L^∞-asymptotic behavior for a finite element approximation to optimal control problems

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Abstract. In this paper, a system of parabolic quasi-variational inequalities relevant to the management of energy production is considered where a quasi-optimal of error estimate on uniform norm is proved, by using semi-implicit scheme combined with Galerkin method. Furthermore, an asymptotic behavior result in the same norm is given, taking into consideration the discrete stability properties.

Keywords: parabolic quasi variational inequalities, Galerkin method, L^∞-asymptotic behavior.

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1. Introduction

In this paper, we are concerned with the numerical approximation in the L^∞ norm for the following problem: find u such that

$$\text{(1.1)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \max\limits_{1\leq i\leq J}\left(\mathcal{A}^i u - f^i\right) = 0 \text{ in } \mathbb{Q}_T := \Omega\times]0,\ T[\,, \\ u\Big|_{t=0} = u_0, \text{ in } \Omega, \\ u = 0 \text{ in } \sum_T :=]0,\ T[\times\Gamma, \\ u\geq 0, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^d$, $d\geq 1$, with smooth boundary Γ, $\mathcal{A}^i$ are J-second-order, uniformly elliptic operators of the form

$$\text{(1.2)}\qquad \mathcal{A}^i = \sum_{j,k=1}^{d} a^i_{jk}(x)\frac{\partial^2}{\partial x_j\partial x_k} + \sum_{j=1}^{d} b^i_j\left(x\right)\frac{\partial}{\partial x_j} + a^i_0\left(x\right),$$

f is a regular function satisfies

$$\text{(1.3)}\qquad f\in L^2\left(0,T,L^\infty\left(\Omega\right)\right)\cap C^1\left(0,T,H^{-1}\left(\Omega\right)\right).$$

It is known (see. [3-5], [9-12]) that the problem (1.1) can be approximated by the following weakly coupled system of parabolic quasi-variational inequalities (QVIs): find a vector $U = \left(u^1,\ u^2,...,\ u^J\right)\in\left(L^2\left(0,\ T;\ H^1_0\left(\Omega\right)\right)\right)^J$ such that

$$\text{(1.4)}\qquad \begin{cases} \dfrac{\partial}{\partial t}(u^i(t),\ v-u^i(t)) + a^i(u^i(t),\ v-u^i(t))\geq (f^i,\ v-u^i(t)), \\ \qquad \forall v\in H^1_0(\Omega), \\ u^i\leq (MU)^i,\ v\leq (MU)^i,\ i=1,\ 2,...,\ J, \\ u^i\geq 0, \end{cases}$$

where $a^i\left(.,.\right)$ are J-elliptic continuous and noncoercive bilinear forms associated $\mathcal{A}^i$ defined as:

$$\text{(1.5)}\qquad a^i\left(u,v\right) = \int\limits_{\Omega}\left(\sum_{j,k=1}^{d} a^i_{jk}(x)\frac{\partial\ u}{\partial\ x_j}\frac{\partial\ v}{\partial\ x_k} + \sum_{k=1}^{d} b^i_k\left(x\right)\frac{\partial u}{\partial x_k}v + a^i_0\left(x\right)uv\right)dx,$$

where $\forall i = 1,...,J,\ a^i_{jk}\left(.\right),\ b^i_j\left(.\right),\ a^i_0\left(.\right)\in C^2\left(\bar{\Omega}\right),\ x\in\bar{\Omega}, 1\leq j,\ k\leq d$ are sufficiently smooth coefficients and satisfy the following conditions:

$$\text{(1.6)}\qquad \begin{cases} a^i_{jk}(x) = a^i_{kj}(x), \\ a^i_0(x)\geq\beta>0,\ \beta \text{ is a constant} \end{cases}$$

and

$$\sum_{j,k=1}^{n} a_{jk}^{i}(x)\xi_j\xi_k \geq \gamma|\xi|^2;\ \xi \in \mathbb{R}^d,\ \gamma > 0,\ x \in \bar{\Omega}. \tag{1.7}$$

In the case studied here, $(MU)^i$ represents a "cost function" and the prototype encountered is

$$(MU)^i = \rho + \inf_{\mu \neq i} u^{\mu},\ i = 1, ...,\ J. \tag{1.8}$$

In (1.8), ρ represents the switching cost. It is positive when the unit is "turned on" and equal to zero when the unit is "turned off". Note also that operator M provides the coupling between the unknowns $u^1, \ldots, u^J$ (see. e.g. [1], [2] and the references therein).

