On the n^*- and γ_n^*- complete fuzzy hypergroups

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Abstract. We extend the fuzzy approach of algebraic hyperstructures to the context of complete fuzzy hypergroups. In this paper we introduce the classes of n^*- complete fuzzy hypergroups and γ_n^*- complete fuzzy hypergroups which they are generalizations of two important classes of hypergroups, also we find some properties of them. Finally, we study 2^*- complete fuzzy hypergroups and give some properties and examples in this regard.

Keywords: fuzzy hypergroup, n^*- complete fuzzy hypergroup, γ_n^*- complete fuzzy hypergroup.

1. Introduction

The study of fuzzy hyperstructures is an interesting research topic for fuzzy sets. There are many works on the connections between fuzzy sets and hyperstructures [3, 10]. In this paper we introduce two types of fuzzy hypergroups: n^*- complete fuzzy hypergroups and γ_n^*- complete fuzzy hypergroups. They are generalizations of two important classes of hypergroups: n^*- complete hypergroups [4] and γ_n^*- complete hypergroups [5]. Notice in a fuzzy hypergroup the elements are combined by a fuzzy hyperoperations, while in a hypergroup, the elements are combined by a crisp hyperoperations. This idea was continuated

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by Sen, Ameri and Chowdhury in [10] where fuzzy semihypergroups are introduced. The fundamental relations are one of the most important and interesting concepts in fuzzy hyperstructures that ordinary algebraic structures are derived from fuzzy hyperstructures by them. Fundamental relations α^* and γ^* on fuzzy hypersemigroups are studied in [1] and [9]. In this paper, by using this relations, we introduce and study two types of fuzzy hypergroups and we present some properties of them. In the last section, we study in particular the 2^*- complete fuzzy hypergroups and we show that some classes of fuzzy hypergroups (fuzzy complete hypergroups, fuzzy join spaces, fuzzy canonical hypergroups, fuzzy steiner hypergroups) are 2^*- complete.

2. Preliminaries

Recall that for a non-empty set S, a fuzzy subset μ of S is a function from S into the real unite interval $[0, 1]$ and $Supp(\mu) = \{x \in S : \mu(x) > 0\}$.We denote the set of all nonzero fuzzy subsets of S by $F^*(S)$. Also for fuzzy subsets μ_1 and μ_2 of S, then μ_1 is *smaller* than μ_2 and write $\mu_1 \leq \mu_2$ iff for all $x \in S$, we have $\mu_1(x) \leq \mu_2(x)$. Define $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ as follows: $\forall x \in S$, $(\mu_1 \vee \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$ and $(\mu_1 \wedge \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$.

A *fuzzy hyperoperation* on S is a mapping $\circ : S \times S \mapsto F^*(S)$ written as $(a, b) \mapsto a \circ b = ab$. The couple $(S, \circ)$ is called a *fuzzy hypergroupoid*.

Definition 2.1 ([10]). A fuzzy hypergroupoid $(S, \circ)$ is called a *fuzzy hypersemigroup* if for all $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset μ of S and all $r \in S$

$$(a \circ \mu)(r) = \begin{cases} \bigvee\limits_{t \in S} ((a \circ t)(r) \wedge \mu(t)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

$$(\mu \circ a)(r) = \begin{cases} \bigvee\limits_{t \in S} (\mu(t) \wedge (t \circ a)(r)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

Definition 2.2. Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$. Then we define $\mu \circ \nu$ by $(\mu \circ \nu)(t) = \bigvee_{p,q \in S}(\mu(p) \wedge (p \circ q)(t) \wedge \nu(q))$, for all $t \in S$.

Definition 2.3 ([10]). A fuzzy hypersemigroup $(S, \circ)$ is called *fuzzy hypergroup* if $x \circ S = S \circ x = \chi_S$, for all $x \in S$, where χ_S is characteristic function of S.

Example 2.4. Consider a fuzzy hyperoperation $\circ$ on a non-empty set S by $a \circ b = \chi_{\{a,b\}}$, for all $a, b \in S$,. Then $(S, \circ)$ is a fuzzy hypersemigroup and fuzzy hypergroup as well.

