Optimization technique for solving fuzzy partial differential equations under strongly generalized differentiability

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Abstract. In this article, we develop and analyze the use of the combined Laplace transform-homotopy perturbation method C(LT-HPM) to find the exact and approximate solutions for fuzzy partial differential equations under strongly generalized differentiability. The C(LT-HPM) allows the solution of the fuzzy partial differential equation to be calculated in the form of an infinite series in which the components can be easily computed. The method is tested on some linear and nonlinear fuzzy partial differential equations with fuzzy initial conditions to show the effectiveness and accuracy of this method.

Keywords: fuzzy partial differential equation, fuzzy derivative, strongly generalized differentiability.

1. Introduction

The study of fuzzy partial differential equations (FPDEs) in both theoretical and numerical calculations of view has been growing in recent years. Generally, FPDEs in the fuzzy setting are a natural way to model dynamical systems when information about its behavior is inadequate. Some problems that lead to FPDEs are found in many applications of which some are mentioned in fields of physics and engineering where often have to solving those as numerical methods. Moreover, some researches present applications of FPDEs with fuzzy parameters that were obtained through fuzzy rule-based system, such as in [1] and [2]. Since L. Zadeh introduced the concept of fuzzy sets in [3] a great amount of research has been developed, including the studies on FPDEs as well as fuzzy set theory. The concept of FPDE was first introduced and investigated by J. Buckley and T. Feuring in [4].

The numerical solutions of FPDEs have been studied by several authors using different approaches. In [5] the author have used an explicit finite difference method (FDM) to solve linear FPDEs based on Seikkala derivative. In [6] also, the authors have developed optimal homotopy asymptotic method (OHAM) to find the approximate-analytical solution for linear partial differential equation involving a fuzzy heat equation based on Seikkala derivative. These approaches have drawbacks; it solves only linear FPDEs but does not discuss the nonlinear FPDEs. Hence, the fuzzy solution becomes fuzzier as time goes by [7, 8], and it behaves quite differently from the crisp solution. The strongly generalized differentiability was first introduced in [9] and studied in [7, 10, 11, 12, 16, 17, 33, 34]. This concept permits us to resolve the above-mentioned drawbacks. Indeed, the strongly generalized differentiability is defined for a larger class of fuzzy-valued functions which is a generalization of the Hukuhara derivative. The purpose of this article is to establish the approximate-analytical solutions for linear and nonlinear FPDE using the C(LT-HPM) under the assumption of strongly generalized differentiability which is as follows:

$$\varphi\left(D_t u(x,t)\right) = \Phi(D_x u(x,t)) + Nu(x,t),\ 0 < x < l,\ t > 0, \tag{1}$$

subject to the fuzzy initial conditions

$$u(x,0) = f(x;\alpha),\ u_t(x,0) = g(x;\alpha),\ 0 \le x \le l, \tag{2}$$

for all $\alpha \in [0,1]$, where the operators $\varphi\left(D_t\right)$ will be a polynomial with a constant coefficient in D_t, $\Phi(D_x)$ is a polynomial with a constant coefficient in D_x, N is the nonlinear operator, $f,\ g : [0,l] \to \mathbb{R}_F$ are continuous fuzzy-valued functions and $u : [0,l] \times (0,\infty) \to \mathbb{R}_F$ is a continuous fuzzy-valued function which is unknown function of independent variables x and t to be determined such that $\mathbb{R}_F$ is the set of fuzzy real numbers on $\mathbb{R}$.

Calculation of the solution of FPDEs is usually very difficult. We can find the exact solution only in a few extraordinary cases. When we are studying in fields of physics and engineering, we often meet the problems of FPDEs. Anyway, by using the parametric form of fuzzy numbers, we employ the C(LT-HPM) to find the exact and approximate solutions for FPDE (1)-(2). The C(LT-HPM) has the following characteristics; first, it is somewhat different from other approximate-analytical methods in that it gives extremely good results for even a large domain with minimal terms of the approximate series solution. The second advantage of this method is its ability to solve other mathematical, physical and engineering issues. Third, it is the first attempt in solving nonlinear FPDEs. Fourth, it is the first attempt gives two locally solutions under strongly generalized differentiability for linear and nonlinear FPDEs. More specifically, we investigate the solution of different types of FPDEs using C(LT-HPM).

This article is organized in six sections including the introduction. In Section 2, we present a few fundamental definitions and preliminary results from the fuzzy calculus theory, including concepts like fuzzy derivative and fuzzy solution. The procedure for converting fuzzy partial differential equation (1)-(2) under strongly generalized differentiability into two systems of crisp partial differential equations is presented in Section 3. In Section 4, the combined Laplace transform-homotopy perturbation technique is built and introduce. The numerical results are reported to illustrate the ability and superiority of the proposed method by considering three numerical examples in Section 5. Finally, the conclusion is drawn in Section 6 with a few concluding comments.

2. Preliminaries

This section present some basic definitions in fuzzy mathematics and introduce the necessary notations which can be used throughout the paper. Hereafter, we adopt strongly generalized differentiability which is a modification of the Hukuhara differentiability and has the advantage of dealing properly with FPDEs.

Definition 2.1 (see [18])**.** A mapping $w : \mathbb{R} \longrightarrow [0,1]$ is called a fuzzy number if the following properties are satisfied:

i) $w(\lambda s + (1-\lambda)t) \geq \min\{w(s), w(t)\}$ for each s, $t \in \mathbb{R}$ and $\lambda \in [0,1]$, which is called a convex property.

ii) $\exists s \in \mathbb{R}$ such that $w(s) = 1$, which is called a normal property.

iii) the set $\{s \in \mathbb{R} \mid w(s) > \alpha\}$ is closed for each $\alpha \in [0,1]$, which is called an upper semicontinuous property.

iv) the set $\overline{\{s \in \mathbb{R} \mid w(s) > 0\}}$ is compact, where $\overline{\{\cdot\}}$ is the closure of $\{\cdot\}$.

For $0 < \alpha \leq 1$, put $[w]_\alpha = \{s \in \mathbb{R} \mid w(s) \geq \alpha\}$, $[w]_0 = \overline{\{s \in \mathbb{R} \mid w(s) > 0\}}$, and $[w]_1 \neq \phi$ (see [15]). Thus, if w is a fuzzy number, then $[w]_\alpha = [\underline{w}(\alpha), \overline{w}(\alpha)]$, where $\underline{w}(\alpha) = \min\{s \mid s \in [w]_\alpha\}$ and $\overline{w}(\alpha) = \max\{s \mid s \in [w]_\alpha\}$ for each $\alpha \in [0,1]$. Hence, the notation $[w]_\alpha$ is called the α-cut representation or parametric form of a fuzzy number w.

Theorem 2.2 (see [15])**.** *A mapping $w : \mathbb{R} \longrightarrow [0,1]$ is a fuzzy number with α-cut representation $[\underline{w}(\alpha), \overline{w}(\alpha)]$ if and only if the following conditions are satisfied:*

i) the function $\underline{w} : [0,1] \to \mathbb{R}$ is a bounded increasing.

ii) the function $\overline{w} : [0,1] \to \mathbb{R}$ is a bounded decreasing.

iii) for each $r \in (0,1]$, $\lim_{\alpha \to r^-} \underline{w}(\alpha) = \underline{w}(r)$ and $\lim_{\alpha \to r^-} \overline{w}(\alpha) = \overline{w}(r)$.

iv) for each $r \in (0,1]$, $\lim_{\alpha \to r^+} \underline{w}(\alpha) = \underline{w}(r)$ and $\lim_{\alpha \to r^+} \overline{w}(\alpha) = \overline{w}(r)$.

v) $\underline{w}(\alpha) \leq \overline{w}(\alpha)$ for all $\alpha \in [0,1]$.

Definition 2.3 (see [32])**.** Let $H_d : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R}^+ \cup \{0\}$. The Hausdorff metric H_d is a function defined by

$$H_d(w,z) = \sup_{\alpha \in [0,1]} \max\{|\underline{w}(\alpha) - \underline{z}(\alpha)|, |\overline{w}(\alpha) - \overline{z}(\alpha)|\}$$

for each w, $z \in \mathbb{R}_F$.

Theorem 2.4 (see [32])**.** $(\mathbb{R}_F, H_d)$ *is a complete fuzzy metric space.*

In what follows, we define some arithmetic operations in fuzzy mathematics as follows:

1. $[w]_\alpha + [z]_\alpha = [\underline{w}(\alpha) + \underline{z}(\alpha), \overline{w}(\alpha) + \overline{z}(\alpha)]$ for each $w, z \in \mathbb{R}_F$ and $0 \leq \alpha \leq 1$.
2. $[w]_\alpha \ominus [z]_\alpha = [\underline{w}(\alpha) - \overline{z}(\alpha), \overline{w}(\alpha) - \underline{z}(\alpha)]$, which is called the H-difference (Hukuhara difference) of w and z.
3. $\lambda^*[w]_\alpha = [\min\{\lambda^*\underline{w}(\alpha), \lambda^*\overline{w}(\alpha)\}, \max\{\lambda^*\underline{w}(\alpha), \lambda^*\overline{w}(\alpha)\}]$ for each $\lambda^* \in \mathbb{R}$.