In the stationary case M. Boulbrachen in [7] studied a particular class of problems related to the management of energy production problems and presented a study of the complete numerical analysis; his approach is based on the concept of subsolutions.

The aim of the present paper is to study the corresponding evolution case and to obtain a quasi-optimal L^∞-asymptotic behavior for a finite element approximation to parabolic quasi-variational inequalities.

The rest of the manuscript is structured as follows. In Section 2, we present the continuous problem. The discrete problem is proposed in Section 3. Then, in Section 4, we prove an error estimate on the uniform norm of the presented problem.

2. Statement of the continuous problem

2.1 The continuous system

2.1.1 Full discretization

In order to obtain a full discretization of (1.4), we consider a uniform mesh for the time variable t and define

$$t_n = n\ \Delta t,\ n = 0, 1, ..., \mathcal{N}, \tag{2.1}$$

$\Delta t > 0$ being the time-step, and $\mathcal{N} = \left[\frac{T}{\Delta t}\right]$, the integral part of $\frac{T}{\Delta t}$.

Next, we replace the time derivative by means of suitable difference quotients, thus constructing a sequence $u^{i,n} \in H_0^1(\Omega)$ that approaches $u^i(t_n,\ x)$.

For simplicity, we confine ourselves to the so-called semi-implicit scheme, which consists of replacing (1.4) by the following scheme: find a vector $U^n =$

$\left(u^{1,n},...,u^{J,n}\right) \in \left(H_0^1(\Omega)\right)^J$ such that

$$\begin{cases} \dfrac{1}{\triangle t}(u^{i,n} - u^{i,n-1},\ v - u^{i,n}) + a^i(u^{i,n},\ v - u^{i,n}) \geq (f^{i,n},\ v - u^{i,n}), \\ \qquad \forall v \in H_0^1(\Omega), \\ u^{i,n} \leq \rho + \inf\limits_{\mu \neq i} u^{\mu,n},\ v \leq \rho + \inf\limits_{\mu \neq i} u^{\mu,n},\ n = 1, ..., \mathcal{N}-1, \\ u^i(0) = u_0^i, \end{cases} \tag{2.2}$$

where

$$\triangle t = \frac{T}{\mathcal{N}}.$$

By adding $(\frac{u^{i,n-1}}{\triangle t},\ v - u^{i,n})$ to both parties of the scheme (2.3), we get

$$\begin{cases} a^i\left(u^{i,n},\ v - u^{i,n}\right) + \dfrac{1}{\triangle t}\left(u^{i,n},\ v - u^{i,n}\right) \\ \qquad \geq \left(f^{i,n} + \dfrac{1}{\triangle t} u^{i,n-1},\ v - u^{i,n}\right), \\ u^{i,n} \leq \rho + \inf\limits_{\mu \neq i} u^{\mu,n},\ v \leq \rho + \inf\limits_{\mu \neq i} u^{\mu,n}, \\ u^i(0) = u_0^i. \end{cases} \tag{2.3}$$

The bilinear form $a^i(.,\ .)$, is a noncoercive in $H_0^1(\Omega)$, and satisfies the following condition: for all $\varphi \in H_0^1(\Omega)$ there exists $\gamma > 0$, such that

$$a^i(\varphi,\ \varphi) + \lambda \|\varphi\|^2_{L^2(\Omega)} \geq \gamma \|\varphi\|^2_{H_0^1(\Omega)}. \tag{2.4}$$

Set

$$b^i(u,\ v) = a^i(u,\ v) + \lambda(u,\ v). \tag{2.5}$$

Thanks to [7] the bilinear $b^i(.,\ .)$ is strongly coercive and (2.3) can be transformed into the following continuous system of elliptic quasi-variational inequalities (QVIs): find a vector $U^n = \left(u^{1,n},...,u^{J,n}\right) \in \left(H_0^1(\Omega)\right)^J$ such that

$$\begin{cases} b^i\left(u^{i,n},\ v - u^{i,n}\right) \geq \left(f^{i,n} + \lambda u^{i,n-1},\ v - u^{i,n}\right),\ \forall v \in H_0^1(\Omega), \\ u^{i,n} \leq \rho + \inf\limits_{\mu \neq i} u^{\mu,n},\ v \leq \rho + \inf\limits_{\mu \neq i} u^{\mu,n},\ n = 1, ..., \mathcal{N}-1, \end{cases} \tag{2.6}$$

where

$$\begin{cases} b^i\left(u^{i,n},\ v - u^{i,n}\right) = a^i\left(u^{i,n},\ v - u^{i,n}\right) + \lambda\left(u^{i,n},\ v - u^{i,n}\right), \\ \lambda = \dfrac{1}{\Delta t} > 0. \end{cases} \tag{2.7}$$

