Bifurcations of Liouville tori of generalized two-fixed center problem

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Abstract. We study the topological type of the level sets of generalized two-fixed center problem. Furthermore, all generic bifurcation of the level sets are presented. We determine the families of periodic solutions by giving the solution in terms of Jacobi's elliptic functions. Finally, the phase portrait is studied, and the singular points are classified.

Keywords: generalized two-fixed center, Hamilton-Jacobi's equations, bifurcations of Liouville tori, topology of the level sets, momentum maps, periodic solution, elliptic functions, phase portrait.

1. Introduction

The first one who investigated the problem of two fixed centers was Euler [11]. From that time, many authors were interested in studying the extensions and generalization of this problem. Lagrange [21] has extended the Euler's solution to the three-dimensional case of motion and made some generalization. Jacobi [15] generalized the three-dimensional problem of two fixed centers to the case of arbitrary number of additional attracting centers located at equal distances from each other on a single straight line with two basic centers of attraction, as well as to the case of presence of an additional force of arbitrary nature acting parallel to this straight line. He has also demonstrated the integrability of the problem under his generalizations.

Thereafter, a large number of papers have been written to generalize Euler problem such as Liouville [22] and Hiltebeitel [14]. In their papers, the integrable cases were determined in the restricted three-body problem with neglecting centrifugal and Coriolis forces in various combinations. In 1901 Darboux [7] presented another generalization of the Eulers problem in the case of a planar motion by introducing complex-conjugated masses and an imaginary distance

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between them. The potential of attraction always assumes real values in this case, and the solution of a problem is also reduced to quadratures. The generalized problem of two fixed centers has achieved a large amount of applications as follows, in [23], a brief of publications on the problem of two fixed centers was given, included its generalizations and astronomical applications. Darboux model was studied by Aksensov et al. [1] and they proved that with a convenient selection of free parameters, such a model can be used for constructing an analytical theory of satellite motion in the gravitational field of an oblate planet. The motion of a star in the stationary stellar system with an axisymmetric nucleus was studied by Kaisin for testing the motion of a spacecraft in the field of an oblate planet with regard to the thrust force of an engine, and to find the solutions to some other problems of astrodynamics [16, 17]. Koman [19, 20] applied the asymmetrical version of the three-dimensional problem of two fixed centers with real masses for studying the motion of artificial satellites of the Moon. Moreover, the model of three Newtonian fixed attracting centers with material masses was used by Arazov to approximate the gravitational potential of Jupiter [4]. Maciejewski and Maria Przybylska [24] studied the non-integrability of the generalized two fixed centers problem. The integrable spherical of the Darboux potential in the planar motion of a particle in the field of two and four fixed Newtonian centers was studied in [3]. Moreover, all results can be applied in theory of artificial Earth satellites. In [33], the bifurcation diagrams for planar motion were analyzed. The motion in 3-dimensions with arbitrary values of the angular motion was studied. Bifurcations in the topology of energy surfaces were discussed in terms of relative equilibria. They also calculated the monodromy matrices from an attempt to construct smooth actions from the natural ones. The quantum version of the two center problem was discussed in symmetric and asymmetric cases. New applications of the generalized two-fixed center problem are introduced in [18]. Thereafter, they used a symmetric version of the problem and the external field of gravitation is approximated. Varvoglis et al. [26] determined the trajectories according to an exhaustive scheme, comprising both periodic and quasi-periodic ones. They also identified the collision orbits and found that collision orbits are of complete measure in a 3-D sub manifold of the phase space while asymptotically collision orbits are of complete measure in the 4-D phase space.

The study of the bifurcation for the problem of two fixed centers was presented by [27],[28], [29]. She constructed the bifurcation set on the plane of values of integrals of motion, classification of domains of possible motion on the configurational space in spaces of constant curvature on a sphere and in Lobachevsky's space, while [30] studied the topological analysis of the two-center problem on the two-dimensional sphere. In [31],[32] the topology of isoenergy surfaces in the integrable problems of celestial mechanics in spaces of constant curvature was introduced, the topological invariants were constructed. El-Sabaa et al. [10] studied the complete description of the real phase topology of a two-fixed center problem.

We will give in this article the qualitative analysis of the generalized problem of two fixed centers by describing the bifurcation of the problem. To describe the real phase, we review the Liouville- Arnold theorem [5] which was stated that the phase space trajectories of a Hamiltonian system with n-degrees of freedom and possessing n-integrals of motion lie on n-dimensional manifold which is topologically equivalent to an n-torus and the regular tori (foliate) the bulk of $2n$-dimensional phase space and its $2n-1$ dimensional energy surface. Fomenko [12] proposed a new approach in the qualitative theory of integrable Hamiltonian system, given the separation of the system, the determination of critical values of the energy momentum map boils down to the analysis of the discriminant surface of a polynomial. Moreover, the hypersurfaces of the constant of energy in the space of the variables of separation can be determined and geometrically represented tori. In quantum mechanics the problem explains non-electron moving in the electric field of two nuclei may have the same charges (symmetric case) or different charges (an asymmetric case) [25].

The current paper is organized as follows, a short summary of the problem is given in Section 2. In Section 3, the topological analysis of the real invariant manifold of the system was studied by using Fomenko's theory [6]. Moreover, the bifurcation diagram of the problem is determined and the complete description of the topology of the level sets of the first integrals was given. The aim of Section 4 is to give the families of periodic solutions, this solution is given in terms of Jacobi's elliptic functions and when the bifurcations of Liouville tori take place, the level set becomes degenerate. The phase portrait is given in Section 5. A concluding remark is given in the last section.

2. Separation of the problem

The classical problem of two fixed centers consists as is known, in the study of the motion of a mass point under the attraction of two fixed mass points P_1 and P_2. Let the coordinate system be $Oxyz$ whose origin is at the center of mass P_1 and P_2 where the line P_1P_2 lies on the z-axis as shown in Figure 1. Then, the equations of motion of the mass points can be written in the form

$$\frac{d^2x}{dt^2} = \frac{\partial V}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial V}{\partial y}, \quad \frac{d^2z}{dt^2} = \frac{\partial V}{\partial z}, \tag{1}$$

where the generalized potential function V is defined by

$$V = f(\frac{M_1}{r_1} + \frac{M_2}{r_2}), \tag{2}$$

where

$r_1 = \sqrt{x^2 + y^2 + (z - a_1)^2},\ r_2 = \sqrt{x^2 + y^2 + (z - a_2)^2}$,

M_1 and M_2 are the masses of P_1 and P_2, a_1 and a_2 are the distances of these points from the coordinate origin.