Definition 2.5 ([10]). Let ρ be an equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, we define two relations $\overline{\rho}$ and $\overline{\overline{\rho}}$ on $F^*(S)$ as follows: for

$\mu, \nu \in F^*(S)$; $\mu\overline{\rho}\nu$ if $\mu(a) > 0$ then there exists $b \in S$ such that $\nu(b) > 0$ and $a\rho b$, also if $\nu(x) > 0$ then there exists $y \in S$, such that $\mu(y) > 0$ and $x\rho y$. $\mu\overline{\overline{\rho}}\nu$ if for all $x \in S$ such that $\mu(x) > 0$ and for all $y \in S$ such that $\nu(y) > 0$, $x\rho y$.

Definition 2.6. An equivalence relation ρ on a fuzzy hypersemigroup $(S, \circ)$ is said to be *(strongly) fuzzy regular* if $a\rho b, a'\rho b'$ implies $a \circ a' \ \overline{\rho} b \circ b' (a \circ a' \ \overline{\overline{\rho}} \ b \circ b')$.

Definition 2.7 ([1])**.** Let $(S, \circ)$ be a fuzzy hypergroup. We define the relation α on S in the following way: $\alpha = \cup_{n \geq 1} \alpha_n$, where $\alpha_1 = \{(x, x) \mid x \in S\}$ and for every $n > 1$ and $(a, b) \in S^2$

$a\alpha_n b \Longleftrightarrow \exists x_1, ..., x_n \in S (n \in \mathbb{N})$: $(x_1 \circ ... \circ x_n)(a) > 0$ and $(x_1 \circ ... \circ x_n)(b) > 0$.

It is clear that α is symetric and reflexive. We take α^* to be the transitive closure of α. Then α^* is an equivalence relation on S.

Theorem 2.8 ([1])**.** *The relation α^* is the smallest equivalence relation on a fuzzy hypergroup $(S, \circ)$ such that S/α^* is a group.*

Definition 2.9 ([9])**.** Let $(S, \circ)$ be a fuzzy hypergroup. The *commutative fundamental relation* on $(S, \circ)$ is the smallest equivalence relation ρ on S such that the quotient structure $(S/\rho, \oplus)$ is a commutative group.

Let $(S, \circ)$ be a fuzzy hypersemigroup. We define the relation γ on S in the following way, $\gamma = \bigcup_{n \geq 1} \gamma_n$ where $\gamma_1 = \{(s, s) : s \in S\}$ and for every $n \geq 2$, $a\gamma_n b$ if $\exists x_1, ..., x_n \in S (n \in \mathbb{N}), \exists \sigma \in S_n$: $(x_1 \circ \ldots \circ x_n)(a) > 0$ and $(x_{\sigma_1} \circ \ldots \circ x_{\sigma_n})(b) > 0$.

It is clear that γ is symetric and reflexive. We take γ^* to be the transitive closure of γ. Then γ^* is an equivalence relation on S.

Proposition 2.10 ([9])**.** *The relation γ^* is a strongly fuzzy regular relation.*

Corollary 2.11 ([9])**.** *Let S be a fuzzy hypersemigroup. Then the quotient S/γ^* is a commutative semigroup.*

3. n^*- and γ_n^*- complete fuzzy hypergroups

For every $n \in \mathbb{N}$ we will write α_n^* and γ_n^* to denote the transitive closure of the relations α_n and γ_n.

Definition 3.1. A fuzzy hypergroup S is said to be $n-$ complete if for every $z_1, ..., z_n, s, t \in S$, we have the following condition:

$$t \in Supp(\prod_{i=1}^{n} z_i), t\alpha s \Rightarrow s \in Supp(\prod_{i=1}^{n} z_i).$$

Proposition 3.2. (i) $\alpha_n \subseteq \alpha_{n+1}, \forall n \in \mathbb{N}$;

(ii) $\alpha_n^* \subseteq \alpha_{n+1}^*, \forall n \in \mathbb{N}$.