2.2 Existence and uniqueness

Next, using the preceding assumptions, we shall prove the existence and uniqueness of a continuous solution for problem (2.6) by means of Banach's fixed point theorem.

Let $\mathbb{H}^{+} = \prod_{i=1}^{J} L_{+}^{\infty}(\Omega) = \{W = (w^1, ..., w^J)$ such that $w^i \in L_{+}^{\infty}(\Omega)\}$, equipped with the norm

$$\|W\|_{\infty} = \max_{1\leq i\leq J} \left\|w^i\right\|_{L^{\infty}(\Omega)}, \tag{2.8}$$

where $L_{+}^{\infty}(\Omega)$ is the positive cone of $L^{\infty}(\Omega)$.

2.2.1 A fixed point mapping associated with the system (2.6)

We consider the following mapping:

$$\begin{aligned} \mathbb{T} \ : \ & \mathbb{H}^{+} \longrightarrow \mathbb{H}^{+}, \\ & W \rightarrow \mathbb{T}W = \zeta^n = \left(\zeta^{1,n}, ..., \zeta^{J,n}\right), \end{aligned} \tag{2.9}$$

where $\zeta^{i,n} = \partial\left(f^{i,n},\ \rho\right) \in H_0^1(\Omega)$ is a solution to following continuous QVIs:

$$\begin{cases} b^i\left(\zeta^{i,n}, v - \zeta^{i,n}\right) \geq \left(f^i + \lambda w^i, v - \zeta^{i,n}\right), \\ \zeta^{i,n} \leq \rho + \inf\limits_{\mu\neq i} \zeta^{\mu,n},\ \ v \leq \rho + \inf\limits_{\mu\neq i} \zeta^{\mu,n}, \qquad n = 1, ..., \mathcal{N}-1. \end{cases} \tag{2.10}$$

The problem (2.10) being a coercive QVIs, thanks to [6], [13] has one and only one solution.

Theorem 1. *Under the preceding hypotheses and notations, the mapping $\mathbb{T}$ is a contraction in $\mathbb{H}^{+}$ with a contraction constant $\frac{1}{\beta\Delta t+1}$. Therefore, $\mathbb{T}$ admits a unique fixed point which coincides with the solution of problem* (2.6).

Proof. We adapt [4]. □

The mapping $\mathbb{T}$ generates the following continuous algorithm.

2.3 A continuous algorithm

Starting from $U^0 = U_0 = \left(u_0^1, ..., u^J\right)$ the solution of the following equation:

$$b^i\left(u_0^i, v\right) = \left(f^i + \lambda u_0^i, v\right), \qquad \forall v \in H_0^1(\Omega). \tag{2.11}$$

we define

$$u^{i,n} = \mathbb{T}u^{i,n-1},\ n = 1, ..., \mathcal{N}-1, \tag{2.12}$$

where $u^{i,n}$ is solution to (2.6).

Proposition 1. *Under the conditions of Theorem 1, we have:*

$$\max_{1\leq i\leq J}\left\|u^{i,n}-u^{i,\infty}\right\|_{\infty}\leq\left(\frac{1}{\beta\Delta t+1}\right)^{n}\max_{1\leq i\leq J}\left\|u_{0}^{i}-u^{i,\infty}\right\|_{\infty}, \tag{2.13}$$

where $u^{i,\infty}$ is the asymptotic solution of the continuous system of QVIs: find a vector $U^{\infty}=\left(u^{1,\infty},...,u^{J,\infty}\right)\in\left(H_{0}^{1}\left(\Omega\right)\right)^{J}$ such that

$$\begin{cases}b^{i}\left(u^{i,\infty},\ v-u^{i,\infty}\right)\geq\left(f^{i}+\lambda u^{i,\infty},\ v-u^{i,\infty}\right),\\ u^{i,\infty}\leq\rho+\inf\limits_{\mu\neq i}u^{\mu,\infty},\ v\leq\rho+\inf\limits_{\mu\neq i}u^{\mu,\infty}.\end{cases} \tag{23}$$