Introducing the quantity a which is represented the distance between P_1 and P_2 such that

$$a_1 = \frac{aM_2}{M_1 + M_2}, \quad a_2 = -\frac{aM_1}{M_1 + M_2}. \tag{3}$$

The inverse distances can be expanded in series of Legendre polynomials:

$$\frac{1}{r_1} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{a_1}{r})^n P_n(\frac{z}{r}), \quad \frac{1}{r_2} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{a_2}{r})^n P_n(\frac{z}{r}), \tag{4}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Then, the potential function become

$$V = \frac{fM}{r}[1 + \sum_{n=0}^{\infty}\frac{\gamma_n}{r^n}P_n(\frac{z}{r})], \tag{5}$$

where

$$\gamma_n = \frac{M_1 a_1^n + M_2 a_2^n}{M}, M = M_1 + M_2.$$

In order to, the potential function V which is given by (5) to be real, it is sufficient for the quantities M and γ_n to be real for any value of n. The function V is real in two case [2]:

1. In the first case: M_1, M_2, a_1 and a_2 are pairs of complex conjugate quantities, i.e.

$$M_1 = \frac{M}{2}(1 + i\sigma), \quad M_2 = \frac{M}{2}(1 - i\sigma), \quad a_1 = c(\sigma + i), \quad a_2 = c(\sigma - i), \tag{6}$$

where σ and c are real constants

2. In the second case: the constants M_1, M_2, a_1 and a_2 are real, i.e.

$$
\begin{aligned}
M_1 &= M(1-\gamma),\\
M_2 &= M\gamma,\\
a_1 &= a\gamma,\\
a_2 &= -a(1-\gamma),
\end{aligned}
\tag{7}
$$

where γ is the ratio of M_2 to the total mass M.

We study the generalized problem of two fixed-center in the first case, so the potential function V has the form

$$V = \frac{fM}{2}[\frac{1+i\sigma}{r_1} + \frac{1-i\sigma}{r_2}], \tag{8}$$

where

$$
\begin{aligned}
r_1 &= \sqrt{x^2+y^2+[z-c(\sigma+i)]^2},\\
r_2 &= \sqrt{x^2+y^2+[z-c(\sigma-i)]^2}.
\end{aligned}
\tag{9}
$$

Expanding V in a series in Legendre polynomial,

$$V = \frac{fM}{r}[1+\sum_{k=2}^{\infty}\frac{\gamma_k}{r^k}P_k(\frac{z}{2})], \tag{10}$$

where

$$\gamma_k = \frac{c^k}{2}[(1+i\sigma)(\sigma+i)^k + (1-i\sigma)(\sigma-i)^k]. \tag{11}$$

Introducing the new coordinates λ, μ, and ω such that

$$
\begin{aligned}
x &= c\sqrt{(1+\lambda^2)(1-\mu^2)}\cos w,\\
y &= c\sqrt{(1+\lambda^2)(1-\mu^2)}\sin w,\\
z &= c\sigma + c\lambda\mu,
\end{aligned}
\tag{12}
$$

then equation (8) became

$$V = \frac{fM}{c}[\frac{\lambda-\sigma\mu}{\lambda^2+\mu^2}]. \tag{13}$$

The kinetic energy can be written as

$$T = \frac{c^2}{2}[(\frac{\dot{\lambda}^2}{1+\lambda^2} + \frac{\dot{\mu}^2}{1-\mu^2})I + \dot{w}^2(1+\lambda^2)(1-\mu^2)] \tag{14}$$

where

$$I = \lambda^2 + \mu^2, \tag{15}$$

then, the lagrange equations are given by

$$(16)\qquad \begin{aligned} &\frac{d}{dt}(\frac{I\dot{\lambda}}{1+\lambda^2}) - 2\lambda(1-\mu^2)\dot{\omega}^2 = -\frac{1}{c^2}\frac{\partial V}{\partial \lambda};\\ &\frac{d}{dt}(\frac{I\dot{\mu}}{1+\mu^2}) + 2\mu(1+\lambda^2)\dot{\omega}^2 = -\frac{1}{c^2}\frac{\partial V}{\partial \mu};\\ &\frac{d}{dt}[\dot{w}(1+\lambda^2)(1-\mu^2)] = 0. \end{aligned}$$

The area integral is got from the last equation in the system (16)

$$(17)\qquad \dot{w}(1+\lambda^2)(1-\mu^2) = c_1.$$

The Hamiltonian function takes the form

$$(18)\qquad H=\frac{1}{2c^2}[\frac{\lambda^2+1}{\lambda^2+\mu^2}p_\lambda^2+\frac{1-\mu^2}{\lambda^2+\mu^2}p_\mu^2]-\frac{c^2c_1^2}{2(\lambda^2+1)(1-\mu^2)}+\frac{fM}{c}(\frac{\lambda-\sigma\mu}{\lambda^2+\mu^2})=h,$$

and from (18) we have

$$(19)\qquad F = (\lambda^2+1)p_\lambda^2 + 2fMc\lambda + \frac{c^4c_1^2}{1+\lambda^2} - 2c^2\lambda^2 h,$$

$$(20)\qquad F = (1-\mu^2)p_\mu^2 - 2fMc\sigma\mu - \frac{c^4c_1^2}{1-\mu^2} - 2c^2\mu^2 h.$$

Now, by using Hamilton Jacobi method we have

$$(21)\qquad \begin{aligned} &\frac{1}{2c^2}[(1+\lambda^2)(\frac{\partial W}{\partial \lambda})^2 + (1-\mu^2)(\frac{\partial W}{\partial \mu})^2] + h(\lambda^2+\mu^2)\\ &+\frac{fM}{c}(\lambda-\sigma\mu) - \frac{c^2c_1^2}{2}(\frac{1}{1-\mu^2} - \frac{1}{1+\lambda^2}) = 0. \end{aligned}$$

The complete integral of this equation is

$$(22)\qquad W = W_1(\lambda) + W_2(\mu).$$

Therefore, the Hamilton-Jacobi equation (21) is satisfied if

$$(23)\quad (\lambda^2+1)(\frac{dW_1}{d\lambda})^2 = \frac{2h}{c^2}\lambda^4 + \frac{2fM}{c^3}\lambda^3 + 2(c_2+\frac{h}{c^2})\lambda^2 + \frac{2fM}{c^3}\lambda + (2c_2+c_1^2),$$

$$(24)\quad (1-\mu^2)(\frac{dW_2}{d\mu})^2 = -\frac{2h}{c^2}\mu^4 + \frac{2fM\sigma}{c^3}\mu^3 + 2(c_2+\frac{h}{c^2})\mu^2 - \frac{2fM\sigma}{c^3}\mu - (2c_2+c_1^2),$$

this yields that W takes the form

$$(25)\qquad W = \int \frac{\sqrt{L(\lambda)}}{\lambda^2+1}d\lambda + \int \frac{\sqrt{M(\mu)}}{1-\mu^2}d\mu,$$

where,

$$L(\lambda) = \frac{2h}{c^2}(\lambda^2+1)(\lambda^2+a\lambda+c_2c^2), \tag{26}$$

$$M(\mu) = \frac{2h}{c^2}(1-\mu^2)(\mu^2-a\sigma\mu-c_2c^2), \tag{27}$$

where $a = \frac{fM}{c}$, σ is a constant and c_2 is a constant of separation. Then,

$$\frac{\partial W}{\partial c_2} = \beta, \frac{\partial W}{\partial h} = t - t_0, \tag{28}$$

where β is a new arbitrary constant, then we have from (28) the following equations:

$$\begin{aligned} &\int \frac{d\lambda}{\sqrt{L(\lambda)}} + \frac{d\mu}{\sqrt{M(\mu)}} = \beta, \\ &\int \frac{\lambda^2 d\lambda}{\sqrt{L(\lambda)}} + \frac{\mu^2 d\mu}{\sqrt{M(\mu)}} = c^2(t-t_0). \end{aligned} \tag{29}$$