Proof. (i) $\forall(x,y) \in S^2, x\alpha_n y \Rightarrow \exists(z_1,...,z_n) \in S^n$:

$$(\prod_{i=1}^{n} z_i)(x) > 0 \text{ and } (\prod_{i=1}^{n} z_i)(y) > 0.$$

Since

$$0 < (\prod_{i=1}^{n} z_i)(x) = \bigvee_{r\in S} [(z_1...z_{n-1})(r) \wedge (rz_n)(x)],$$

there exists $r \in S$ such that $(z_1...z_{n-1})(r) > 0$ and $(rz_n)(x) > 0$. Also since S is a fuzzy hypergroup $\exists(t_1,t_2) \in S^2$ such that $(t_1t_2)(z_n) > 0$. Now

$$(z_1....z_{n-1}t_1t_2)(x) = \bigvee_{p,q\in S} [(\prod_{i=1}^{n-1} z_i)(p) \wedge (t_1t_2)(q) \wedge (pq)(x)].$$

Let $p = r$ and $q = z_n$, then $(z_1....z_{n-1}t_1t_2)(x) > 0$. In the similar way, we can show $(z_1....z_{n-1}t_1t_2)(y) > 0$, Therefore $x\alpha_{n+1}y$ and $\alpha_n \subseteq \alpha_{n+1}$.

(ii) It follows from (i). □

Proposition 3.3. $\forall(a,b,x) \in S^3$, $a\alpha_n^* b \Rightarrow (ax)\overline{\overline{\alpha_{n+1}^*}}(bx), (xa)\overline{\overline{\alpha_{n+1}^*}}(xb)$.

Proof. If $a\alpha_n^* b$, then $\exists(z_1,...,z_m) \in S^m : a = z_0\alpha_n z_1\alpha_n....z_m\alpha_n z_{m+1} = b$. Thus $\forall j, 0 \le j \le m, \exists(\alpha_1^j,\alpha_2^j,...,\alpha_n^j) \in S^n : \{z_j,z_{j+1}\} \subseteq Supp(\prod_{i=1}^{n}\alpha_i^j)$. Therefore $\forall j \in \{0,1,...,m\}, \forall t \in S$ such that $(z_jx)(t) > 0$, since $(\alpha_1^j...\alpha_n^j x)(t) = \bigvee_{p\in S}[(\prod_{i=1}^{n}\alpha_i^j)(p) \wedge (px)(t)]$, we let $p = z_j$ and obtain $t \in Supp(\alpha_1^j...\alpha_n^j x)$. Also for every $s \in S$, if $(z_{j+1}x)(s) > 0$, then $s \in Supp(\alpha_1^j...\alpha_n^j x)$. Therefore $(z_jx)\overline{\overline{\alpha_{n+1}}}(z_{j+1}x)$ and so $(ax)\overline{\overline{\alpha_{n+1}^*}}(bx)$. In a analogous way, we can prove the rest. □

Proposition 3.4. $\alpha_n^* = \alpha_{n+1}^* \Rightarrow \alpha_{n+1}^* = \alpha_{n+2}^*$.

Proof. It is sufficient to prove that $\alpha_{n+2} \subseteq \alpha_{n+1}^*$.

If $\{x,y\} \subseteq Supp(\prod_{i=1}^{n+2} z_i)$, then there exist $\{t_1,t_2\} \subseteq Supp(\prod_{i=1}^{n+1} z_i)$ such that $(t_1z_{n+2})(x) > 0$ and $(t_2z_{n+2})(y) > 0$. Obviously $t_1\alpha_{n+1}t_2$ and $t_1\alpha_n^* t_2$. Thus from 3.3, $(t_1z_{n+2})\overline{\overline{\alpha_{n+1}^*}}(t_2z_{n+2})$. It implies that $x\alpha_{n+1}^* y$. □

Proposition 3.5. *If there exists $n \in \mathbb{N}$ such that $\alpha_n^* = \alpha_{n+1}^*$ then $\alpha = \alpha_n^*$.*

Proof. It follows from 3.4 and Definition of the relation α. □

Definition 3.6. A fuzzy hypergroup S is said to be n^*- complete if there exists $n \in \mathbb{N}$ such that $\alpha_n^* = \alpha$ and $\alpha_n^* \neq \alpha_{n-1}^*$.

Remark 1. A fuzzy hypergroup S is n^*- complete if and only if $\alpha_{n+1} \subseteq \alpha_n^* \neq \alpha_{n-1}^*$.

Remark 2. S is n^*- complete if and only if n is the minimum integer such that S/α_n^* is a group.