Definition 2.5 (see [14])**.** Let $u : [a, b] \longrightarrow \mathbb{R}_F$ be a fuzzy-valued function and $t_0 \in [a, b]$. We say u is strongly generalized differentiability at t_0, if there exists an element $u'(t_0) \in \mathbb{R}_F$ such that either:

i) for all $h > 0$ sufficiently near to 0, the H-differences $u(t_0 + h) \ominus u(t_0)$, $u(t_0) \ominus u(t_0 - h)$ exist and

$$u'(t_0) = \lim_{h \longrightarrow 0^+} \frac{u(t_0 + h) \ominus u(t_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{u(t_0) \ominus u(t_0 - h)}{h}.$$

In this part of the definition, we denote $u'(t_0)$ by $D_1^1 u(t_0)$, or

ii) for all $h < 0$ sufficiently near to 0, the H-differences $u(t_0 + h) \ominus u(t_0)$, $u(t_0) \ominus u(t_0 - h)$ exist and

$$u'(t_0) = \lim_{h \longrightarrow 0^-} \frac{u(t_0 + h) \ominus u(t_0)}{h} = \lim_{h \longrightarrow 0^-} \frac{u(t_0) \ominus u(t_0 - h)}{h}.$$

In this part of the definition, we denote $u'(t_0)$ by $D_2^1 u(t_0)$.

Definition 2.6 (see [10])**.** Let $u : [a, b] \longrightarrow \mathbb{R}_F$ be a fuzzy-valued function. We say that u is (1)-differentiable on $[a, b]$ if u is differentiable in the first form (i) of Definition (2.5). Similarly, we say that u is (2)-differentiable on $[a, b]$ if u is differentiable in the second form (ii) of Definition (2.5).

Theorem 2.7 (see [11])**.** *Let $u : [a, b] \longrightarrow \mathbb{R}_F$ be a fuzzy-valued function, where $[u(t)]_\alpha = [\underline{u}(t;\alpha), \overline{u}(t;\alpha)]$ for each $\alpha \in [0, 1]$.*

i) if u is (1)-differentiable, then $\underline{u}$ and $\overline{u}$ are differentiable functions and $[D_1^1 u(t)]_\alpha = [\underline{u}'(t;\alpha), \overline{u}'(t;\alpha)]$.

ii) if u is (2)-differentiable, then $\underline{u}$ and $\overline{u}$ are differentiable functions and $[D_2^1 u(t)]_\alpha = [\overline{u}'(t;\alpha), \underline{u}'(t;\alpha)]$.

Theorem 2.8 (see [13])**.** *Let $D_1^1 u : [a, b] \longrightarrow R_F$ or $D_2^1 u : [a, b] \longrightarrow R_F$ be fuzzy-valued functions, where $[u(t)]_\alpha = [\underline{u}(t;\alpha), \overline{u}(t;\alpha)]$ for each $\alpha \in [0, 1]$.*

i) if $D_1^1 u$ is (1)-differentiable, then $\underline{u}'$ and $\overline{u}'$ are differentiable functions and $[u''(t)]_\alpha = [\underline{u}''(t;\alpha), \overline{u}''(t;\alpha)]$.

ii) if $D_1^1 u$ is (2)-differentiable, then $\underline{u}'$ and $\overline{u}'$ are differentiable functions and $[u''(t)]_\alpha = [\overline{u}''(t;\alpha), \underline{u}''(t;\alpha)]$.

iii) if $D_2^1 u$ *is (1)-differentiable, then* $\underline{u}'$ *and* $\overline{u}'$ *are differentiable functions and* $[u''(t)]_\alpha = [\overline{u}''(t;\alpha), \underline{u}''(t;\alpha)]$.

iv) if $D_2^1 u$ *is (2)-differentiable, then* $\underline{u}'$ *and* $\overline{u}'$ *are differentiable functions and* $[u''(t)]_\alpha = [\underline{u}''(t;\alpha), \overline{u}''(t;\alpha)]$.

According to Theorem (2.8) that is showing us the way to translate FPDE (1)-(2) into two systems of crisp partial differential equations (PDEs), we might use the numerical method directly on the obtained crisp partial differential systems instead of rewriting this method for PDEs in a fuzzy setting as in the next Section.

3. The fuzzy partial differential equations

It is well known that all researchers in fuzzy mathematics transfer any fuzzy problem in mathematics to the system of PDEs or ODEs because there is no method to solve it without convert it to the system. Now, we study the FPDEs using the concept of a fuzzy derivative under strongly generalized differentiability in each step of differentiation.

Let $u : [0,l] \times (0,\infty) \longrightarrow \mathbb{R}_F$ be a continuous fuzzy-valued function such that $[u(x,t)]_\alpha = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$, and consequently the fuzzy functions f and g in Eq. (2) can be obtained via the Zadeh extension principle. If u satisfy FPDE (1)-(2), then we say that u is a fuzzy solution of FPDE (1)-(2). Now from Section 2 the defuzzification of FPDE (1)-(2) for all $\alpha \in [0,1]$ and to determine the lower and upper functions of the solution of FPDE (1)-(2), we discuss the following two cases:

Case 1. If we consider $\mu\,(D_x, D_t)\,u(x,t)$ by using the derivative in (1)-differentiable, then we have

(3) $$[\mu\,(D_x, D_t)\,u(x,t)]_\alpha = [\mu\,(D_x, D_t)\,\underline{u}(x,t;\alpha), \mu\,(D_x, D_t)\,\overline{u}(x,t;\alpha)],$$

where μ is a polynomial with a constant coefficient in D_x and D_t, and we should solve the system of crisp PDEs

(4) $$\varphi\,(D_t\underline{u}(x,t)) = \Phi(D_x\underline{u}(x,t)) + N\underline{u}(x,t;\alpha),\ 0 < x < l,\ t > 0,$$

(5) $$\varphi\,(D_t\overline{u}(x,t)) = \Phi(D_x\overline{u}(x,t)) + N\overline{u}(x,t;\alpha),\ 0 < x < l,\ t > 0,$$

(6) $$\underline{u}(x,0;\alpha) = \underline{f}(x;\alpha),\ \underline{u}_t(x,0;\alpha) = \underline{g}(x;\alpha),\ 0 \leq x \leq l,$$

(7) $$\overline{u}(x,0;\alpha) = \overline{f}(x;\alpha),\ \overline{u}_t(x,0;\alpha) = \overline{g}(x;\alpha),\ 0 \leq x \leq l.$$

Case 2. If we consider $\mu\,(D_x, D_t)\,u(x,t)$ by using the derivative in (2)-differentiable, then we have

(8) $$[\mu\,(D_x, D_t)\,u(x,t)]_\alpha = [\mu\,(D_x, D_t)\,\overline{u}(x,t;\alpha), \mu\,(D_x, D_t)\,\underline{u}(x,t;\alpha)],$$

and we should solve the system of crisp PDEs

(9) $\varphi\left(D_t\underline{u}(x,t)\right) = \Phi(D_x\overline{u}(x,t)) + N\overline{u}(x,t;\alpha),\ 0 < x < l,\ t > 0,$

(10) $\varphi\left(D_t\overline{u}(x,t)\right) = \Phi(D_x\underline{u}(x,t)) + N\underline{u}(x,t;\alpha),\ 0 < x < l,\ t > 0,$

(11) $\underline{u}(x,0;\alpha) = \underline{f}(x;\alpha),\ \underline{u}_t(x,0;\alpha) = \underline{g}(x;\alpha),\ 0 \leq x \leq l,$

(12) $\overline{u}(x,0;\alpha) = \overline{f}(x;\alpha),\ \overline{u}_t(x,0;\alpha) = \overline{g}(x;\alpha),\ 0 \leq x \leq l.$

Consequently, we use the strongly generalized differentiability in the present work. Under fitting conditions, the FPDE (1)-(2) under this interpretation has locally two solutions.

4. Analysis of the method

To illustrate the essential thoughts of the new method for solving linear and nonlinear FPDEs, we take the Laplace transform $\mathcal{L}$ on both sides of all equations in cases (1) and (2) as follows:
first system

(13) $\mathcal{L}\left\{\varphi\left(D_t\underline{u}(x,t)\right)\right\} = \mathcal{L}\left\{\Phi(D_x\underline{u}(x,t)) + N\underline{u}(x,t;\alpha)\right\},\ 0 < x < l,\ t > 0,$

(14) $\mathcal{L}\left\{\varphi\left(D_t\overline{u}(x,t)\right)\right\} = \mathcal{L}\left\{\Phi(D_x\overline{u}(x,t)) + N\overline{u}(x,t;\alpha)\right\},\ 0 < x < l,\ t > 0,$

second system

(15) $\mathcal{L}\left\{\varphi\left(D_t\underline{u}(x,t)\right)\right\} = \mathcal{L}\left\{\Phi(D_x\overline{u}(x,t)) + N\overline{u}(x,t;\alpha)\right\},\ 0 < x < l,\ t > 0,$

(16) $\mathcal{L}\left\{\varphi\left(D_t\overline{u}(x,t)\right)\right\} = \mathcal{L}\left\{\Phi(D_x\underline{u}(x,t)) + N\underline{u}(x,t;\alpha)\right\},\ 0 < x < l,\ t > 0.$