Introducing a new time defined by

$$d\tau = (\lambda^2+\mu^2)dt. \tag{30}$$

Therefore, the differential equations satisfied by λ and μ are:

$$\begin{aligned} &\int \frac{d\lambda}{\sqrt{L(\lambda)}} = \tau - \tau_0, \\ &\int \frac{d\mu}{\sqrt{M(\mu)}} = \tau - \tau_0. \end{aligned} \tag{31}$$

3. Topological analysis

First, we give the following definitions as in [8]:

1. The smooth mapping
$$F : M^{2n} \to R^n,$$
where $F(x) = (f_1(x), ..., f_n(x))$ is said to be the momentum mapping, M^{2n} is a simplistic manifold in the integrable Hamiltonian system, and $f_1,\ f_2, ..., f_n$ its independent integrals.

2. If rank $dF(x) < n$, and its image $F(x)$ in R^n is a critical value, then the point $x \in M$ is a critical point of the momentum mapping.

3. If K is the set of all critical points of the momentum mapping such that $K \subset M$, then the set $\Sigma = F(k) \subset R^n$ is the bifurcation diagram, where the whole of Σ is the union of several pieces Σ^k.

The topology of the level sets is introduced as

$$L_S = (x, y, \dot{x}, \dot{y}) \in R^4 : H = h, F = c_2 \subset R^4. \tag{32}$$

The energy-momentum mapping is determined by getting the set of critical point

$$(x,\ y,\ \dot{x},\ \dot{y}) \to (H,\ F)$$

this means the bifurcation diagram Σ, where Σ is the discriminant of the polynomials $L(\lambda)$ and $M(\mu)$:

$$\Sigma{=}\Sigma_1{\cup}\Sigma_2{=}[(h, c_2){\in}R^2/disc(L(\lambda)){=}0]{\cup}[(h, c_2){\in}R^2/disc(M(\mu)){=}0]. \tag{33}$$

On the point (h, c_2), the topological type of L_S can be change. The set R^2/Σ consists of 8 connected parts as shown in Figure 2. So, in each connected portion of the set R^2/Σ, the topological type of L_S is similar.

The Arnold-Liouville's theorem [5] state that, for noncritical values of H and F the level set L_s is a limited union of low dimensional tori, whose number depends only on the number and the location of the allowed ovals on the Riemann surface connected to the elliptic curve Γ_1 and Γ_2 where

$$\Gamma_1 : \omega_1 = \sqrt{L(\lambda)}\ \text{and}\Gamma_2 : \omega_2 = \sqrt{M(\mu)}.$$

In order to obtain the ovals of of Γ_1 and Γ_2 (see Table 2), the real roots of the polynomials $L(\lambda)$ and $M(\mu)$ must be studied which shown in Table 1.

The topological type of L_S is either a torus, two-tori $2T$, or empty as shown in Table 2 (see [13]).

For getting the generic bifurcations of the system (1) (see Table 3), we must use the bifurcation of the roots of the polynomials $L(\lambda)$ and $M(\mu)$ as shown in Figures (3−4).

4. The solution of the problem

In this section, the elliptic functions and the Jacobi elliptic function of motion are used. Through the study of topology on the problem, we found that there is a periodic solution on the curve C_2 where $1 - 4hc_2 = 0$, the torus T contracted to one axial circle S and then disappeared as shown in Figure 4. It found that $\lambda_{1,2} = b$ where b is a constant and the μ parameter takes values in the period $[-1, \mu_1]$.

Returning to the second equation of (31), the function $M(\mu)$ is a polynomial of fourth degree with four real roots $1,\ -1,\ \mu_1$ and μ_2 as shown in Table 4, such that

$$M(\mu) = (\mu + 1)(\mu - 1)(\mu_1 - \mu)(\mu - \mu_2). \tag{34}$$

If $h < 0$ and $c_2 < 0$ the real motion is bounded where $(-1 \leq\ \mu\ \leq \mu_1)$, let

$$\mu = \frac{-(1-\mu_1)+(1+\mu_1)\sin^2\phi}{(1-\mu_1)+(1+\mu_1)\sin^2\phi}, sin^2\phi = \frac{1-\mu_1}{1+\mu_1}\frac{\mu+1}{1-\mu}, \tag{35}$$

and

$$d\mu = \frac{4(1-\mu_1)(1+\mu_1)\sin\phi\cos\phi}{[(1-\mu_1)+(1+\mu_1)\sin^2\phi]^2}d\phi. \tag{36}$$

By substituting from (34-36) in the second equation of (31), we have

$$t - t_0 = \frac{1}{d}\int_0^{\phi}\frac{d\phi}{\sqrt{1-k_1^2\sin^2\phi}}, \tag{37}$$

then, the solution is

$$\mu(\tau) = \frac{-(1-\mu_1)+(1+\mu_1)sn^2[d(\tau-\tau_0),k_1]}{(1-\mu_1)+(1+\mu_1)sn^2[d(\tau-\tau_0),k_1]}, \tag{38}$$

and the period T of $\mu(\tau)$ is

$$T = \frac{1}{d}sn^{-1}(1,k_1) = \frac{1}{d}K(k_1), \tag{39}$$

where,

$$d = \frac{\sqrt{(1+\mu_2)(1-\mu_1)}}{2}, k_1^2 = \frac{(\mu_1+1)(1-\mu_2)}{(1+\mu_2)(\mu_1-1)}. \tag{40}$$

Similarly, we get the periodic solution on the curve C_3 where $1 + 4hc_2 = 0$, $\mu_{1,2} = e$ and the λ parameter takes values on the interval $[0, \lambda_2]$. By solving the first equation of (31) the function $L(\lambda)$ is a polynomial of fourth degree with two real roots λ_1 and λ_2 and two complex roots i, $-i$ such that

$$L(\lambda) = (\lambda^2+1)(\lambda-\lambda_1)(\lambda_2-\lambda). \tag{41}$$

If $h < 0$ and $c_2 > 0$ the real motion is bounded where $(\lambda_1 \leq \lambda \leq \lambda_2)$,

let

$$\lambda = (\frac{\lambda_2+\lambda_1}{2} - \frac{\lambda_2-\lambda_1}{2})(\frac{n-\cos\phi}{1-n\cos\phi}),\ \tan^2\frac{\phi}{2} = (\frac{\cos\sigma_1}{\cos\sigma_2})(\frac{\lambda_2-\lambda}{\lambda-\lambda_1}), \tag{42}$$

where

$$\begin{aligned} \tan\sigma_1 &= \lambda_2, \\ \tan\sigma_2 &= \lambda_1, \\ n &= \tan\frac{\sigma_1-\sigma_2}{2}\tan\frac{\sigma_1+\sigma_2}{2}, \end{aligned} \tag{43}$$

and

$$d\lambda = -\frac{(\lambda_2 - \lambda_1)[1 - (\lambda_2^2 - \lambda_1^2)^2]}{[1 - (\lambda_2^2 - \lambda_1^2)\cos\phi]^2} \sin\phi d\phi. \tag{44}$$

