Rings with strongly algebraically closed lattices

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Abstract. In this article, we prove that if the central idempotents lattice of a Baer ring and the projection lattice of a $*$-Baer ring center and the set of all saturated subsets of a Noetherian regular ring are q'-compact, then they are strongly algebraically closed lattice. Also, for a commutative ring R, it is shown that if the set of idempotents of a Specker R-algebra is q'-compact, then it is a strongly algebraically closed lattice.

Keywords: strongly algebraically closed lattices, equationally Noetherian lattice, $*$-Baer rings, projection lattice.

1. Introduction

Universal algebraic geometry is a branch of mathematics and it deals with the solutions of systems of equations over an arbitrary algebraic structure (algebra for short). The main part of the investigations in this area are due to E. Daniyarova, A. Miasnikov, V. Remeslennikov, and the obtained results can be applied to algebraic geometry over an arbitrary algebra ([6]-[9]). In this paper, for a commutative ring R, we find some relationship among strongly algebraically closed lattices, Baer and $*$-Baer rings, and Specker R-algebra.

In section 2, we recall some basic notations and definitions from universal algebraic geometry. In section 3, it is proved that if central idempotents of a Baer ring and projection lattice of a $*$-Baer ring center are q'-compact, then the central idempotents and the projection lattice are strongly algebraically closed lattice. In addition, it is shown that the set of all saturated subsets of a Noetherian regular ring is also true. Finally, in section 4, we prove that if the set of idempotents of a Baer ring S is q'-compact, then it is a strongly algebraically closed lattice, which S is a Specker R-algebra.

2. Strongly algebraically closed lattices

Let S be a system of equations in an algebra A. The set of all logical consequences of S over A is the radical $Rad_A(S)$, which $V_A(S)$ is the sets of solutions of S in A. In other words, $Rad_A(S)$ is the set of all lattice equations $f \approx g$ such that $V_A(S) \subseteq V_A(f \approx g)$.

Definition 2.1. We say that two lattices A and B are geometrically equivalent, if for any system S, we have $Rad_A(S) = Rad_B(S)$. A lattice A is q'-compact, if it is geometrically equivalent to any of its elementary extensions.

The problem of geometric equivalence was posed in [15]. In [13] this problem was solved for equationally Noetherian groups. Now, in this section of paper we provide examples of geometric equivalence and q'-compact.

Example 2.2. For the first example of geometric equivalence, we have that two irreducible and faithful representations of finite groups over the same field are geometrically equivalent if and only if they are isomorphic. For the second example of geometric equivalence, we know that if two algebras are logically Noetherian, then they are geometrically equivalent if and only if they have the same quasi-identities. Also, consider a field and two its extensions F_1 and F_2. If both F_1 and F_2 are algebraically closed, then they are geometrically equivalent. Then they have the same equational theories. Actually, it is known that even their elementary theories coincide. Therefore, if two algebras are geometrically equivalent in universal logic, then they have the same universal theory.

Example 2.3. For example of q'-compact, it is clear that nontrivial lattices are geometrically equivalent. It is sufficient to prove that any nontrivial lattice L is geometrically equivalent to the two-element lattice $\{0, 1\}$ and finite lattice has no proper elementary extension.

By a Boolean lattice, we mean a complemented distributive lattice. By a Boolean algebra, we mean a Boolean lattice together with the unary operation of complementation (see [4]).

A lattice A is called algebraically closed, if any finite consistent system of equations with coefficients from A, has a solution in A. A system S with coefficients in A is called consistent, if there is an extension B, such that S has a solution in B.

Definition 2.4. A lattice A in a class of lattices is said to be strongly algebraically closed if every system (not necessarily finite) of equations with parameters in A which has a solution in some extension B of A in the class, has already a solution in A.