Using the differentiation rule of the Laplace transform, we have $\underline{U}(x,s,\underline{f},\underline{g},\Phi\left(D_x\underline{u}\right),N\underline{u})$ and $\overline{U}(x,s,\overline{f},\overline{g},\Phi\left(D_x\overline{u}\right),N\overline{u})$ in the first system (13)-(14), and $\underline{U}(x,s,\underline{f},\underline{g},\Phi\left(D_x\overline{u}\right),N\overline{u})$ and $\overline{U}(x,s,\overline{f},\overline{g},\Phi\left(D_x\underline{u}\right),N\underline{u})$ in the second system (15)-(16). After that, taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides gives:

(17) $\underline{u}(x,t;\alpha) = \mathcal{L}^{-1}\left\{\underline{U}(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha),\Phi\left(D_x\underline{u}(x,s;\alpha)\right),N\underline{u}(x,s;\alpha))\right\},$

(18) $\overline{u}(x,t;\alpha) = \mathcal{L}^{-1}\left\{\overline{U}\left(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha),\Phi\left(D_x\overline{u}(x,s;\alpha)\right),N\overline{u}(x,s;\alpha)\right)\right\},$

in the first system (13)-(14), and

(19) $\underline{u}(x,t;\alpha) = \mathcal{L}^{-1}\left\{\underline{U}\left(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha),\Phi\left(D_x\overline{u}(x,s;\alpha)\right),N\overline{u}(x,s;\alpha)\right)\right\},$

(20) $\overline{u}(x,t;\alpha) = \mathcal{L}^{-1}\left\{\overline{U}\left(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha),\Phi\left(D_x\underline{u}(x,s;\alpha)\right),N\underline{u}(x,s;\alpha)\right)\right\},$

in the second system (15)-(16). Now, we construct a HPM to obtain approximate-analytical solutions of FPDE (1)-(2). Obviously, from Eqs. (3) and (8) we consider Eq. (1) as:

(21) $$L(\underline{v}) = \underline{v} - N(\underline{v}) = 0,$$

(22) $$L(\overline{v}) = \overline{v} - N(\overline{v}) = 0,$$

where L is a linear operator and N is a nonlinear operator with solutions $\underline{u}$ and $\overline{u}$, respectively. By the homotopy technique (see [20, 21, 22, 23, 25]), we construct a homotopy $\hat{H} : [0, l] \times (0, \infty) \times [0, 1] \to \mathbb{R}_F$ which satisfies

$$\hat{H}(\underline{v}, \lambda) = (1 - \lambda)(L(\underline{v}) - L(\underline{u}_0)) + \lambda L(\underline{v}) = 0, \tag{23}$$

$$\hat{H}(\overline{v}, \lambda) = (1 - \lambda)(L(\overline{v}) - L(\overline{u}_0)) + \lambda L(\overline{v}) = 0, \tag{24}$$

where $\lambda \in [0, 1]$ is an embedding parameter, $\underline{u}_0$ and $\overline{u}_0$ are initial approximations of Eqs. (21) and (22) which satisfies the fuzzy initial conditions. It is obvious that

$$\hat{H}(\underline{v}, 0) = L(\underline{v}) - L(\underline{u}_0) = 0, \tag{25}$$

$$\hat{H}(\overline{v}, 0) = L(\overline{v}) - L(\overline{u}_0) = 0, \tag{26}$$

$$\hat{H}(\underline{v}, 1) = L(\underline{v}) = 0, \tag{27}$$

$$\hat{H}(\overline{v}, 1) = L(\overline{v}) = 0, \tag{28}$$

the changing process of λ from zero to unity is just that of $\hat{H}(\underline{v}, \lambda)$ and $\hat{H}(\overline{v}, \lambda)$ from $\underline{u}_0(x, t; \alpha)$, $\overline{u}_0(x, t; \alpha)$ to $\underline{u}(x, t; \alpha)$, $\overline{u}(x, t; \alpha)$, respectively. The embedding parameter $\lambda \in [0, 1]$ can be considered as an expanding parameter (see [19, 20, 26]), and assume that the solutions of Eqs. (21) and (22) can be written as a power series in λ:

$$\underline{v}(x, t; \alpha) = \sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x, t; \alpha), \tag{29}$$

$$\overline{v}(x, t; \alpha) = \sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x, t; \alpha). \tag{30}$$

Definition 4.1 (see [21])**.** The He polynomials is defined as follows:

$$H_n(v_0, v_1, ..., v_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} N \left(\sum_{k=0}^{n} \lambda^k v_k \right), \; n = 0, 1, 2, ... \, .$$

If the terms $N\underline{v}(x, t; \alpha)$ and $N\overline{v}(x, t; \alpha)$ are nonlinear functions, then they can be decomposed as:

$$N\underline{v}(x, t; \alpha) = \sum_{n=0}^{\infty} \lambda^n H_n \left(\underline{v}(x, t; \alpha) \right), \tag{31}$$

$$N\overline{v}(x, t; \alpha) = \sum_{n=0}^{\infty} \lambda^n H_n \left(\overline{v}(x, t; \alpha) \right), \tag{32}$$

where the H_n are He's polynomials and are calculated by the last definition.

Theorem 4.2. Suppose that N is an increasing nonlinear fuzzy-valued function, and $v = [\sum_{k=0}^{\infty} \lambda^k \underline{v}_k, \sum_{k=0}^{\infty} \lambda^k \overline{v}_k]$, then for an embedding parameter $\lambda \in [0, 1]$ we have

$$\frac{\partial^n}{\partial \lambda^n} N\left(v\right)_{|\lambda=0} = \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k\right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k\right)_{|\lambda=0} \right]. \tag{33}$$

Proof. Since N is an increasing function, then

$$\begin{aligned} N(v) &= \left[N\left(\sum_{k=0}^{\infty} \lambda^k \underline{v}_k\right), N\left(\sum_{k=0}^{\infty} \lambda^k \overline{v}_k\right) \right] \\ &= \left[N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k + \sum_{k=n+1}^{\infty} \lambda^k \underline{v}_k\right), N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k + \sum_{k=n+1}^{\infty} \lambda^k \overline{v}_k\right) \right], \end{aligned}$$

by using the derivative in (1)-differentiable, we have such result as following:

$$\begin{aligned} &\frac{\partial^n}{\partial \lambda^n} N\left(v\right)_{|\lambda=0} = \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k \underline{v}_k\right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k \overline{v}_k\right)_{|\lambda=0} \right] \\ &= \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k + \sum_{k=n+1}^{\infty} \lambda^k \underline{v}_k\right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k + \sum_{k=n+1}^{\infty} \lambda^k \overline{v}_k\right)_{|\lambda=0} \right] \\ &= \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k\right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k\right)_{|\lambda=0} \right]. \end{aligned}$$

So, the proof of the theorem is complete. □

Dependence on Theorem (4.2), if N is a decreasing nonlinear fuzzy-valued function, then

$$\frac{\partial^n}{\partial \lambda^n} N\left(v\right)_{|\lambda=0} = \left[\frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \overline{v}_k\right)_{|\lambda=0}, \frac{\partial^n}{\partial \lambda^n} N\left(\sum_{k=0}^{n} \lambda^k \underline{v}_k\right)_{|\lambda=0} \right]. \tag{34}$$

Now, substituting Eqs. (29), (31) in (17) and (30), (32) in (18) respectively, we get the solution of the first system (13)-(14) as:

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x, t; \alpha) = \mathcal{L}^{-1} \Big\{ &\underline{U}(x, s, \underline{f}(x; \alpha), \underline{g}(x; \alpha), \\ &\lambda \Phi(D_x \sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x, s; \alpha)), \lambda \sum_{n=0}^{\infty} \lambda^n H_n(\underline{v}(x, s; \alpha))) \Big\}, \end{aligned} \tag{35}$$

$$\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,t;\alpha)=\mathcal{L}^{-1}\Big\{\overline{U}(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha),\\ \lambda\Phi(D_x\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,s;\alpha)),\lambda\sum_{n=0}^{\infty}\lambda^n H_n(\overline{v}(x,s;\alpha)))\Big\}. \tag{36}$$

Similarly, substituting Eqs. (29), (30) and (32) in (19) and (29), (30) and (31) in (20) respectively, we get the solution of the second system (15)-(16) as:

$$\sum_{n=0}^{\infty}\lambda^n\underline{v}_n(x,t;\alpha)=\mathcal{L}^{-1}\Big\{\underline{U}(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha),\\ \lambda\Phi(D_x\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,s;\alpha)),\lambda\sum_{n=0}^{\infty}\lambda^n H_n(\overline{v}(x,s;\alpha)))\Big\}, \tag{37}$$

$$\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,t;\alpha)=\mathcal{L}^{-1}\Big\{\overline{U}(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha),\\ \lambda\Phi(D_x\sum_{n=0}^{\infty}\lambda^n\underline{v}_n(x,s;\alpha)),\lambda\sum_{n=0}^{\infty}\lambda^n H_n(\underline{v}(x,s;\alpha)))\Big\}, \tag{38}$$

which is the combination of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of λ, the following approximations in Eqs. (35) and (36) are obtained

$$\begin{aligned}
\lambda^0 &: \underline{v}_0=\mathcal{L}^{-1}\left\{\underline{U}(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha))\right\},\\
&\quad \overline{v}_0=\mathcal{L}^{-1}\left\{\overline{U}(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha))\right\},\\
\lambda^1 &: \underline{v}_1=\mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\underline{v}_0(x,s;\alpha)),H_0(\underline{v}(x,s;\alpha)))\right\},\\
&\quad \overline{v}_1=\mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\overline{v}_0(x,s;\alpha)),H_0(\overline{v}(x,s;\alpha)))\right\},\\
\lambda^2 &: \underline{v}_2=\mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\underline{v}_1(x,s;\alpha)),H_1(\underline{v}(x,s;\alpha)))\right\},\\
&\quad \overline{v}_2=\mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\overline{v}_1(x,s;\alpha)),H_1(\overline{v}(x,s;\alpha)))\right\},\\
\lambda^3 &: \underline{v}_3=\mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\underline{v}_2(x,s;\alpha)),H_2(\underline{v}(x,s;\alpha)))\right\},\\
&\quad \overline{v}_3=\mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\overline{v}_2(x,s;\alpha)),H_2(\overline{v}(x,s;\alpha)))\right\},\\
&\quad \vdots
\end{aligned}$$