By substituting from (41, 42, 44) in the first equation of (31), we have

$$t - t_0 = \frac{1}{g}\int_0^\phi \frac{d\phi}{\sqrt{1 - k_1^2 \sin^2\phi}}, \tag{45}$$

then, the solution is

$$\lambda(\tau) = \frac{\lambda_1 + \lambda_2}{2} - \frac{\lambda_1 - \lambda_2}{2} \frac{n\, cn[g(\tau - \tau_0), k_2]}{1 - n\, cn[g(\tau - \tau_0), k_2]}, \tag{46}$$

where

$$\begin{aligned} n &= \lambda_2^2 - \lambda_1^2, g = -[(1 + \lambda_1^2)(1 + \lambda_2^2)]^{\frac{1}{4}}, \\ k_2 &= \frac{1}{2}[1 - \frac{1 - \lambda_1\lambda_2}{\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}}], \end{aligned} \tag{47}$$

and the period T of $\lambda(\tau)$ is

$$T = \frac{1}{g} sn^{-1}(1, k_2) = \frac{1}{g}K(k_2). \tag{48}$$

5. Phase portrait of the separated functions

In this section, we use the phase portrait to find the topological translation of the path. El-Sabaa found the singular points and its types for separated functions of Kovaleveskaya top by using Kolsoff variables [9].

Consider the function

$$F_1 = h(q_1^2 - q_1^4) + \frac{1}{2c^2}(1 - 2q_1^2 + q_1^4)p_1^2 - k_3(q_1^2 - q_1^3) - \alpha q_1^2 - k_2, \tag{49}$$

where $k_2 = 2c_2 + c_1^2$, $\alpha = c_2$ and $k_3 = \frac{2fM}{c^3}$.

To construct the lines of constant F_1, we first study the singular points of F_1. These points can be found from the equations

$$\frac{\partial F_1}{\partial p_1} = \frac{1}{c^2}(1 - 2q_1^2 + q_1^4)p_1 = 0, \tag{50}$$

$$\frac{\partial F_1}{\partial q_1} = 2h(q_1 - 2q_1^3) + \frac{2}{c^2}p_1^2(q_1^3 - q_1) - k_3(1 - 3q_1^2) - 2\alpha q_1 = 0, \tag{51}$$

and hence we have the following where $p = 0$ we get

$$-4hq_1^3 + 3q_1^2 k_3 + 2(h - \alpha)q_1 - k_3 = 0, \tag{52}$$

then from (50) we have $q_1 = \pm 1$,

we get the two equations

$$k_3 - h - \alpha = 0, \quad k_3 + h + \alpha = 0. \tag{53}$$

The positive regions of the functions

$$\begin{aligned} f_1 &= 128h^4 + 36h^2k_3^2 + 108k_3^4 - 384h^3\alpha \\ &\quad - 504k_3^2\alpha + 384h^2\alpha^2 + 36k_3^2\alpha^2 - 128h\alpha^3, \\ f_2 &= k_3 - h - \alpha, \\ f_3 &= k_3 + h + \alpha. \end{aligned} \tag{54}$$

are shown in Figure (5).

It is clear that the curve f_1 is tangent to the curves f_2 and f_3 at the points $\frac{1}{2}$ and $\frac{-1}{2}$.

We study the motion in domain $D_i (i = 1, 2, 3, ..., 8)$

1. The first region D_1: $f_1 < 0,\ f_2 < 0,\ f_3 > 0$.

At $p_1 = 0$ we have one point with q_1 coordinate

$$q_1^* = \frac{k_3}{4h} + \frac{-9k_3^2 - 24h(h-\alpha)}{[6\times 2^{\frac{2}{3}} h(216h^2k_3 - 54k_3^3 + 216hk_3\alpha + \sqrt{4(-9k_3^2 - 24h(h-\alpha))^3 + (216h^2k_3 - 54k_3^2 + 216hk_3\alpha)^2})]^{\frac{1}{3}}}$$
$$- \frac{(216h^2k_3 - 54k_3^3 + 216hk_3\alpha + \sqrt{4(-9k_3^2 - 24h(h-\alpha))^3 + (216h^2k_3 - 54k_3^2 + 216hk_3\alpha)^2})^{\frac{1}{3}}}{12\times 2^{\frac{1}{3}} h}.$$