A lattice A is called equationally Noetherian, if any system of equations with coefficient in A is equivalent with a finite subsystem. If any system of equations over A is equivalent with a finite system then it is said weakly equationally Noetherian. Recall that, equationally and weak equationally Noetherian Boolean algebras (with coefficients) are characterized by Shevlyakov in [17]. Suppose $\mathcal{L}$ is an algebraic language and A is an algebra of type $\mathcal{L}$. If we attach the elements of A as constants to $\mathcal{L}$, then the new language will be denoted by $\mathcal{L}(A)$. We say that the algebra A is finitary equational Noetherian, if every finitary system of equations in the language $\mathcal{L}(A)$ is reducible over A to a finite system. A Boolean algebra A is complete if every subset B of A has a least upper bound $\bigvee B$ and a greatest lower bound $\bigwedge B$. Here, we state the following theorem:

Theorem 2.5 ([14])**.** *Let A be a complete Boolean lattice which is q'-compact. Then A is strongly algebraically closed in the class of distributive lattices.*

3. Baer and $*$-Baer rings

In this section, we present some basic notations and definitions that we use in this paper. For more detailed information, we refer the reader to [1] and [10]. Recall that the study of Baer rings has its roots in functional analysis and various authors have investigated properties of the star order (introduced by Drazin in 1978) on algebras of matrices and of bounded linear operators on a Hilbert space.

Definition 3.1. A commutative ring R is a Baer ring if the annihilator ideal of each subset of R is a principal ideal generated by an idempotent.

Theorem 3.2. *Let B be central idempotents of a Baer ring R. If B is q'-compact, then B is a strongly algebraically closed lattice.*

Proof. Let x and y be arbitrary elements of the idempotents of the center of R. We have that

$$x \cap y = xy, \quad x \cup y = x + y + xy, \quad x' = 1 - x,$$

form a Boolean algebra, which $x \leq y$ is defined by $xy = x$. Thus, B is Baer ring and Boolean algebra. Now, we prove that the center idempotents of a Baer ring form a complete lattice. For doing this work, assume that $S = \{x_i \mid i \in I\}$ subset of B and

$$S^r = \{x \in R \mid \forall s \in S \ (xs = 0)\}.$$

Since the center of a Baer ring is a Baer ring. So, there exists a central idempotent $y \in B$ such that $S^r = yR$. We set $z = 1 - y$ and claim that z is a supremum of the x_i. Suppose $i \in I$ and $t \in R$. If $x_i t = 0$, then $t \in yR$, $yt = t$, $zt = 0$. Now, assume that e is an arbitrary element of R that is idempotent, then $x_i(1-e) = 0$ and we conclude $z \leq e$. We observe that $z = \sup x_i$. For infimum suffices we set $1 - \sup(1 - x_i)$. So, B is a complete Boolean algebra. By applying theorem 2.3, since B is q'-compact and a complete Boolean algebra, then B a strongly algebraically closed lattice (see [1]). □

Kaplansky's axiomatic approach for studying simultaneously the classical equivalence relations on projection lattices is developed in detail, culminating in the construction of a dimension function in that context.

Definition 3.3. An element p of a $*$-ring is called a projection if p is a self-adjoint ($p^* = p$) idempotent ($p^2 = p$).

An associative unital ring R is a $*$-ring (or ring with involution) if there exists an operation $* : R \longrightarrow R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^* x^*$, and $(x^*)^* = x$ for all $x, y \in R$.

Definition 3.4. A $*$-ring is called Baer $*$-ring if the right annihilator of every nonempty subset is generated by a projection.

Obviously, 0 and 1 are projections of any $*$-ring. Projection lattices naturally arise in the context of lattice packings. Now let us recall the following definition of Rickart $*$-ring of [3].

Definition 3.5. A Rickart $*$-ring is a $*$-ring such that the right annihilator of each element is the principal right ideal generated by a projection (a self-adjoint idempotent).

Theorem 3.6. *Let $P(Z)$ be the projection lattice of the center of a $*$-Baer ring. If $P(Z)$ is q'-compact, then $P(Z)$ is a strongly algebraically closed lattice.*

Proof. It is not hard to verify that any any Baer $*$-ring is a Rickart $*$-ring and a Baer ring. Recall that the projections of a Rickart $*$-ring form a lattice, with $e \cup f = f + RP[e(1-f)]$, $e \cap f = e - LP[e(1-f)]$ (see [12], Lemma 5.3). We observe that in a Rickart $*$-ring, every central idempotent is a projection. Because if u is a arbitrary idempotent element of center of R, then $u^* = u$. By ([11], Prop. 2.1), $uf = fR$ with a projection f, whence $u = fu = uf = f$. Since, we saw in Theorem 3.2, the central idempotents of a Baer ring form a complete Boolean algebra. So, the projection lattice of the center of $*$-Baer ring is a complete Boolean algebra. Here, $P(Z)$ is q'-compact. By theorem 2.1, $P(Z)$ is a strongly algebraically closed lattice (see [1]). $\square$

A non empty subset F of a commutative ring R with identity is said to be saturated if for any x and y of R we have

$$xy \in F \Longleftrightarrow x, y \in F.$$

Now in the following Corollary we consider relationship between Noetherian regular rings and strongly algebraically closed lattices.