Proposition 3.7. *Every finite fuzzy hypergroup is n^*- complete.*

Proof. It follows from $\alpha_1^* \subseteq \alpha_2^* \subseteq ... \subseteq \alpha_n^* \subseteq ....$ $\square$

Remark 3. For every finite fuzzy hypergroup, there exists $m \in \mathbb{N}$ such that $\alpha = \alpha_m$.

Proposition 3.8. *If S is n-complete fuzzy hypergroup then $\exists m \leq n$ such that S is m^*- complete.*

Definition 3.9. A fuzzy hypergroup S is said to be γ_n- complete if for every $z_1, ..., z_n, s, t \in S$, and every $\sigma \in S_n$ the following condition holds:

$$t \in Supp(\prod_{i=1}^{n} z_{\sigma(i)}), t\gamma s \Rightarrow s \in Supp(\prod_{i=1}^{n} z_i).$$

Proposition 3.10. (i) $\gamma_n \subseteq \gamma_{n+1}, \forall n \in \mathbb{N}$*;*
(ii) $\gamma_n^* \subseteq \gamma_{n+1}^*, \forall n \in \mathbb{N}$.

Proof. (i) If $x\gamma_n y$, then $\exists(z_1, ..., z_n) \in S^n, \exists\sigma \in S_n$:

$$(\prod_{i=1}^{n} z_i)(x) > 0 \ \text{ and } \ (\prod_{i=1}^{n} z_{\sigma(i)})(y) > 0.$$

Since S is a fuzzy hypergroup, so $\exists(t_1, t_2) \in S^2$ such that $(t_1t_2)(z_n) > 0$. Let $z_{i'} = z_i$, for $1 \leq i \leq n-1$ and $z_{n'} = t_1, z_{n+1'} = t_2$. Thus $(\prod_{i=1}^{n+1} z_{i'})(x) > 0$. Let $\sigma(k) = n$, now since $(z_{\sigma(1)}...z_{\sigma(k)}z_{\sigma(k+1)}...z_{\sigma(n)})(y) = \bigvee_{p,q\in S}[(z_{\sigma(1)}...z_{\sigma(k)})(p) \wedge (z_{\sigma(k+1)}...z_{\sigma(n)})(q) \wedge (pq)(y)]$, there exist $p, q \in S$ such that $(z_{\sigma(1)}...z_{\sigma(k)})(p) > 0$, $(z_{\sigma(k+1)}...z_{\sigma(n)})(q) > 0$ and $(pq)(y) > 0$. But $(z_{\sigma(1)}...z_{\sigma(k)})(p) > 0$ implies that there exists $r \in S$ such that $(z_{\sigma(1)}...z_{\sigma(k-1)})(r) > 0$ and $(rz_n)(p) > 0$. Now, $(z_{\sigma(1)}...z_{\sigma(k-1)}t_1t_2z_{\sigma(k+1)}...z_{\sigma(n)})(y) = \bigvee_{p',q'\in S}[(z_{\sigma(1)}...z_{\sigma(k-1)}t_1t_2)(p') \wedge (z_{\sigma(k+1)}...z_{\sigma(n)})(q') \wedge (p'q')(y)]$.

Let $p' = p$ and $q' = q$. Since

$$(z_{\sigma(1)}...z_{\sigma(k-1)}t_1t_2)(p) = \bigvee_{r',s'\in S} [(z_{\sigma(1)}...z_{\sigma(k-1)})(r') \wedge (t_1t_2)(s') \wedge (r's')(p)].$$

Let $r' = r$ and $s' = z_n$. Therefore $(z_{\sigma(1)}...z_{\sigma(k-1)}t_1t_2z_{\sigma(k+1)}...z_{\sigma(n)})(y) > 0$ and $x\gamma_{n+1}y$.

(ii) It follows from (i). $\square$

Proposition 3.11. $\forall(a, b, x) \in S^3$,

$$a\gamma_n^* b \Rightarrow (ax)\overline{\overline{\gamma_{n+1}^*}}(bx) \ \ and \ \ (xa)\overline{\overline{\gamma_{n+1}^*}}(xb).$$

Definition 3.12. A fuzzy hypergroup S is said to be γ_n^*- complete if there exists $n \in \mathbb{N}$ and n is the smallest integer such that $\gamma_n^* = \gamma$ and $\gamma_n^* \neq \gamma_{n-1}^*$.