Then, to get the type of this point, we put

$$q_1 = q_1^* + y, p_1 = x, \tag{55}$$

in the function F_1

$$F_1 = [\tfrac{1}{4} + \frac{23}{512h^4} - \frac{5}{16h^3} + \frac{23}{96h^2} + \frac{1}{3h} - \frac{\rho}{6912h^4} + \frac{1115}{2^{\frac{2}{3}} u^{\frac{4}{3}}} + \frac{81}{128\times 2^{\frac{2}{3}} h^4 u^{\frac{4}{3}}} - \frac{27}{2^{\frac{2}{3}} h^3 u^{\frac{4}{3}}} +$$
$$\frac{27}{4\times 2^{\frac{2}{3}} h^2 u^{\frac{4}{3}}} + \frac{54\times 2^{\frac{1}{3}}}{h^2 u^{\frac{4}{3}}} - \frac{246\times 2^{\frac{1}{3}}}{h u^{\frac{4}{3}}} - \frac{656\times 2^{\frac{1}{3}} h}{u^{\frac{4}{3}}} + \frac{408\times 2^{\frac{1}{3}} h^2}{u^{\frac{4}{3}}} - \frac{128\times 2^{\frac{1}{3}} h^3}{u^{\frac{4}{3}}} + \frac{16\times 2^{\frac{1}{3}} h^4}{u^{\frac{4}{3}}} + \frac{105}{u} +$$
$$\frac{27}{64h^4u} - \frac{27}{4h^3u} + \frac{315}{8h^2u} + \frac{100}{hu} + \frac{48h}{u} - \frac{8h^2}{u} - \frac{3}{2^{\frac{4}{3}} u^{\frac{2}{3}}} + \frac{45}{128\times 2^{\frac{1}{3}} u^{\frac{2}{3}}} - \frac{9}{2\times 2^{\frac{1}{3}} h^3 u^{\frac{2}{3}}} + \frac{9}{8\times 2^{\frac{1}{3}} h^2 u^{\frac{2}{3}}} +$$
$$\frac{9\times 2^{\frac{2}{3}}}{h^2 u^{\frac{2}{3}}} - \frac{41\times 2^{\frac{2}{3}}}{3hu^{\frac{2}{3}}} + \frac{8\times 2^{\frac{2}{3}} h}{u^{\frac{2}{3}}} - \frac{8\times 2^{\frac{2}{3}} h^2}{3u^{\frac{2}{3}}} + \frac{1}{2^{\frac{2}{3}} u^{\frac{1}{3}}} - \frac{3}{16\times 2^{\frac{2}{3}} h^4 u^{\frac{1}{3}}} + \frac{7}{4\times 2^{\frac{2}{3}} h^3 u^{\frac{1}{3}}} - \frac{1}{8\times 2^{\frac{2}{3}} h^2 u^{\frac{1}{3}}} - \frac{2^{\frac{4}{3}}}{h^2 u^{\frac{1}{3}}} -$$
$$\frac{17u^{\frac{1}{3}}}{1536\times 2^{\frac{1}{3}} h^4} + \frac{3u^{\frac{1}{3}}}{64\times 2^{\frac{1}{3}} h^3} + \frac{3u^{\frac{1}{3}}}{128\times 2^{\frac{1}{3}} h^2} + \frac{\rho u^{\frac{1}{3}}}{82944\times 2^{\frac{1}{3}} h^4} + \frac{5u^{\frac{2}{3}}}{2304\times 2^{\frac{2}{3}} h^4} - \frac{u^{\frac{2}{3}}}{216\times 2^{\frac{2}{3}} h^3} - \frac{u^{\frac{2}{3}}}{216\times 2^{\frac{2}{3}} h^2}]x^2 +$$
$$[+2 - \frac{3}{8h} - h + \frac{36\times 2^{\frac{2}{3}}}{u^{\frac{2}{3}}} - \frac{27}{4\times 2^{\frac{1}{3}} hu^{\frac{2}{3}}} - \frac{114\times 2^{\frac{2}{3}} h}{u^{\frac{2}{3}}} + \frac{96\times 2^{\frac{2}{3}} h^2}{u^{\frac{2}{3}}} - \frac{24\times 2^{\frac{2}{3}} h^3}{u^{\frac{2}{3}}} - \frac{u^{\frac{2}{3}}}{24\times 2^{2/3} h}]y^2 + A_0$$

where A_0 contains the zeros terms of x and y,
$[\sqrt{4(-9 - 24(-2+h)h)^3 + (-54 + 432h + 216h^2)^2}] = \rho$ and $[(-54 + 432h + 216h^2) + \sqrt{4(-9 - 24(-2+h)h)^3 + (-54 + 432h + 216h^2)^2}] = u$.

The singular point is hyperbolic point, where

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}_{x=y=0} < 0. \tag{56}$$

In the same manner, we get the type of points in the domains D_i where, $i = 1, ..., 8$.

Table 5 shows the points and its types for all domains.

6. Summary-conclusions

In the current paper, the generalized two-fixed center problem provided us to get a complete picture of the dynamics of its potential:

1. The complete characterization of the real phase topology.
2. The periodic solution, where the variables of motion can be described through the Jacobi elliptic function which is a periodic function, the solution with Jacobi elliptic function has been explained in details in [30, 31].
3. Phase portrait: the singular points of the separated functions were determined. The type of these points is either elliptic or hyperbolic point. The elliptic points in the figures were stable while the hyperbolic points were unstable.

Table 1: Topological type of L_S and real roots of the polynomials $L(\lambda)$ and $M(\mu)$ for $(h, c_2) \in R^2/\Sigma$.

Domain	*Roots of* $L(\lambda)$	*Roots of* $M(\mu)$
1	$\lambda_2 < \lambda_1 < 0$	$-1 < \mu_2 < 0 < \mu_1 < 1$
2	$\lambda_2 < 0 < \lambda_1$	$-1 < 0 < \mu_2 < \mu_1 < 1$
3	$0 < \lambda_2 < \lambda_1$	$-1 < \mu_1 < 0 < \mu_2 < 1$
4	$\lambda_1 < 0 < \lambda_2$	$-1 < \mu_2 < \mu_1 < 0 < 1$
5	0	$-1 < \mu_2 < 0 < \mu_1 < 1$
6	$\lambda_2 < 0 < \lambda_1$	0
7	0	$-1 < \mu_1 < 0 < \mu_2 < 1$
8	$\lambda_1 < 0 < \lambda_2$	0

Table 2: Admissible ovals on diagram Σ.

Domain	$\lambda - plane\ \triangle_1$	$\mu - plane\ \triangle_2$	*Topological type*
1	$\emptyset$	$[-1, \mu_2]$	$\emptyset$
2	$[0, \lambda_1]$	$[-1, 0]$	T
3	$[\lambda_2, \lambda_1]$	$[-1, \mu_1]$	T
4	$[0, \lambda_2]$	$[\mu_2, \mu_1] \cup [-1, 0]$	$2T$
5	$\emptyset$	$[-1, \mu_2]$	$\emptyset$
6	$[0, \lambda_1]$	$\emptyset$	$\emptyset$
7	$\emptyset$	$[-1, \mu_1]$	$\emptyset$
8	$[0, \lambda_2]$	$\emptyset$	$\emptyset$

Table 3: Generic bifurcations of the level set L_S passing from domain i to domain j.

$2 \to 5$	$4 \to 5$	$2 \to 3$	$4 \to 2$
$2 \to 1$	$4 \to 1$		$4 \to 3$
$2 \to 6$	$4 \to 6$		
$2 \to 7$	$4 \to 7$		
$2 \to 8$	$4 \to 8$		
$3 \to 1$			
$3 \to 5$			
$3 \to 6$			
$3 \to 7$			
$3 \to 8$			
$T \to \emptyset$	$2T \to \emptyset$	$T \to T$	$2T \to T$

Table 4: Topological type of L_S for $(h, c_2) \in \Sigma$.

Domain	$\lambda - plane\ \triangle_1$	$\mu - plane\ \triangle_2$	*Topological type*
C_1	$[\lambda_2, \lambda_1]$	$[-1, 0]$	S
C_2	$[\lambda_2 = \lambda_1]$	$[-1, \mu_1]$	S
C_3	$[0, \lambda_2]$	$[-1, \mu_2 = \mu_1]$	$2S$
C_4	$[\lambda_1 = 0]$	$[-1, \mu_2 = 0]$	S
C_5	$[0, \lambda_2]$	$[-1, \mu_1]$	$S \times (S \wedge S)$

Table 5: The type of points in the domains D_i

Domain	The points	Types of points	*Figures*
$D_1 : f_1 < 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-hyperbolic point	Figure 6
$D_2 : f_1 > 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-hyperbolic point	Figure 7
$D_3 : f_1 > 0, f_2 < 0, f_3 < 0$	$(q_1^*, 0)$	One-hyperbolic point	Figure 8
$D_4 : f_1 > 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-elliptic point	Figure 9
$D_5 : f_1 < 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-elliptic point	Figure 10
$D_6 : f_1 > 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-elliptic point	Figure 11
$D_7 : f_1 < 0, f_2 < 0, f_3 > 0$	$(q_1^*, 0)$	One-elliptic point	Figure 12
$D_8 : f_1 > 0, f_2 > 0, f_3 < 0$	$(q_1^*, 0)$	One-hyperbolic point	Figure 13