Corollary 3.7. *Suppose that $S(R)$ is the set of all saturated subsets of a Noetherian regular ring R. If $S(R)$ is q'-compact, then $S(R)$ is a strongly algebraically closed lattice.*

Proof. We know that if R is a Noetherian regular ring, then R is a direct sum of fields and will have

$$R = F_1 \oplus \ldots \oplus F_n.$$

Also, the prime ideals are

$$P_j = \Pi_{i \neq j} F_i, j = 1, 2, \ldots, n$$

and $S(R)$ is isomorphic to the Boolean algebra of subsets of $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\}$. Thus, $S(R)$ is a Boolean algebra. On the other hand, it is clear that the set of all saturated subsets of a commutative ring with identity form a complete lattice. Hence, $S(R)$ is a complete Boolean algebra and q'-compact. From theorem 2.3, we immediately obtain that is a strongly algebraically closed lattice. □

4. Specker R-algebra of a commutative ring

Throughout this section, R will be a commutative ring with 1. A R-algebra is a ring with identity together with a ring homomorphism $f : R \longrightarrow A$ such that the subring $f(R)$ of A is contained within the center of A. Let S be a commutative R-algebra and $Id(S)$ be the set of idempotents of S. We call a nonzero idempotent e of S faithful if for each $a \in R$, whenever $ae = 0$, then $a = 0$. Let B be a Boolean subalgebra of $Id(S)$ that generates S. We say that B is a faithful generating algebra of idempotents of S if each nonzero $e \in B$ is faithful. We recall that an R-algebra S is Specker R-algebra if S is a commutative R-algebra that has a faithful generating algebra of idempotents [5]. To build Specker R-algebras from Boolean algebras we introduce a construction which has its roots in the work of Bergman [2] and Rota [16].

Theorem 4.1. *Let S be a Specker R-algebra. If S is Baer and $Id(S)$ is q'-compact, then $Id(S)$ is a strongly algebraically closed lattice.*

Proof. We know that if S is a commutative R-algebra, then S is a commutative ring with 1, it is well known that the set $Id(S)$ of idempotents of S is a Boolean algebra via the operations

$$e \vee f = e + f - ef, \;\; e \wedge f = ef, \;\; \neg e = 1 - e.$$

In order to prove that $Id(S)$ is a strongly algebraically closed lattice, it remains to show that $Id(S)$ is a complete lattice. In the other words, we show that for every subset $E = \{e_i \,|\, i \in I\}$ of idempotents of S has a largest element. One can easily prove that if $K = \{1 - e_i \,|\, i \in I\}$, then $ann_S(1 - e_i) = e_i S$ and $ann_S(K) = \cap e_i S$. We have S is Baer, so that $ann_S(K) = eS$ for some $e \in Id(S)$. Now, we will prove $e = \wedge e_i$. It is easy to see $e \in ann_S(K)$, we have $ee_i = e$ and then $e \leq e_i$. But this says e is a lower bound of the e_i. First note that if $f \in Id(S)$ be a lower bound of the e_i, then $fe_i = f$, as a result $(1 - e_i)f = 0$. Therefore, $f \in ann_S(K) = eS$. This shows that $fe = f$, so $f \leq e$. Thus, $e = \wedge_i e_i$. It can be observe that, $Id(S)$ is a complete Boolean algebra and is q'-compact. By theorem 2.3, $Id(S)$ is a strongly algebraically closed lattice (see [1]). □

Corollary 4.2. *Let R be indecomposable and S be a Specker R-algebra. If S is Baer and $Id(S)$ is q'-compact, then $Id(S)$ is a strongly algebraically closed lattice.*

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