We know $\alpha^* = \gamma^*$ in commutative fuzzy hypergroups, thus we obtain the following:

Proposition 3.13. *A commutative fuzzy hypergroup S is γ_n^*- complete if and only if S is n^*- complete fyzzy hypergroup.*

Proposition 3.14. *S is γ_n^*- complete fuzzy hypergroup if and only if S/γ_n^* is an abelian group.*

Proposition 3.15. *If S is γ_n- complete fuzzy hypergroup then $\exists m \leq n$ such that S is γ_m^*- complete.*

Proof. If S is γ_n- complete, then $\gamma_n = \gamma$, so $\gamma_n^* = \gamma$ and there exists $m \leq n$ such that $\gamma_m^* = \gamma$ and $\gamma_{m-1}^* \neq \gamma_m^*$. □

Proposition 3.16. *Every finite fuzzy hypergroup is γ_n^*- complete.*

Let $\phi : S \to S/\alpha^*$ be the canonical projection, then we denote $\omega_S = \phi^{-1}(1_{S/\alpha^*})$.

Proposition 3.17. *We have:*
(i) If $\forall(v, w) \in \omega_S^2$, $v\alpha_n w$ then $\alpha = \alpha_{n+1}$;
(ii) If $\forall(v, w) \in \omega_S^2$, $v\alpha_n^* w$ then $\alpha = \alpha_{n+1}^*$.

Proof. (i) If $x\alpha y$ then $\exists(v, w) \in \omega_S^2$ such that $(xv)(y) > 0$ and $(xw)(x) > 0$. But for the hypothesis $v\alpha_n w$ and so $(xv)\overline{\overline{\alpha_{n+1}}}(xw)$, whence $x\alpha_{n+1}y$, therefore $\alpha \subseteq \alpha_{n+1}$.

(ii) It follows from (i) and 3.3. □

Remark 4. Both of two parts of last proposition are verifiable, when we use γ instead of α.

Corollary 3.18. *If $\forall(u, w) \in \omega_S^2$, $u\alpha_n^* w$ and $\exists(u', w') \in \omega_S^2$ such that $u' \notin \alpha_{n-1}^*(w')$, then S is n^*-complete or $(n + 1)^*$-complete.*

Remark 5. Both of the two possibilities of corollary are verifiable, as the following examples:

Example 3.19. Let $(S, \circ)$ be a fuzzy hypergroup, where is defined by:

$$(a \circ a)(a) = 0.3, \ \ (b \circ a)(b) = (a \circ b)(b) = 0.1,$$
$$(a \circ c)(c) = (c \circ a)(c) = (c \circ b)(c) = 0.2,$$
$$(b \circ b)(a) = (b \circ b)(b) = (b \circ c)(c) = 0.4, \ (c \circ c)(a) = (c \circ c)(b) = 0.5$$

The remaining binary products are zero. In this case it is easy to verify that S is n^*-complete.

Example 3.20. Let $(S' = \{a, b, c, d\}, \circ)$ be a fuzzy hypergroup with the hyperoperation is defined by:

$$\begin{aligned}
&(a \circ a)(a) = (a \circ b)(b) = (a \circ c)(c) = (a \circ d)(d) = 0.1 = (b \circ a)(b)\\
&\qquad = (c \circ a)(c) = (d \circ a)(d),\\
&(b \circ b)(a) = (b \circ c)(d) = (b \circ d)(c) = 0.2,\\
&(c \circ c)(a) = (c \circ c)(b) = (c \circ d)(a) = 0.4,\\
&(c \circ d)(b) = (c \circ b)(d) = (d \circ b)(c) = 0.3,\\
&(d \circ c)(a) = (d \circ c)(b) = (d \circ d)(a) = (d \circ d)(b) = 0.5.
\end{aligned}$$

It is easy to verify that S' is $(n+1)^*$-complete.