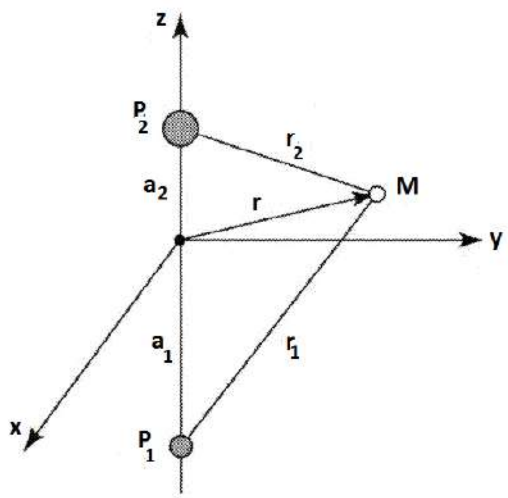

Figure 1: The description of the problem.

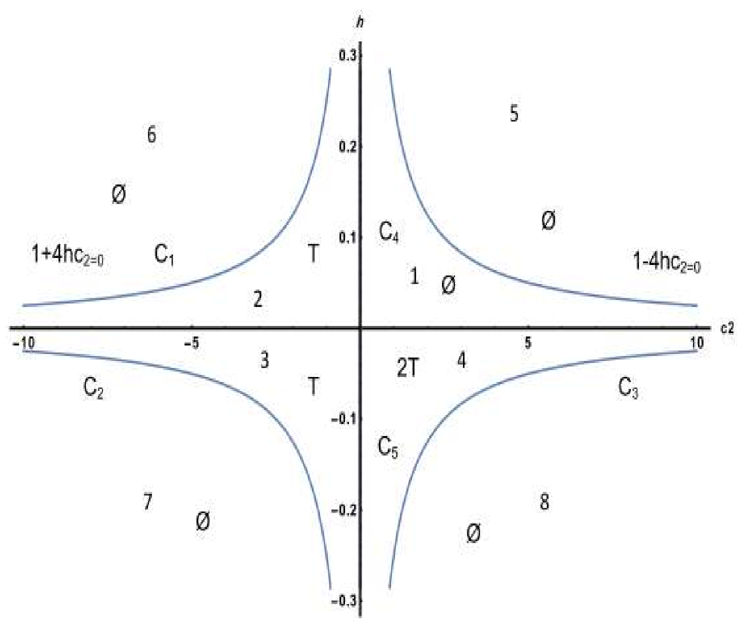

Figure 2: Diagram of bifurcation $\Sigma = \Sigma_1 \cup \Sigma_2$.

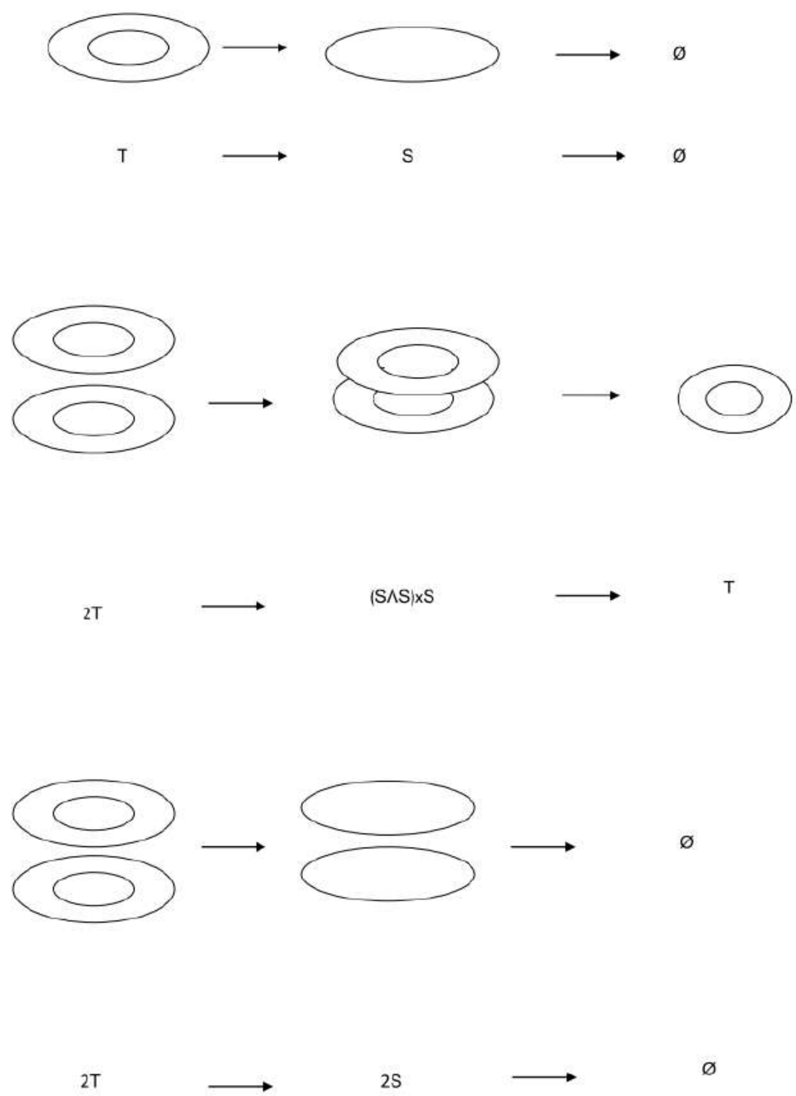

Figure 3: The bifurcation of liouvile tori, where a torus spirals twice around a torus and become twice, as a result, it is created $S \times (S \wedge S)$ which is a circle and two other circles that are above each other but not in the same level and have one common point.