Furthermore, the following approximations in Eqs. (37) and (38) are obtained

$$\begin{aligned}
\lambda^0 &: \underline{v}_0=\mathcal{L}^{-1}\left\{\underline{U}(x,s,\underline{f}(x;\alpha),\underline{g}(x;\alpha))\right\},\\
&\quad \overline{v}_0=\mathcal{L}^{-1}\left\{\overline{U}(x,s,\overline{f}(x;\alpha),\overline{g}(x;\alpha))\right\},
\end{aligned}$$

$$\begin{aligned}
\lambda^1 \quad &: \quad \underline{v}_1 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\overline{v}_0(x,s;\alpha)), H_0(\overline{v}(x,s;\alpha)))\right\},\\
&\quad\ \overline{v}_1 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\underline{v}_0(x,s;\alpha)), H_0(\underline{v}(x,s;\alpha)))\right\},\\
\lambda^2 \quad &: \quad \underline{v}_2 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\overline{v}_1(x,s;\alpha)), H_1(\overline{v}(x,s;\alpha)))\right\},\\
&\quad\ \overline{v}_2 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\underline{v}_1(x,s;\alpha)), H_1(\underline{v}(x,s;\alpha)))\right\},\\
\lambda^3 \quad &: \quad \underline{v}_3 = \mathcal{L}^{-1}\left\{\underline{U}(\Phi(D_x\overline{v}_2(x,s;\alpha)), H_2(\overline{v}(x,s;\alpha)))\right\},\\
&\quad\ \overline{v}_3 = \mathcal{L}^{-1}\left\{\overline{U}(\Phi(D_x\underline{v}_2(x,s;\alpha)), H_2(\underline{v}(x,s;\alpha)))\right\},\\
&\quad\ \vdots
\end{aligned}$$

When $\lambda \to 1$, Eqs. (23) and (24) corresponds to Eqs. (21) and (22), and also Eqs. (29) and (30) becomes the approximate solutions of Eqs. (21) and (22), i.e.,

$$\underline{u}(x,t;\alpha) = \lim_{\lambda\to 1}\underline{v}(x,t;\alpha) = \sum_{n=0}^{\infty}\underline{v}_n(x,t;\alpha), \tag{39}$$

$$\overline{u}(x,t;\alpha) = \lim_{\lambda\to 1}\overline{v}(x,t;\alpha) = \sum_{n=0}^{\infty}\overline{v}_n(x,t;\alpha). \tag{40}$$

The series (39) and (40) are convergent for most cases, and as well the rate of convergence depends on $L(\underline{v})$ and $L(\overline{v})$ (see [23]). Now will be discuss the convergence on nonlinear operator N.

Theorem 4.3. *Suppose that (R_F, H_d) be a Banach space and $N : R_F \to R_F$ is a contraction nonlinear mapping with a contractivity $\rho \in (0,1)$, that is*

$$H_d\left(N(v), N(v^*)\right) \le \rho H_d\left(v, v^*\right),$$

for all $v = [\underline{v}, \overline{v}]$, $v^* = [\underline{v}^*, \overline{v}^*] \in R_F$. The sequence generated by the homotopy perturbation method will be regarded as

$$V_n = N(V_{n-1}),\ V_{n-1} = \sum_{i=0}^{n-1} u_i,\ n = 1,2,3,...,$$

and suppose that $V_0 = v_0 = u_0 \in B_r(u) = \{u^* \in R_F \mid H_d\left(u, u^*\right) < r\}$, then we have the following statements:

(*i*) $H_d\left(V_n, u\right) \le \rho^n H_d\left(v_0, u\right)$.

(*ii*) $V_n \in B_r(u)$.

(*iii*) $\lim_{n\to\infty} V_n = u$.

Proof. (i) Since $(\mathbb{R}_F, H_d)$ is a complete metric space and N is a contraction on $(\mathbb{R}_F, H_d)$, then according to Banach's fixed point theorem the mapping N

has precisely one fixed point $u = [\underline{u}, \overline{u}] \in \mathbb{R}_F$ such that $N(u) = u$, that is, $[N(\underline{u}), N(\overline{u})] = [\underline{u}, \overline{u}]$. Now by the induction method, if $n = 1$ we have

$$H_d(V_1, u) = H_d(N(V_0), N(u)) = H_d(N(v_0), N(u)) \leq \rho H_d(v_0, u).$$

Again, if $n = 2$ we have

$$H_d(V_2, u) = H_d(N(V_1), N(u)) \leq \rho H_d(V_1, u) \leq \rho^2 H_d(v_0, u).$$

Assume that $H_d(V_{n-1}, u) \leq \rho^{n-1} H_d(v_0, u)$. In the same way, it is easy to see that

$$H_d(V_n, u) = H_d(N(V_{n-1}), N(u)) \leq \rho H_d(V_{n-1}, u) \leq \rho^n H_d(v_0, u).$$

(ii) Since $0 < \rho < 1$ and from (i), we have

$$H_d(V_n, u) \leq \rho^n H_d(v_0, u) < \rho^n r < r.$$

(iii) We know $\lim_{n\to\infty} \rho^n = 0$. Using (i), we have $\lim_{n\to\infty} H_d(V_n, u) = 0$, that is, $\lim\limits_{n\to\infty} V_n = u$. So, the proof of the theorem is complete. □

5. Numerical experiments

In this section, we provide three numerical examples to demonstrate the application of the C(LT-HPM) for solving the FPDEs. More precisely, in Examples (5.1) and (5.2), we give two locally solutions under strongly generalized differentiability for linear FPDEs unlike the previous papers (see [5, 6]) which give only one locally solution under Seikkala derivative. To test the C(LT-HPM) upon nonlinear FPDEs, we initially apply this method on two linear FPDEs.

Example 5.1. Consider the following linear fuzzy partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \; 0 < x < 1, \; t > 0, \tag{41}$$

subject to the fuzzy initial condition

$$u(x,0) = K(\alpha)\sin(\pi x), \; 0 \leq x \leq 1, \tag{42}$$

where $K(\alpha) = [\alpha - 1, 1 - \alpha]$ for all $\alpha \in [0,1]$.

According to Section 3, the FPDE (41)-(42) is equivalent to the following systems of crisp partial differential equations:
first system

$$\frac{\partial \underline{u}(x,t;\alpha)}{\partial t} = \frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}, \; 0 < x < 1, \; t > 0, \tag{43}$$

$$\frac{\partial \overline{u}(x,t;\alpha)}{\partial t} = \frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}, \; 0 < x < 1, \; t > 0, \tag{44}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = (\alpha-1)\sin(\pi x),\ 0 \le x \le 1, \tag{45}$$

$$\overline{u}(x,0;\alpha) = (1-\alpha)\sin(\pi x),\ 0 \le x \le 1, \tag{46}$$

second system

$$\frac{\partial \underline{u}(x,t;\alpha)}{\partial t} = \frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{47}$$

$$\frac{\partial \overline{u}(x,t;\alpha)}{\partial t} = \frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{48}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = (\alpha-1)\sin(\pi x),\ 0 \le x \le 1, \tag{49}$$

$$\overline{u}(x,0;\alpha) = (1-\alpha)\sin(\pi x),\ 0 \le x \le 1. \tag{50}$$

Firstly, we take the Laplace transform $\mathcal{L}$ on both sides of Eqs. (43) and (44):

$$\underline{U}(x,s;\alpha) = \frac{(\alpha-1)\sin(\pi x)}{s} + \frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}\right\},\ 0 < x < 1, \tag{51}$$

$$\overline{U}(x,s;\alpha) = \frac{(1-\alpha)\sin(\pi x)}{s} + \frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}\right\},\ 0 < x < 1. \tag{52}$$

Taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides of Eqs. (51) and (52) gives

$$\underline{u}(x,t;\alpha) = (\alpha-1)\sin(\pi x) + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}\right\}\right\}, \tag{53}$$
$$0 < x < 1,\ t > 0,$$

$$\overline{u}(x,t;\alpha) = (1-\alpha)\sin(\pi x) + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}\right\}\right\}, \tag{54}$$
$$0 < x < 1,\ t > 0.$$

Now, applying the HPM method

$$\sum_{n=0}^{\infty} \lambda^n \underline{v}_n(x,t;\alpha) = (\alpha-1)\sin(\pi x)$$
$$+ \lambda\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n \underline{v}_n(x,t;\alpha)\right)\right\}\right\}, \tag{55}$$

$$\sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x,t;\alpha) = (1-\alpha)\sin(\pi x)$$
$$+ \lambda\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n \overline{v}_n(x,t;\alpha)\right)\right\}\right\}. \tag{56}$$