4. On 2^*-complete fuzzy hypergroup

Lemma 4.1. *If $\mathcal{A}$ denotes the family of the fuzzy hyperproducts of two elements of S and there exists a family $\mathcal{M} = \{M_1, M_2, \dots, M_p\} \subseteq \mathcal{A}$ such that $M_i \wedge M_{i+1} \neq 0$ $(i \in \{1, 2, \dots, p-1\})$ and $\bigvee_{i=1}^{p} M_i = \chi_S$ then S is 2^*-complete and $\omega_S = S$.*

Proof. It is sufficient to prove that $\alpha_3 \subseteq \alpha_2^*$. If $x\alpha_3 y$, then $\exists (z_1, z_2, z_3) \in S^3$ such that $(\prod_{i=1}^{3} z_i)(x) > 0$ and $(\prod_{i=1}^{3} z_i)(y) > 0$ since $(x, y) \in S^2$ and $\chi_S = \bigvee_{i=1}^{p} M_i$. Thus:

A: $\exists M_i$ such that $M_i(x) > 0$ and $M_i(y) > 0$ then $x\alpha_2^* y$.

B: $\exists M_i$ such that $M_i(x) > 0$ and $\exists M_{i+1}$ such that $M_{i+1}(y) > 0$, since $M_i \wedge M_{i+1} \neq 0 \to \exists t \in S$ such that $M_i(t) > 0$ and $M_{i+1}(t) > 0$ thus $x\alpha_2 t$ and $t\alpha_2 y$ then $x\alpha_2^* y$.

C: $\exists M_i$ such that $M_i(x) > 0$ and $\exists M_j$ $(j \neq i,\ j \neq i+1)$ such that $M_j(y) > 0$ since $M_i \wedge M_{i+1} \neq 0$ then there exists $t_1 \in S$; $M_i(t_1) > 0$ and $M_{i+1}(t_1) > 0$ then $x\alpha_2 t_1$ and since $M_{i+1} \wedge M_{i+2} \neq 0$ thus there exists $t_2 \in S$; $M_{i+1}(t_2) > 0$ and $M_{i+2}(t_2) > 0$ then $t_1\alpha_2 t_2$. So as a consequence one obtains $t_j\alpha_2 y$ and so $x\alpha_2^* y$. □

Definition 4.2. An equivalence relation R on a fuzzy hypergroup S is called fuzzy feebly regular to the right if for every $x, y \in S$ and $\forall a \in S$, $\exists (u, v) \in S^2$ such that $(x \circ a)(u) > 0$ and $(y \circ a)(v) > 0$ and uRv. Analogously, we define the fuzzy feebly regularity to the left. An equivalence fuzzy feebly regular to the right and to the left is called fuzzy feebly regular.

Proposition 4.3. *Let $S = (S, \circ)$ be a fuzzy hypergroup. The following statements are equivalent:*

(i) *S is 2^*-complete;*

(ii) *α_2^* is fuzzy strongly regular;*

(*iii*) α_2^* *is fuzzy regular;*

(*iv*) α_2^* *is fuzzy feebly regular;*

(*v*) $\forall(x,y) \in S^2$, $x\alpha_2^* y \Longrightarrow \forall a \in S$, $[\alpha_2^*(x \circ a) \cap \alpha_2^*(y \circ a) \neq \emptyset]$ *and* $[\alpha_2^*(a \circ x) \cap \alpha_2^*(a \circ y) \neq \emptyset]$.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate.

We show that $(iv) \Rightarrow (v)$. Since $\exists \alpha \in S$; $(x \circ a)(\alpha) > 0$ and $\exists \gamma \in S$; $(y \circ a)(\gamma) > 0$ and $\alpha\alpha_2^*\gamma$, it follows that $\alpha_2^*(\alpha) \subseteq \alpha_2^*(x \circ a)$ and $\alpha_2^*(\gamma) \subseteq \alpha_2^*(y \circ a)$. But $\beta_2^*(\alpha) = \beta_2^*(\gamma)$ then we obtain the implication.

We prove now $(v) \Rightarrow (ii)$. Let $x\alpha_2^*y$, certainly $\forall a \in S$, $\exists \alpha \in S$; $(x \circ a)(\alpha) > 0$ and $\exists \gamma \in S$; $(y \circ a)(\gamma) > 0$ and $\alpha\alpha_2^*\gamma$. Thus it follows that $\alpha_2^*(x \circ a) = \alpha_2^*(\alpha) = \alpha_2^*(\gamma) = \alpha_2^*(y \circ a)$ whence $\forall z \in S$; $(x \circ a)(z) > 0$ and $\forall t \in S$; $(y \circ a)(t) > 0$, $z\alpha_2^*t$. It shows that fuzzy strong regularity to the right. Similarly it proves fuzzy strong regularity to the left.