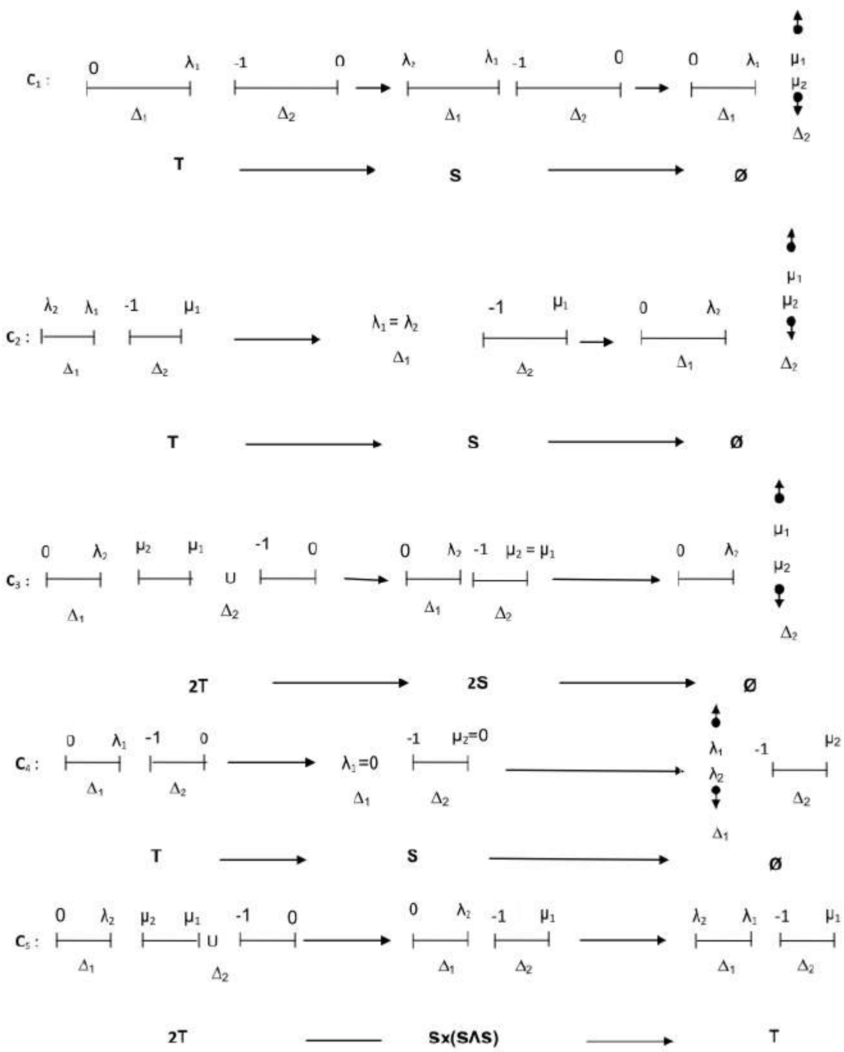

Figure 4: Correspondence between bifurcation of roots of polynomials $L(\lambda)$ *and* $M(\mu)$ and bifurcation of invariant Liouville tori.

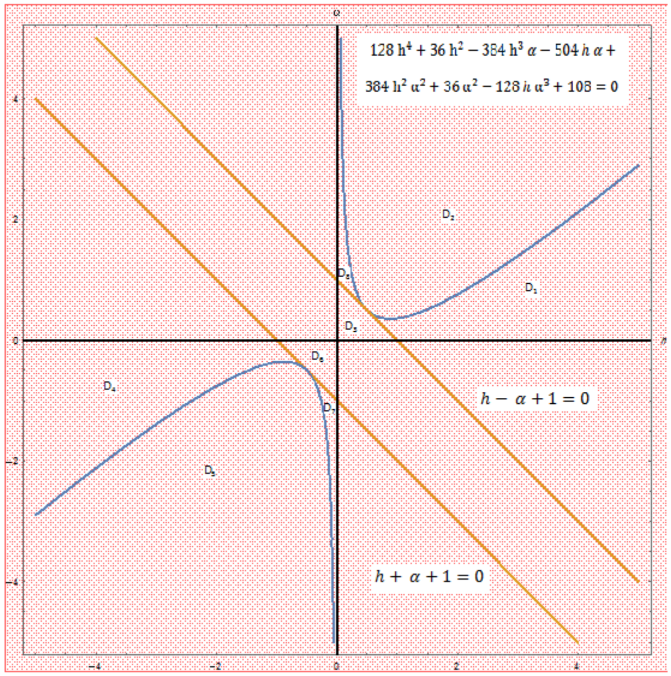

Figure 5: The regions D_i of the real motion on the (h,α) plane.

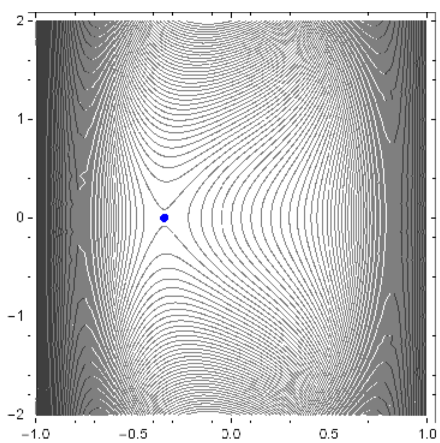

Figure 6: The one-hyperbolic point in domain D_1.

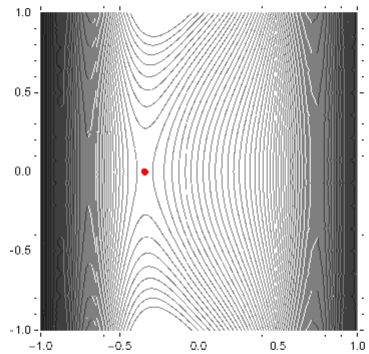

Figure 7: The one-hyperbolic point in domain D_2.

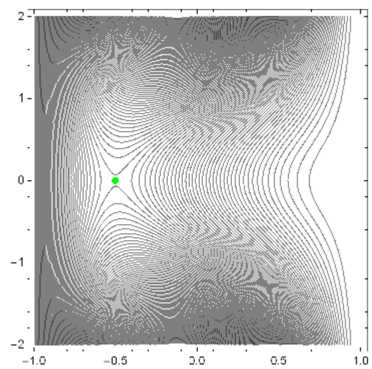

Figure 8: The one-hyperbolic point in domain D_3.

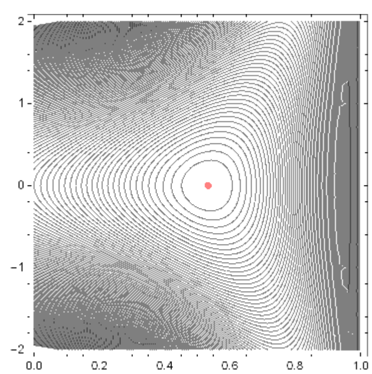

Figure 9: The one-elliptic point in domain D_4.

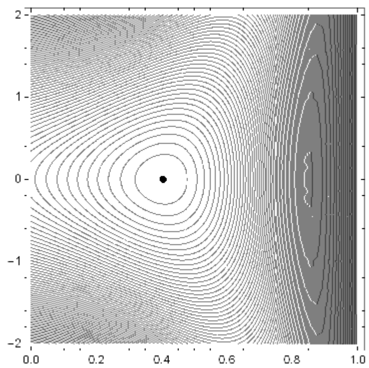

Figure 10: The one-elliptic point in domain D_5.

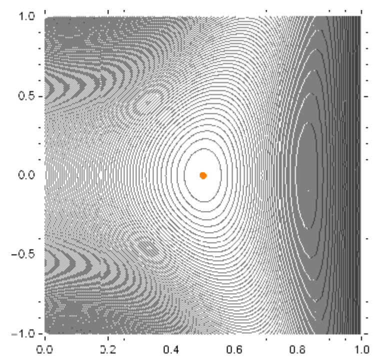

Figure 11: The one-elliptic point in domain D_6.