Comparing the coefficient of like powers of λ, the following approximations are obtained

$$\lambda^0: \quad \underline{v}_0(x,t;\alpha) = (\alpha-1)\sin(\pi x),\ \overline{v}_0(x,t;\alpha) = (1-\alpha)\sin(\pi x),$$

$$\lambda^1: \quad \begin{aligned} \underline{v}_1(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_0(x,t;\alpha)}{\partial x^2}\right\}\right\} = -\pi^2(\alpha-1)t\sin(\pi x),\\ \overline{v}_1(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_0(x,t;\alpha)}{\partial x^2}\right\}\right\} = -\pi^2(1-\alpha)t\sin(\pi x),\end{aligned}$$

$$\lambda^2: \quad \begin{aligned} \underline{v}_2(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_1(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{\pi^4(\alpha-1)t^2}{2!}\sin(\pi x),\\ \overline{v}_2(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_1(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{\pi^4(1-\alpha)t^2}{2!}\sin(\pi x),\end{aligned}$$

$$\lambda^3: \quad \begin{aligned} \underline{v}_3(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_2(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{-\pi^6(\alpha-1)t^3}{3!}\sin(\pi x),\\ \overline{v}_3(x,t;\alpha) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_2(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{-\pi^6(1-\alpha)t^3}{3!}\sin(\pi x),\end{aligned}$$

$$\vdots$$

From Eqs. (39) and (40), the approximate solution of the first system is given by

$$\begin{aligned}\underline{u}(x,t;\alpha) &= (\alpha-1)\sin(\pi x) - \pi^2(\alpha-1)t\sin(\pi x) + \frac{\pi^4(\alpha-1)t^2}{2!}\sin(\pi x) - ...\\ &= (\alpha-1)\sin(\pi x)\left(1-\pi^2 t + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!} + ...\right),\end{aligned}$$

$$\begin{aligned}\overline{u}(x,t;\alpha) &= (1-\alpha)\sin(\pi x) - \pi^2(1-\alpha)t\sin(\pi x) + \frac{\pi^4(1-\alpha)t^2}{2!}\sin(\pi x) - ...\\ &= (1-\alpha)\sin(\pi x)\left(1-\pi^2 t + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!} + ...\right).\end{aligned}$$

These series have the closed form as $n \to \infty$. Therefore, the exact solution of the first system is given by

$$u(x,t) = \left[(\alpha-1)e^{-\pi^2 t}\sin(\pi x), (1-\alpha)e^{-\pi^2 t}\sin(\pi x)\right].$$

Secondly, we take the Laplace transform $\mathcal{L}$ on both sides of Eqs. (47) and (48):

$$\underline{U}(x,s;\alpha) = \frac{(\alpha-1)\sin(\pi x)}{s} + \frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}\right\},\ 0<x<1, \tag{57}$$

$$\overline{U}(x,s;\alpha) = \frac{(1-\alpha)\sin(\pi x)}{s} + \frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}\right\},\ 0<x<1. \tag{58}$$

Taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides of Eqs. (57) and (58) gives

$$\underline{u}(x,t;\alpha) = (\alpha-1)\sin(\pi x) + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}\right\}\right\},\ 0<x<1,\ t>0, \tag{59}$$

$$\overline{u}(x,t;\alpha) = (1-\alpha)\sin(\pi x) + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}\right\}\right\},\ 0<x<1,\ t>0. \tag{60}$$

Now, applying the HPM method

$$\sum_{n=0}^{\infty}\lambda^n \underline{v}_n(x,t;\alpha) = (\alpha-1)\sin(\pi x) + \lambda\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n \overline{v}_n(x,t;\alpha)\right)\right\}\right\}, \tag{61}$$

$$\sum_{n=0}^{\infty}\lambda^n \overline{v}_n(x,t;\alpha) = (1-\alpha)\sin(\pi x) + \lambda\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n \underline{v}_n(x,t;\alpha)\right)\right\}\right\}. \tag{62}$$

Comparing the coefficient of like powers of λ, the following approximations are obtained

$$\lambda^0: \quad \underline{v}_0(x,t;\alpha) = (\alpha-1)\sin(\pi x),\ \overline{v}_0(x,t;\alpha) = (1-\alpha)\sin(\pi x),$$

$$\lambda^1: \quad \underline{v}_1(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_0(x,t;\alpha)}{\partial x^2}\right\}\right\} = -\pi^2(1-\alpha)t\sin(\pi x),$$
$$\overline{v}_1(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_0(x,t;\alpha)}{\partial x^2}\right\}\right\} = -\pi^2(\alpha-1)t\sin(\pi x),$$

$$\lambda^2: \quad \underline{v}_2(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_1(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{\pi^4(\alpha-1)t^2}{2!}\sin(\pi x),$$
$$\overline{v}_2(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_1(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{\pi^4(1-\alpha)t^2}{2!}\sin(\pi x),$$

$$\lambda^3: \quad \underline{v}_3(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_2(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{-\pi^6(1-\alpha)t^3}{3!}\sin(\pi x),$$
$$\overline{v}_3(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_2(x,t;\alpha)}{\partial x^2}\right\}\right\} = \frac{-\pi^6(\alpha-1)t^3}{3!}\sin(\pi x),$$

$$\vdots$$

From Eqs. (39) and (40), the approximate solution of the second system is given by

$$\begin{aligned}\underline{u}(x,t;\alpha) &= (\alpha-1)\sin(\pi x) - \pi^2(1-\alpha)t\sin(\pi x) + \frac{\pi^4(\alpha-1)t^2}{2!}\sin(\pi x) - ...\\ &= (\alpha-1)\sin(\pi x)\left(1+\pi^2 t + \frac{\pi^4 t^2}{2!} + \frac{\pi^6 t^3}{3!} + ...\right),\end{aligned}$$

$$\begin{aligned}\overline{u}(x,t;\alpha) &= (1-\alpha)\sin(\pi x) - \pi^2(\alpha-1)t\sin(\pi x) + \frac{\pi^4(1-\alpha)t^2}{2!}\sin(\pi x) - ...\\ &= (1-\alpha)\sin(\pi x)\left(1+\pi^2 t + \frac{\pi^4 t^2}{2!} + \frac{\pi^6 t^3}{3!} + ...\right).\end{aligned}$$

These series have the closed form as $n \to \infty$. Therefore, the exact solution of the second system is given by

$$u(x,t) = \left[(\alpha-1)e^{\pi^2 t}\sin(\pi x), (1-\alpha)e^{\pi^2 t}\sin(\pi x)\right].$$

Example 5.2. Consider the following linear fuzzy partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 4\frac{\partial^2 u(x,t)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{63}$$

subject to the fuzzy initial conditions

$$u(x,0) = K(\alpha)\sin(\pi x),\ u_t(x,0) = 0,\ 0 \le x \le 1, \tag{64}$$

where $K(\alpha) = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$ for all $\alpha \in [0,1]$.

According to Section 3, the FPDE (63)-(64) is equivalent to the following systems of crisp partial differential equations:
first system

$$\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial t^2} = 4\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{65}$$

$$\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial t^2} = 4\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{66}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = (0.75 + 0.25\alpha)\sin(\pi x),\ \underline{u}_t(x,0;\alpha) = 0,\ 0 \le x \le 1, \tag{67}$$

$$\overline{u}(x,0;\alpha) = (1.25 - 0.25\alpha)\sin(\pi x),\ \overline{u}_t(x,0;\alpha) = 0,\ 0 \le x \le 1, \tag{68}$$

second system

$$\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial t^2} = 4\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{69}$$

$$\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial t^2} = 4\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{70}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = (0.75 + 0.25\alpha)\sin(\pi x),\ \underline{u}_t(x,0;\alpha) = 0,\ 0 \le x \le 1, \tag{71}$$

$$\overline{u}(x,0;\alpha) = (1.25 - 0.25\alpha)\sin(\pi x),\ \overline{u}_t(x,0;\alpha) = 0,\ 0 \le x \le 1. \tag{72}$$

Again, by using the same procedure as mentioned in Section 4, the first few components in the first system are given by

$$\begin{aligned}
\lambda^0 \quad : \quad & \underline{v}_0(x,t;\alpha) = (0.75 + 0.25\alpha)\sin(\pi x), \\
& \overline{v}_0(x,t;\alpha) = (1.25 - 0.25\alpha)\sin(\pi x), \\
\lambda^1 \quad : \quad & \underline{v}_1(x,t;\alpha) = -2\pi^2(0.75 + 0.25\alpha)t^2\sin(\pi x), \\
& \overline{v}_1(x,t;\alpha) = -2\pi^2(1.25 - 0.25\alpha)t^2\sin(\pi x), \\
\lambda^2 \quad : \quad & \underline{v}_2(x,t;\alpha) = \frac{16\pi^4(0.75 + 0.25\alpha)t^4}{4!}\sin(\pi x), \\
& \overline{v}_2(x,t;\alpha) = \frac{16\pi^4(1.25 - 0.25\alpha)t^4}{4!}\sin(\pi x), \\
\lambda^3 \quad : \quad & \underline{v}_3(x,t;\alpha) = \frac{-64\pi^6(0.75 + 0.25\alpha)t^6}{6!}\sin(\pi x), \\
& \overline{v}_3(x,t;\alpha) = \frac{-64\pi^6(1.25 - 0.25\alpha)t^6}{6!}\sin(\pi x), \\
& \vdots
\end{aligned}$$