Finally, we prove that $(ii) \Rightarrow (i)$, being $\alpha_2^* \subseteq \alpha$ and considering that α is the smallest fuzzy strongly regular equivalence on a fuzzy hypergroup S. $\square$

Corollary 4.4. *In every fuzzy hypergroup S, the following conditions, are equivalent:*

(*i*) S *is* 2^**-complete;*

(*ii*) $(S/\alpha_2^*, \otimes)$ *is a group;*

(*iii*) $(S/\alpha_2^*, \otimes)$ *is a hypergroup.*

Proposition 4.5. *In every fuzzy hypergroup $S = (S, \circ)$ we have*

$$\alpha_2^* = \alpha \Longleftrightarrow \forall(x,y) \in S^2, \exists z \in S; \alpha_2^*(x) \circ \alpha_2^*(y) \subseteq \alpha_2^*(z).$$

Proof. Let $\mathcal{C}(a)$ be the complete closure of a in S [1]. The right implication is a consequence of the fact that $\forall a \in S$, $\alpha_2^*(a) = \mathcal{C}(a)$. Now we prove that α_2^* is fuzzy strongly regular. If $x\alpha_2^*y$ then $\alpha_2^*(x) = \alpha_2^*(y)$ and so $\forall a \in S$, $\alpha_2^*(x) \circ \alpha_2^*(a) = \alpha_2^*(y) \circ \alpha_2^*(a)$. For the hypothesis, there exists an element $z \in S$ such that $\alpha_2^*(x) \circ \alpha_2^*(a) = \alpha_2^*(y) \circ \alpha_2^*(a) \subseteq \alpha_2^*(z)$. Thus $\forall t \in S$; $(x \circ a)(t) > 0$ and $\forall u \in S$; $(y \circ a)(u) > 0$. We obtain $\{t, u\} \subseteq \alpha_2^*(z)$ and finally $t\alpha_2^*u$. In the analogous way we can prove the strong regularity to the left. $\square$

Definition 4.6. A fuzzy hypergroup $S = (S, \circ)$ is called 1-fuzzy hypergroups if w_S is a singleton.

Proposition 4.7. *If S is* 1*-fuzzy hypergroup, then S is* 2^**-complete.*

Proof. Let w_S be a singleton $\{e\}$, we have that the classes modulo α are the fuzzy hyperproducts $e \circ a$, $\forall a \in S$. It follows at once that $\alpha = \alpha_2 = \alpha_2^*$. $\square$

Definition 4.8. A fuzzy hypergroup S is called fuzzy steiner hypergroup if $\forall(x,y) \in S^2$: $(x \circ y)(x) > 0$ and $(x \circ y)(y) > 0$.

Proposition 4.9. *Every fuzzy steiner hypergroup is* 2^**-complete.*

Proof. Since in such fuzzy hypergroup $\forall(x,y) \in S^2$, we have $(x \circ y)(x) > 0$ and $(x \circ y)(y) > 0$, thus $x\alpha_2 y$. Therefore $\alpha = \alpha_2 = \alpha_2^*$. $\square$

Definition 4.10 ([7])**.** If $\circ : S \times S \longrightarrow F^*(S)$ is a fuzzy hypercomposition, then S is called mimic fuzzy hypergroup (fuzzy$_M$-hypergroup), if the following two axioms are valid:

i. $(a \circ b) \circ c = a \circ (b \circ c)$, $\forall(a,b,c) \in S^3$ (associativity)

ii. $a/b \neq 0_H$ and $a \setminus b \neq 0_H$, $(a,b) \in S^2$.

Proposition 4.11 ([7])**.** *In a fuzzy$_M$-hypergroup S, it holds that $a \circ b \neq 0_S$ ($\forall(a,b) \in S^2$).*

Example 4.12. Every fuzzy$_M$-hypergroup S is a fuzzy steiner hypergroup. Thus S is 2^*-complete.

