence of partial sums defined by $f_m(z) = z$ and $f_m(z) = z - \sum_{k=2}^{m} a_k z^k$, when the coefficients of $f(z)$ are sufficiently small to satisfy the condition (1.9). We will determine sharp lower bounds, for $\Re \int(\frac{f(z)}{f_m(z)} \int), \Re \int(\frac{f_m(z)}{f(z)} \int), \Re \int(\frac{f'(z)}{f'_m(z)} \int)$ and $\Re \int(\frac{f'_m(z)}{f'(z)} \int)$.

In what follows, we will use the well known result

$$\Re\left(\frac{1-w(z)}{1+w(z)}\right) \qquad (z \in U)\,,$$

if and only if

$$w(z) = \sum_{k=1}^{\infty} D_k z^k,$$

satisfies the inequality $|w(z)| \le |z|$.

Theorem 6. *Let $f \in M_{\mu}^{a,c}(\alpha; A, B)$, then*

$$\Re\left(\frac{f(z)}{f_m(z)}\right) \ge 1 - \frac{1}{D_{m+1}} \qquad (z \in U, m \in \mathbb{N})\,, \tag{4.1}$$

and

$$\Re\left(\frac{f_m(z)}{f(z)}\right) \ge \frac{D_{m+1}}{1 + D_{m+1}} \qquad (z \in U, m \in \mathbb{N})\,, \tag{4.2}$$

where

$$D_k = \frac{\Gamma(c+\mu)[(1-B)(k-1) + (A-B)(1-\alpha)]\Gamma(a+k\mu)}{\Gamma(a+\mu)\Gamma(c+k\mu)(A-B)(1-\alpha)}. \tag{4.3}$$

The estimates in (4.1) *and* (4.2) *are sharp.*

Proof. Employing the same technique used by Silverman [12]. The function $f \in M_{\mu}^{a,c}(\alpha; A, B)$ if and only if $\sum_{k=1}^{\infty} D_k z^k \le 1$. It is easy to verify that $D_{k+1} > D_k > 1$. Thus

$$\sum_{k=1}^{m} a_k + D_{m+1} \sum_{k=m+1}^{\infty} a_k \le \sum_{k=2}^{\infty} D_k a_k < 1. \tag{4.4}$$

Now, setting

$$D_{m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{D_{m+1}}\right)\right\}$$
$$= \frac{1 - \sum\limits_{k=2}^{m} a_k z^{k-1} - D_{m+1} \sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1 - \sum\limits_{k=1}^{m} a_k z^{k-1}} = \frac{1+E(z)}{1+Y(z)},$$

and $\frac{1+E(z)}{1+Y(z)} = \frac{1-w(z)}{1+w(z)}$, then we have

$$w(z) = \frac{Y(z)-E(z)}{2+E(z)+Y(z)} = \frac{D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{2-2\sum\limits_{k=2}^{m} a_k z^{k-1} - D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}$$

which implies

$$|w(z)| \leq \frac{D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k}{2-2\sum\limits_{k=2}^{m} a_k - D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k}.$$

Hence $|w(z)| \leq 1$ if and only if

$$\sum_{k=2}^{m} a_k + D_{m+1}\sum_{k=m+1}^{\infty} a_k \leq 1$$

which is true by (4.4). This readily yields (4.1).

Now consider the function

$$f(z) = 1 - \frac{z^{m+1}}{D_{m+1}} \tag{4.5}$$

Thus $\frac{f(z)}{f_m(z)} = 1 - \frac{z^m}{D_{m+1}}$. Letting $z \longrightarrow 1^-$, then $f(z) = 1 - \frac{1}{D_{m+1}}$. So $f(z)$ given by (4.5) satisfies the sharp result in (4.1). This shows that the bounds in (4.1) are best possible for each $m \in \mathbb{N}$.

Similarly, setting

$$\begin{aligned}(1+D_{m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{D_{m+1}}{1+D_{m+1}}\right\} &= \frac{1-\sum\limits_{k=2}^{m} a_k z^{k-1} + D_{m+1}\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1-\sum\limits_{k=2}^{m} a_k z^{k-1}} \\ &\equiv \frac{1-w(z)}{1+w(z)},\end{aligned}$$

where

$$|w(z)| \leq \frac{(1+D_{m+1})\sum\limits_{k=m+1}^{\infty} a_k}{2-2\sum\limits_{k=2}^{m} a_k + (1-D_{m+1})\sum\limits_{k=m+1}^{\infty} a_k}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{k=2}^{m} a_k + D_{m+1}\sum_{k=m+1}^{\infty} a_k \leq 1,$$

which readily implies the assertion (4.2). The estimate in (4.2) is sharp with the extremal function $f(z)$ given by (4.5). This completes the proof of the theorem.

Following similar steps to that followed in Theorem 6, we can state the following theorem

Theorem 7. *Let* $f \in M_{\mu}^{a,c}(\alpha; A, B)$, *then*

$$\Re\left(\frac{f'(z)}{f_m'(z)}\right) \geq 1 - \frac{m+1}{D_{m+1}} \qquad (z \in U, m \in \mathbb{N}), \tag{4.6}$$

and

$$\Re\left(\frac{f_m'(z)}{f'(z)}\right) \geq \frac{D_{m+1}}{m+1+D_{m+1}} \qquad (z \in U, m \in \mathbb{N}), \tag{4.7}$$

where $D_k, k \in \mathbb{N}$ is given by (4.3). The estimates in (4.6) and (4.7) are sharp with the extremal function $f(z)$ is as defined in (4.5).

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Accepted: 28.08.2018