From Eqs. (39) and (40), the approximate solution of the first system is given by

$$\underline{u}(x,t;\alpha) = (0.75 + 0.25\alpha)\sin(\pi x)\left(1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \frac{(2\pi t)^6}{6!} + ...\right),$$

$$\overline{u}(x,t;\alpha) = (1.25 - 0.25\alpha)\sin(\pi x)\left(1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \frac{(2\pi t)^6}{6!} + ...\right).$$

These series have the closed form as $n \to \infty$. Therefore, the exact solution of the first system is given by

$$u(x,t) = \left[(0.75 + 0.25\alpha)\sin(\pi x)\cos(2\pi t), (1.25 - 0.25\alpha)\sin(\pi x)\cos(2\pi t)\right].$$

After that, the first few components in the second system are given by

$$\begin{aligned}
\lambda^0 \quad &: \quad \underline{v}_0(x,t;\alpha) = (0.75+0.25\alpha)\sin(\pi x),\\
&\quad\ \overline{v}_0(x,t;\alpha) = (1.25-0.25\alpha)\sin(\pi x),\\
\lambda^1 \quad &: \quad \underline{v}_1(x,t;\alpha) = -2(\pi t)^2(1.25-0.25\alpha)\sin(\pi x),\\
&\quad\ \overline{v}_1(x,t;\alpha) = -2(\pi t)^2(0.75+0.25\alpha)\sin(\pi x),\\
\lambda^2 \quad &: \quad \underline{v}_2(x,t;\alpha) = \frac{16(\pi t)^4(0.75+0.25\alpha)}{4!}\sin(\pi x),\\
&\quad\ \overline{v}_2(x,t;\alpha) = \frac{16(\pi t)^4(1.25-0.25\alpha)}{4!}\sin(\pi x),\\
\lambda^3 \quad &: \quad \underline{v}_3(x,t;\alpha) = \frac{-64(\pi t)^6(1.25-0.25\alpha)}{6!}\sin(\pi x),\\
&\quad\ \overline{v}_3(x,t;\alpha) = \frac{-64(\pi t)^6(0.75+0.25\alpha)}{6!}\sin(\pi x),\\
\lambda^4 \quad &: \quad \underline{v}_4(x,t;\alpha) = \frac{256(\pi t)^8(0.75+0.25\alpha)}{8!}\sin(\pi x),\\
&\quad\ \overline{v}_4(x,t;\alpha) = \frac{256(\pi t)^8(1.25-0.25\alpha)}{8!}\sin(\pi x),\\
\lambda^5 \quad &: \quad \underline{v}_5(x,t;\alpha) = \frac{-1024(\pi t)^{10}(1.25-0.25\alpha)}{10!}\sin(\pi x),\\
&\quad\ \overline{v}_5(x,t;\alpha) = \frac{-1024(\pi t)^{10}(0.75+0.25\alpha)}{10!}\sin(\pi x),\\
\lambda^6 \quad &: \quad \underline{v}_6(x,t;\alpha) = \frac{4096(\pi t)^{12}(0.75+0.25\alpha)}{12!}\sin(\pi x),\\
&\quad\ \overline{v}_6(x,t;\alpha) = \frac{4096(\pi t)^{12}(1.25-0.25\alpha)}{12!}\sin(\pi x),\\
\lambda^7 \quad &: \quad \underline{v}_7(x,t;\alpha) = \frac{-16\,384(\pi t)^{14}(1.25-0.25\alpha)}{14!}\sin(\pi x),\\
&\quad\ \overline{v}_7(x,t;\alpha) = \frac{-16\,384(\pi t)^{14}(0.75+0.25\alpha)}{14!}\sin(\pi x),\\
&\quad\ \vdots
\end{aligned}$$

Therefore, the approximate solution of the second system is given by

$$\begin{aligned}
\underline{u}(x,t;\alpha) &= (0.75+0.25\alpha)\sin(\pi x)\left(1+\frac{(2\pi t)^4}{4!}+\frac{(2\pi t)^8}{8!}+\frac{(2\pi t)^{12}}{12!}+...\right)\\
&-(1.25-0.25\alpha)\sin(\pi x)\left(\frac{(2\pi t)^2}{2!}+\frac{(2\pi t)^6}{6!}+\frac{(2\pi t)^{10}}{10!}+\frac{(2\pi t)^{14}}{14!}+...\right)\\
&=\left((0.75+0.25\alpha)\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n}}{(4n)!}-(1.25-0.25\alpha)\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n+2}}{(4n+2)!}\right)\sin(\pi x),
\end{aligned}$$

$$\overline{u}(x,t;\alpha) = (1.25-0.25\alpha)\sin(\pi x)\left(1+\frac{(2\pi t)^4}{4!}+\frac{(2\pi t)^8}{8!}+\frac{(2\pi t)^{12}}{12!}+...\right)$$
$$-(0.75+0.25\alpha)\sin(\pi x)\left(\frac{(2\pi t)^2}{2!}+\frac{(2\pi t)^6}{6!}+\frac{(2\pi t)^{10}}{10!}+\frac{(2\pi t)^{14}}{14!}+...\right)$$
$$=\left((1.25-0.25\alpha)\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n}}{(4n)!}-(0.75+0.25\alpha)\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n+2}}{(4n+2)!}\right)\sin(\pi x).$$

By the ratio test, the series $\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n}}{(4n)!}$ and $\sum_{n=0}^{\infty}\frac{(2\pi t)^{4n+2}}{(4n+2)!}$ are convergent for all $t\in(0,\infty)$.

To allow a clear overview of our work and to demonstrate the discussed method, we present the first attempt to solve nonlinear FPDEs under strongly generalized differentiability.

Example 5.3. Consider the following nonlinear fuzzy partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\sin(u(x,t))=\frac{\partial^2 u(x,t)}{\partial x^2},\ 0<x<1,\ t>0, \tag{73}$$

subject to the fuzzy initial conditions

$$u(x,0)=0,\ u_t(x,0)=K(\alpha)sechx,\ 0\le x\le 1, \tag{74}$$

where $K(\alpha)=[\alpha,2-\alpha]$ for all $\alpha\in[0,1]$.

It is noted here that $\sin x$ is a continuous increasing function on $(0,1)$. By using Zadeh's extension principle, we get $[\sin(u(x,t))]_\alpha=[\sin(\underline{u}(x,t)),\sin(\overline{u}(x,t))]$ for all $\alpha\in[0,1]$. According to Section 3, the FPDE (73)-(74) is equivalent to the following systems of crisp partial differential equations:
first system

$$\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial t^2}+\sin(\underline{u}(x,t))=\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0<x<1,\ t>0, \tag{75}$$

$$\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial t^2}+\sin(\overline{u}(x,t))=\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0<x<1,\ t>0, \tag{76}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha)=0,\ \underline{u}_t(x,0;\alpha)=\alpha sechx,\ 0\le x\le 1, \tag{77}$$

$$\overline{u}(x,0;\alpha)=0,\ \overline{u}_t(x,0;\alpha)=(2-\alpha)sechx,\ 0\le x\le 1, \tag{78}$$

second system

$$\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial t^2}+\sin(\overline{u}(x,t))=\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2},\ 0<x<1,\ t>0, \tag{79}$$

$$\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial t^2} + \sin(\underline{u}(x,t)) = \frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2},\ 0 < x < 1,\ t > 0, \tag{80}$$

subject to the initial conditions

$$\underline{u}(x,0;\alpha) = 0,\ \underline{u}_t(x,0;\alpha) = \alpha sechx,\ 0 \le x \le 1, \tag{81}$$

$$\overline{u}(x,0;\alpha) = 0,\ \overline{u}_t(x,0;\alpha) = (2-\alpha) sechx,\ 0 \le x \le 1. \tag{82}$$

Firstly, we take the Laplace transform $\mathcal{L}$ on both sides of Eqs. (75) and (76):

$$\underline{U}(x,s;\alpha) = \frac{\alpha sechx}{s^2} + \frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2} - \sin(\underline{u}(x,t))\right\},\ 0 < x < 1, \tag{83}$$

$$\overline{U}(x,s;\alpha) = \frac{(2-\alpha) sechx}{s^2} + \frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2} - \sin(\overline{u}(x,t))\right\}, 0<x<1. \tag{84}$$

Taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides of Eqs. (83) and (84) gives

$$\underline{u}(x,t;\alpha) = \alpha t sechx + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2} - \sin(\underline{u}(x,t))\right\}\right\}, \tag{85}$$

$$\overline{u}(x,t;\alpha) = (2-\alpha) t sechx + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2} - \sin(\overline{u}(x,t))\right\}\right\}. \tag{86}$$