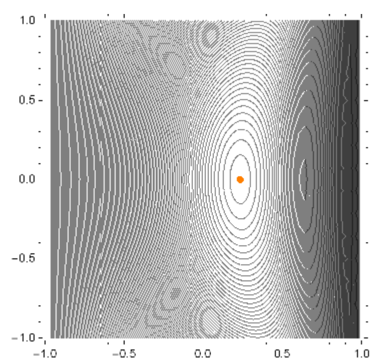

Figure 12: The one-hyperbolic point in domain D_7.

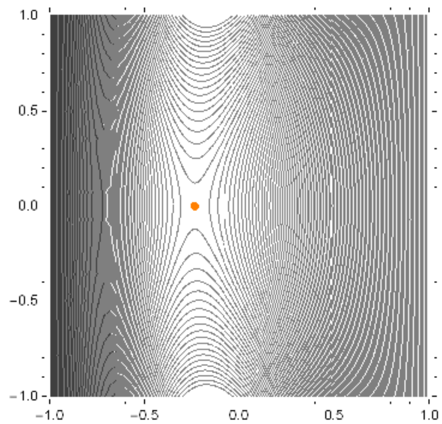

Figure 13: The one-hyperbolic point in domain D_8.

References

[1] E. P. Aksenov, E.A. Grebenikov, V.G. Demin, *General solution to the problem of satellite motion in the normal field of Earth's attraction*, Planet Space Science, 9 (1962), 491-498.

[2] E. P. Aksenov, E. A. Grebenikov, V. G. Demin, *The generalized problem of motion about two fixed centers and its application to the theory of artificial Earth satellites*, Soviet Astronomy-AJ, 7 (1963), 276-282.

[3] V. Alexey, Borisov, S. Ivan Mamaev, *Generalized problem of two and four Newtonian center*, Celestial Mechanics and Dynamical Astronomy, 92 (2005), 371-380.

[4] G. T. Arazov, *Investigation of motion of a satellite of spheroidal planet*, Astron. Zh., 52 (1975), 891-894.

[5] V. I. Aronlod, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, Springer, 60 (1989).

[6] A. V. Bolsinov, A. T. Fomenko, *Integrable Hamiltionian systems*, Geometry, Topology, Classification, CRC Press, 2004.

[7] G. Darboux, *Sur un probleme de mechanique*, Archives Neerlandaises des Sciences, Exact et Naturel, 6 (1901).

[8] B. A. Dubrovin, A.T. Fomenko, S.P. Novikov, *Modern geometry methods and applications*, Part II: The Geometry and Topology of Manifolds, Springer-Verlag New York Inc, 1985.

[9] F. M. El-Sabaa, *Bifurcation of Kovaleskaya polynomial*, International Journal of Theoretical Physics, 34 (1995), 2071-2083.

[10] F.M. El-Sabaa, M. Hosny, S.K. Zakria, *Bifurcations of Liouville tori of a two fixed center problem*, Astrophys Space Sci, 363 (2018), 1-10.

[11] L. Euler, *Un corps etant attire an raison reciproque quarree des distances vers deux points fixes donnes*, Mem. Berlin, 16 (1767), 228-249.

[12] A. T. Fomenko, *Integrablility and nonintegrability in geometry and mechanics*, Kluwer Academic Publishers, 1988.

[13] A. T. Fomenko, *Visual geometry and topology*, Springer-Verlag, Berlin Heidelberg, 1994.

[14] A. M. Hiltebeitel, *On the problem of two fixed centers and certain of its generalizations*, Am. J. Math., 33 (1911), 337-362.

[15] K. Jacobi, *Lektsii po dinamike* (Lectures in Dynamics), Moscow: ONTI, 1936.

[16] V. K. Kaisin, *Spacecraft motion in the normal gravity field of the earth under the action of additional forces*, Kosm. Issled., 7 (1969), 686-693.

[17] V. K. Kaisin, *One case of generalization of the problem of two immobile centers*, Byull. Inst. Teor. Astron., 12 (1970), 163-171.

[18] Yu. Kochetkova, L. G. Luḱyanov, *Some new applications of the generalized problem of two fixed centers*, Cosmic Research, 45 (2007), 354-358.

[19] G. G. Koman, *Intermediate orbits of artificial lunar satellites*, Soobshcheniya Gos. Astron. Inst. Shternberga, 186 (1973), 3-45.

[20] G. G. Koman, *One form of differential equations of motion of artificial lunar satellites*, Astron. Zh., 52 (1975), 207-209.

[21] J. Lagrange, *Recherhes sur le mouvement d́un corps qui est attiré vers deux centres fixes*, Auc. Mem. de Turin, 4 (1769), 67-121.

[22] J. Liouville, *Sur quelques cas particuliers où les équations du mouvement d́un point matériel peuvent s'intégrer*, Journal de Mathḿatiques Pures et Appliqúes, 11 (1846), 345-378.

[23] L. G. Lukyanov, N. V. Emeljanov, G. I. Shirmin, *Generalized problem of two fixed centers or the Darboux-Gredeaks problem,* Cosmic Research, 43 (2005), 186-191.

[24] J. Maciejewski, Maria Przybylska, *The non-integrability of the generalized two fixed centers problem*, Celestial Mechanics and Dynamical Astronomy, 89 (2004), 145-164.

[25] W. Pauli, *Uber das modell des wasserstoffmolekulions*, Annalen der Physik, 68 (1922), 177-240.

[26] H. Varvoglis, CH. Vozikis, K. Wodnar, *The two fixed centers: an exceptional integrable system,* Celestial Mechanics and Dynamical Astronomy, 89 (2004), 343-356.

[27] T. G. Vozmischeva, *The two center and Lagrange problem in the Lobachevsky space,* Proc. Int. Conf. Geometry, Integrability, and Quantization, Bulgaria, 283-298, 1999.

[28] T. G. Vozmischeva, *Classification of motions for generalization of the two center problem on a sphere,* Cel. Mech. and Dyn. Astr., 77 (2000), 37-48.

[29] T.G. Vozmischeva, *The Lagrange and two-center problems in the Lobachevsky space,* Cel. Mech. and Dyn. Astr., 84 (2002), 65-85.

[30] T. G. Vozmischeva and A. A. Oshemkov, *The topological analysis of the two-center problem on the two-dimensional sphere,* Sbornik: Mathematics, 193 (2002), 3-38.

[31] T. G. Vozmischeva, *Integrable problems of celestial mechanics in spaces of constant curvature*, Astrophysics and Space Science Library, Berlin, 180, 2003.

[32] T. G. Vozmishcheva, *Integrable problems of celestial mechanics in spaces of constant curvature*, Journal of Mathematical Sciences, 125 (2005), 419-532.

[33] H. Waalkens, R. Holger Dullina, H. Peter Richter, *The problem of two fixed centers: bifurcations, actions, monodromy*, Physica D., 196 (2004), 265-310.

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