Definition 4.13. A commutative fuzzy hypergroup S is called a fuzzy join space if $\forall(a,b,c,d) \in S^4$,

$$a/b \wedge c/d \neq 0 \implies (a \circ d) \wedge (b \circ c) \neq 0.$$

Proposition 4.14. *Every fuzzy join space S is 2^*-complete.*

Proof. Suppose $S = (S, \circ)$ is a fuzzy join space which is not group. We prove that $\alpha_3 \subseteq \alpha_2^*$. If $t\alpha_3 u$ then $\exists(z_1,z_2,z_3) \in S^3$ such that $(z_1 \circ z_2 \circ z_3)(t) > 0$ and $(z_1 \circ z_2 \circ z_3)(u) > 0$ and $z_1 \circ z_2 \circ z_3 = z_1 \circ z_3 \circ z_2$. It follows that $\exists(a_1,a_2) \in S^2$, such that $(z_1 \circ z_2)(a_1) > 0$ and $(z_1 \circ z_2)(a_2) > 0$ and $\exists(b_1,b_2) \in S^2$; $(z_1 \circ z_3)(b_1) > 0$ and $(z_1 \circ z_3)(b_2) > 0$. Therefore $(a_1 \circ z_3)(t) > 0$ and $(b_1 \circ z_2)(t) > 0$ too $(a_2 \circ z_3)(u) > 0$ and $(b_2 \circ z_2)(u) > 0$. Therefore $a_1/z_2 \wedge b_2/z_3 \neq 0$ and since S is a fuzzy join space, one obtains that $(a_1 \circ z_3) \wedge (b_2 \circ z_2) \neq 0$. Therefore, $\exists w \in S$, such that $(a_1 \circ z_3)(w) > 0$ and $(b_2 \circ z_2)(w) > 0$ and since $(a_1 \circ z_3)(t) > 0$ and $(b_2 \circ z_2)(u) > 0$ whence $t\alpha_2^* u$. $\square$

Definition 4.15. A fuzzy hypergroup S is fuzzy regular if it has at least one identity and every elements has at least one inverse.

Definition 4.16. A fuzzy regular hypergroup is said to be fuzzy reversible if, denoting $\forall x \in S$ with $i(x)$ the set of the inverses of x, one has $\forall(a,b,x) \in S^3$

$$\begin{aligned}(b \circ x)(a) > 0 \implies & \exists x' \in i(x);\ (a \circ x')(b) > 0, (x \circ b)(a) > 0 \\ \implies & \exists x'' \in i(x); (x'' \circ a)(b) > 0.\end{aligned}$$

Definition 4.17. A fuzzy commutative reversible hypergroup is called canonical if it has a scalar identity and $\forall x$, $i(x)$ is a singleton.

Corollary 4.18. *Canonical fuzzy hypergroups are 2^*-complete.*

Example 4.19. We give now a fuzzy commutative regular hypergroup S with scalar identity which is not reversible so is not 2^*-complete.

$$(d \circ d)(b) = (a \circ a)(a) = (a \circ b)(b) = (a \circ c)(c) = (a \circ d)(d) = 0.1$$
$$(d \circ c)(d) = (b \circ a)(b) = (c \circ a)(c) = (d \circ a)(d) = (b \circ b)(a) = 0.2$$
$$(d \circ c)(a) = (b \circ c)(d) = (b \circ d)(c) = (c \circ b)(d) = (d \circ b)(c) = 0.3$$
$$(d \circ d)(c) = (c \circ c)(b) = (c \circ c)(c) = 0.4$$
$$(c \circ d)(a) = (c \circ d)(d) = 0.5.$$

The remaining binary products are zero. S is not reversible, since $(c \circ d)(d) > 0$ and $(d \circ d^{-1})(c) = 0$, $(i(d) = \{a, d\})$, one has $\alpha_2^*(a) = \{a, d\}$, $\alpha_2^*(b) = \{b, c\}$ and so $\alpha_2^* \neq \alpha$. Thus S is not 2^*-complete.

Conclusions. We introduced the concepts of n^*- complete fuzzy hypergroups and γ_n^*- complete fuzzy hypergroups by using fundamental relation and commutative fundamental relation of a fuzzy hypergroup and we determined some properties of them. We will study about these topics in fuzzy hyperrings.

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