Since

$$\begin{aligned}
\sin(\underline{u}(x,t)) &= \underline{u}(x,t) - \frac{(\underline{u}(x,t))^3}{3!} + ... + \frac{(-1)^n}{(2n+1)!}(\underline{u}(x,t))^{2n+1} + ... \\
&= \sum_{n=0}^{\infty}\lambda^n \underline{u}_n(x,t) - \frac{1}{3!}\left(\sum_{n=0}^{\infty}\lambda^n \underline{u}_n(x,t)\right)^3 + \frac{1}{5!}\left(\sum_{n=0}^{\infty}\lambda^n \underline{u}_n(x,t)\right)^5 - ... \\
&= \sum_{n=0}^{\infty}\lambda^n \underline{u}_n(x,t) - \frac{1}{3!}\sum_{n=0}^{\infty}\lambda^n A_n(\underline{u}(x,t)) + \frac{1}{5!}\sum_{n=0}^{\infty}\lambda^n B_n(\underline{u}(x,t)) - ... \\
&= \sum_{n=0}^{\infty}\lambda^n H_n(\underline{u}(x,t)),
\end{aligned} \tag{87}$$

where A_n, B_n are Adomain polynomials (see [23, 24]) and H_n are He's polynomials. From Section 4, then Eqs. (85) and (86) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}\lambda^n \underline{v}_n(x,t;\alpha) = \alpha t sechx \\
&+ \lambda\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n \underline{v}_n(x,t;\alpha)\right) - \sum_{n=0}^{\infty}\lambda^n H_n(\underline{v}(x,t;\alpha))\right\}\right\},
\end{aligned} \tag{88}$$

$$\sum_{n=0}^{\infty} \lambda^n \overline{v}_n(x,t;\alpha) = (2-\alpha)tsechx$$
$$+\lambda\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\lambda^n\overline{v}_n(x,t;\alpha)\right)-\sum_{n=0}^{\infty}\lambda^n H_n(\overline{v}(x,t;\alpha))\right\}\right\}. \tag{89}$$

From Theorem (4.2) and Eq. (87), we have

$$\frac{\partial^n}{\partial\lambda^n}\left(A_0+\lambda A_1+...+\lambda^n A_n\right)_{|\lambda=0}$$
$$=\frac{\partial^n}{\partial\lambda^n}\left(\underline{u}_0(x,t)+\lambda\underline{u}_1(x,t)+...+\lambda^n\underline{u}_n(x,t)\right)^3_{|\lambda=0}, \tag{90}$$

$$\frac{\partial^n}{\partial\lambda^n}\left(B_0+\lambda B_1+...+\lambda^n B_n\right)_{|\lambda=0}$$
$$=\frac{\partial^n}{\partial\lambda^n}\left(\underline{u}_0(x,t)+\lambda\underline{u}_1(x,t)+...+\lambda^n\underline{u}_n(x,t)\right)^5_{|\lambda=0}. \tag{91}$$

Now, if $n=0$, we get

$$A_0=\underline{u}_0^3(x,t),\ B_0=\underline{u}_0^5(x,t), \tag{92}$$

if $n=1$, we get

$$A_1=3\underline{u}_0^2(x,t)\underline{u}_1(x,t),\ B_1=5\underline{u}_0^4(x,t)\underline{u}_1(x,t), \tag{93}$$

if $n=2$, we get

$$A_2=3\underline{u}_0^2(x,t)\underline{u}_2(x,t)+3\underline{u}_1^2(x,t)\underline{u}_0(x,t),$$
$$B_2=5\underline{u}_0^4(x,t)\underline{u}_2(x,t)+10\underline{u}_0^3(x,t)\underline{u}_1^2(x,t), \tag{94}$$

$$\vdots$$

From Eqs. (87), (92), (93) and (94), we have

$$H_0(\underline{u}(x,t))=\underline{u}_0(x,t)-\frac{\underline{u}_0^3(x,t)}{3!}+\frac{\underline{u}_0^5(x,t)}{5!}-..., \tag{95}$$
$$H_1(\underline{u}(x,t))=\underline{u}_1(x,t)-\frac{3\underline{u}_0^2(x,t)\underline{u}_1(x,t)}{3!}+\frac{5\underline{u}_0^4(x,t)\underline{u}_1(x,t)}{5!}-..., \tag{96}$$
$$H_2(\underline{u}(x,t))=\underline{u}_2(x,t)-\frac{3\underline{u}_0^2(x,t)\underline{u}_2(x,t)+3\underline{u}_1^2(x,t)\underline{u}_0(x,t)}{3!}$$
$$+\frac{5\underline{u}_0^4(x,t)\underline{u}_2(x,t)+10\underline{u}_0^3(x,t)\underline{u}_1^2(x,t)}{5!}-..., \tag{97}$$

$$\vdots$$

Similarly, we find $H_0(\overline{u}(x,t))$, $H_1(\overline{u}(x,t))$, $H_2(\overline{u}(x,t))$,... . The first few components in the first system are given by

$$\begin{aligned}
\lambda^0 \quad : \quad & \underline{v}_0(x,t;\alpha) = \alpha t sechx, \\
& \overline{v}_0(x,t;\alpha) = (2-\alpha) t sechx, \\
\lambda^1 \quad : \quad & \underline{v}_1(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_0(x,t;\alpha)}{\partial x^2} - H_0(\underline{u}(x,t))\right\}\right\} \\
& = \frac{-\alpha t^3 sechx \tanh x}{3!} - \frac{\alpha t^3 sechx}{3!} + \frac{(\alpha sechx)^3 t^5}{5!} - \frac{(\alpha sechx)^5 t^7}{7!} + ..., \\
& \overline{v}_1(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_0(x,t;\alpha)}{\partial x^2} - H_0(\overline{u}(x,t))\right\}\right\} \\
& = \frac{-(2-\alpha) t^3 sechx \tanh x}{3!} - \frac{(2-\alpha) t^3 sechx}{3!} \\
& + \frac{((2-\alpha) sechx)^3 t^5}{5!} - \frac{((2-\alpha) sechx)^5 t^7}{7!} + ..., \\
\lambda^2 \quad : \quad & \underline{v}_2(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \underline{v}_1(x,t;\alpha)}{\partial x^2} - H_1(\underline{u}(x,t))\right\}\right\} \\
& = \frac{330(\alpha sechx)^9 t^{13}}{13!} - \frac{162(\alpha sechx)^7 t^{11}}{11!} \\
& + \frac{(\alpha sechx)^5 t^9}{9!}(35\tanh x + 62 - 30\tanh^2 x) \\
& + \frac{(\alpha sechx)^3 t^7}{7!}(12\tanh^2 x - 14 - 10\tanh x) \\
& + \frac{(\alpha sechx) t^5}{5!}(2 - 6\tanh^3 x - 2\tanh^2 x + 6\tanh x) + ..., \\
& \overline{v}_2(x,t;\alpha) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2 \overline{v}_1(x,t;\alpha)}{\partial x^2} - H_1(\overline{u}(x,t))\right\}\right\} \\
& = \frac{330((2-\alpha) sechx)^9 t^{13}}{13!} - \frac{162((2-\alpha) sechx)^7 t^{11}}{11!} \\
& + \frac{((2-\alpha) sechx)^5 t^9}{9!}(35\tanh x + 62 - 30\tanh^2 x) \\
& + \frac{((2-\alpha) sechx)^3 t^7}{7!}(12\tanh^2 x - 14 - 10\tanh x) \\
& + \frac{((2-\alpha) sechx) t^5}{5!}(2 - 6\tanh^3 x - 2\tanh^2 x + 6\tanh x) + ..., \\
& \vdots
\end{aligned}$$

Therefore, the approximate solution of the first system is given by

$$\begin{aligned}
\underline{u}(x,t) \quad = \quad & \alpha t sechx - \frac{\alpha t^3 sechx \tanh x}{3!} - \frac{\alpha t^3 sechx}{3!} + \frac{(\alpha sechx)^3 t^5}{5!} \\
& - \frac{(\alpha sechx)^5 t^7}{7!} + \frac{330(\alpha sechx)^9 t^{13}}{13!} - \frac{162(\alpha sechx)^7 t^{11}}{11!}
\end{aligned}$$

$$+\frac{(\alpha sechx)^5t^9}{9!}(35\tanh x+62-30\tanh^2 x)$$
$$+\frac{(\alpha sechx)^3t^7}{7!}(12\tanh^2 x-14-10\tanh x)$$
$$+\frac{(\alpha sechx)t^5}{5!}(2-6\tanh^3 x-2\tanh^2 x+6\tanh x)+...,$$
$$\overline{u}(x,t)=(2-\alpha)tsechx-\frac{(2-\alpha)t^3sechx\tanh x}{3!}-\frac{(2-\alpha)t^3sechx}{3!}$$
$$+\frac{((2-\alpha)sechx)^3t^5}{5!}-\frac{((2-\alpha)sechx)^5t^7}{7!}+\frac{330((2-\alpha)sechx)^9t^{13}}{13!}$$
$$-\frac{162((2-\alpha)sechx)^7t^{11}}{11!}+\frac{((2-\alpha)sechx)^5t^9}{9!}(35\tanh x+62-30\tanh^2 x)$$
$$+\frac{((2-\alpha)sechx)^3t^7}{7!}(12\tanh^2 x-14-10\tanh x)$$
$$+\frac{((2-\alpha)sechx)t^5}{5!}(2-6\tanh^3 x-2\tanh^2 x+6\tanh x)+\dots .$$

The approximate fuzzy solution to the first system in parametric form for $\alpha = 0.5$, various x in $[0,1]$ and various t in $[0,5]$ is given in Figure 1. It is clear from the Figure 1 that, the numerical results obtained by C(LT-HPM) satisfy the convex symmetric triangular fuzzy number. Here, we use the first four terms in $\underline{u}(x,t)$ and $\overline{u}(x,t)$ to sketch the approximate fuzzy solution.

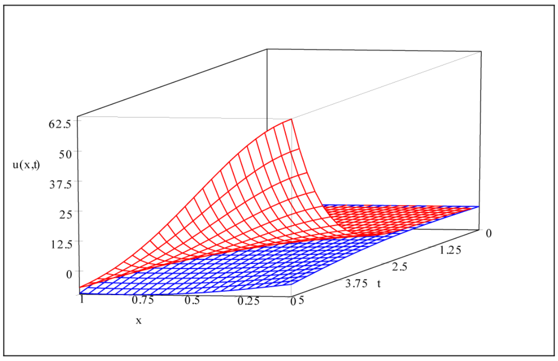

Figure 1: blue $\underline{u}(x,t)$ and red $\overline{u}(x,t)$

Secondly, we take the Laplace transform $\mathcal{L}$ on both sides of Eqs. (79) and (80):

$$(98)\qquad \underline{U}(x,s;\alpha)=\frac{\alpha sechx}{s^2}+\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2\overline{u}(x,t;\alpha)}{\partial x^2}-\sin(\overline{u}(x,t))\right\},\ 0<x<1,$$

$$(99)\qquad \overline{U}(x,s;\alpha)=\frac{(2-\alpha)sechx}{s^2}+\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2\underline{u}(x,t;\alpha)}{\partial x^2}-\sin(\underline{u}(x,t))\right\},0<x<1.$$

Taking the Laplace inverse $\mathcal{L}^{-1}$ on both sides of Eqs. (98) and (99) gives

$$(100)\qquad \underline{u}(x,t;\alpha)=\alpha tsechx+\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2\overline{u}(x,t;\alpha)}{\partial x^2}-\sin(\overline{u}(x,t))\right\}\right\},$$

$$(101)\quad \overline{u}(x,t;\alpha)=(2-\alpha)tsechx+\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\frac{\partial^2\underline{u}(x,t;\alpha)}{\partial x^2}-\sin(\underline{u}(x,t))\right\}\right\}.$$

Again, using the same procedure in Section 4, the first few components in the second system are given by

$$\begin{aligned}
\lambda^0\ :\ & \underline{v}_0(x,t;\alpha)=\alpha tsechx,\\
& \overline{v}_0(x,t;\alpha)=(2-\alpha)tsechx,\\
\lambda^1\ :\ & \underline{v}_1(x,t;\alpha)=\frac{-2(2-\alpha)t^3sech^3x}{3!}+\frac{(2-\alpha)^3t^5sech^3x}{5!}\\
& -\frac{(2-\alpha)^5t^7sech^5x}{7!}+...,\\
& \overline{v}_1(x,t;\alpha)=\frac{-2\alpha t^3sech^3x}{3!}+\frac{\alpha^3t^5sech^3x}{5!}-\frac{\alpha^5t^7sech^5x}{7!}+...,\\
\lambda^2\ :\ & \underline{v}_2(x,t;\alpha)=\left(\frac{\alpha t^5}{15}-\frac{\alpha^3t^7}{1260}\right)sech^3x+\frac{\alpha^5t^9sech^5x}{60\,480}\\
& +\left(\frac{\alpha^3t^7}{420}-\frac{\alpha t^5}{5}\right)\tanh^2 xsech^3x\\
& -\frac{\alpha^5t^9\tanh^2 xsech^5x}{12\,096}+\frac{1}{10\,080}\left(\frac{7\alpha^3(\alpha-2)^2t^9}{12}-40\alpha(\alpha-2)^2t^7\right)sech^5x\\
& -\frac{\alpha^5(\alpha-2)^4t^{13}sech^9x}{18\,869\,760}\\
& +\left(\frac{\alpha^3(\alpha-2)^4t^{11}}{316\,800}-\frac{\alpha(\alpha-2)^4t^9}{5184}-\frac{\alpha^5(\alpha-2)^2t^{11}}{1108\,800}\right)sech^7x+...,\\
& \overline{v}_2(x,t;\alpha)=\left(\frac{(2-\alpha)\,t^5}{15}-\frac{(2-\alpha)^3\,t^7}{1260}\right)sech^3x+\frac{(2-\alpha)^5\,t^9sech^5x}{60\,480}\\
& +\left(\frac{(2-\alpha)^3\,t^7}{420}-\frac{(2-\alpha)\,t^5}{5}\right)\tanh^2 xsech^3x-\frac{(2-\alpha)^5\,t^9\tanh^2 xsech^5x}{12\,096}\\
& +\frac{1}{10\,080}\left(\frac{7\alpha^2\,(2-\alpha)^3\,t^9}{12}-40\alpha^2\,(2-\alpha)\,t^7\right)sech^5x-\frac{\alpha^4\,(2-\alpha)^5\,t^{13}sech^9x}{18\,869\,760}\\
& +\left(\frac{\alpha^4\,(2-\alpha)^3\,t^{11}}{316\,800}-\frac{\alpha^4\,(2-\alpha)\,t^9}{5184}-\frac{\alpha^2\,(2-\alpha)^5\,t^{11}}{1108\,800}\right)sech^7x+...,\\
& \vdots
\end{aligned}$$

Therefore, the approximate solution of the second system is given by

$$\begin{aligned}
\underline{u}(x,t)=&\ \alpha tsechx-\frac{2(2-\alpha)}{3!}t^3sech^3x+\frac{(2-\alpha)^3}{5!}t^5sech^3x\\
&-\frac{(2-\alpha)^5}{7!}t^7sech^5x+\left(\frac{\alpha}{15}t^5-\frac{\alpha^3}{1260}t^7\right)sech^3x+\frac{\alpha^5}{60\,480}t^9sech^5x
\end{aligned}$$

$$+\left(\frac{\alpha^3}{420}t^7-\frac{\alpha}{5}t^5\right)\tanh^2 xsech^3x-\frac{\alpha^5}{12\,096}t^9\tanh^2 xsech^5x$$
$$+\frac{1}{10\,080}\left(\frac{7}{12}\alpha^3(\alpha-2)^2t^9-40\alpha(\alpha-2)^2t^7\right)sech^5x-\frac{\alpha^5(\alpha-2)^4}{18\,869\,760}t^{13}sech^9x$$
$$+\left(\frac{\alpha^3(\alpha-2)^4}{316\,800}t^{11}-\frac{\alpha(\alpha-2)^4}{5184}t^9-\frac{\alpha^5(\alpha-2)^2}{1108\,800}t^{11}\right)sech^7x+\ldots .$$
$$\overline{u}(x,t)=(2-\alpha)tsechx-\frac{2\alpha}{3!}t^3sech^3x+\frac{\alpha^3}{5!}t^5sech^3x-\frac{\alpha^5}{7!}t^7sech^5x$$
$$+\left(\frac{(2-\alpha)}{15}t^5-\frac{(2-\alpha)^3}{1260}t^7\right)sech^3x+\frac{(2-\alpha)^5}{60\,480}t^9sech^5x$$
$$+\left(\frac{(2-\alpha)^3}{420}t^7-\frac{(2-\alpha)}{5}t^5\right)\tanh^2 xsech^3x-\frac{(2-\alpha)^5}{12\,096}t^9\tanh^2 xsech^5x$$
$$+\frac{1}{10\,080}\left(\frac{7}{12}\alpha^2(2-\alpha)^3t^9-40\alpha^2(2-\alpha)t^7\right)sech^5x-\frac{\alpha^4(2-\alpha)^5}{18\,869\,760}t^{13}sech^9x$$
$$+\left(\frac{\alpha^4(2-\alpha)^3}{316\,800}t^{11}-\frac{\alpha^4(2-\alpha)}{5184}t^9-\frac{\alpha^2(2-\alpha)^5}{1108\,800}t^{11}\right)sech^7x+\ldots,$$

The approximate fuzzy solution to the second system in parametric form for $\alpha=0.2$, various x in $[0,1]$ and various t in $[0,5]$ is given in Figure 2. It is clear from the Figure 2 that, the numerical results obtained by C(LT-HPM) satisfy the fuzzy numbers properties by taking the triangular fuzzy numbers shape. Here, we use the first seven terms in $\underline{u}(x,t)$ and $\overline{u}(x,t)$ to sketch the approximate fuzzy solution.

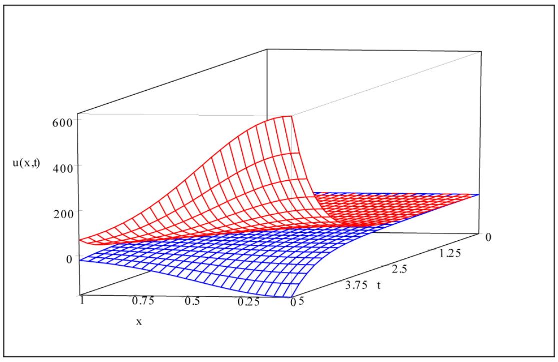

Figure 2: blue $\underline{u}(x,t)$ and red $\overline{u}(x,t)$

6. Conclusion

The primary objective of this paper is to determine an approximate-analytical solutions for the fuzzy partial differential equations. We have accomplished this objective by applying C(LT-HPM). The results are very encouraging, demonstrating the unwavering quality and proficiency of the proposed strategy with less computational work and time. This strategy is based on the definition of strongly generalized differentiability.

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