Proof. We adapt [4]. □

3. Statement of the discrete problem

Let Ω be decomposed into triangles and let τ_h denote the set of all those elements; $h>0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider $\phi_l, l=1,2,...,m\left(h\right)$, the usual basis of affine functions defined by $\phi_l\left(M_s\right)=\delta_{l,\ s}$ where M_s is a vertex of the considered triangulation.

Let us $\mathbb{V}_h$ denote the standard piecewise linear finite element space such that

$$\mathbb{V}_{h}=\left\{\begin{array}{l}v_{h}\in C^{0}\left(\bar{\Omega}\right),\ v_{h}=0\text{ on }\partial\Omega\text{ such that:}\\ v_{h}\left|_{K^{i}}\right.\in P_{1},\ K\in\tau_{h},v_{h}\leq r_{h}\psi,v_{h}\left(.,0\right)=v_{0h}\text{ in }\Omega.\end{array}\right\} \tag{3.1}$$

Let also r_h be the usual interpolation operator defined by

$$v_{h}\in L^{2}\left(\left[0,T\right];\ H_{0}^{1}\left(\Omega\right)\right)\cap C\left(\left[0,T\right];\ H_{0}^{1}\left(\bar{\Omega}\right)\right),\ r_{h}v_{h}=\sum_{l=1}^{m(h)}v\left(M_{l}\right)\phi_{l}\left(x\right), \tag{3.2}$$

and $\mathbb{B}^i,\ 1\leq i\leq J$ be the matrix with generic entries

$$\left(\mathbb{B}^{i}\right)_{l,\ s}=b^{i}\left(\phi_{l},\ \phi_{s}\right)=a^{i}\left(\phi_{l},\ \phi_{s}\right)+\lambda\int\limits_{\Omega}\phi_{l}\ \phi_{s}\ dx,\ 1\leq l,s\leq m\left(h\right). \tag{3.3}$$

In the sequel of the paper, we shall use the discrete maximum assumption (*d.m.p.*). In other words, we shall assume that the matrix $\mathbb{B}^i,\ 1\leq i\leq J$ is an M-matrix (cf. [14]).

Remark 1. Under the d.m.p., we shall achieve a similar study to that devoted to the continuous problem, therefore the qualitative properties and results stated in the continuous case are conserved in the discrete case.

3.1 The discrete system

As in the continuous situation, one can tackle the discrete system by considering the equivalent formulation: find a vector $U_h^n = (u_h^{1,n}, ..., u_h^{J,n}) \in (\mathbb{V}_h)^J$ such that

$$(3.4)\quad \begin{cases} b^i\left(u_h^{i,n},\ v_h - u_h^{i,n}\right) \geq \left(f^{i,n} + \lambda u_h^{i,n},\ v_h - u_h^{i,n}\right), \text{for all } v_h \in \mathbb{V}_h, \\ u_h^{i,n} \leq r_h\left(\rho + \inf_{\mu \neq i} u_h^{\mu,n}\right),\ v_h \leq r_h\left(\rho + \inf_{\mu \neq i} u_h^{\mu,n}\right),\ n = 1, ..., \mathcal{N}-1. \end{cases}$$

Existence and uniqueness of a solution of system (3.5) can be shown similar to that of the continuous case provided the discrete maximum principle is satisfied.

3.2 Existence and uniqueness

3.2.1 A fixed point mapping associated with discrete problem (3.5)

We consider the following mapping:

$$(3.5)\quad \begin{aligned} \mathbb{T}_h \ &: \ \mathbb{H}^+ \longrightarrow (\mathbb{V}_h)^J, \\ &W \mapsto \mathbb{T}_h W = \zeta_h^n = \left(\zeta_h^{1,n}, ..., \zeta_h^{J,n}\right), \end{aligned}$$

where $\zeta_h^{i,n} = \partial_h\left(f^{i,n},\ \rho\right) \in \mathbb{V}_h$ is a solution to following discrete coercive QVIs:

$$(3.6)\quad \begin{cases} b^i\left(\zeta_h^{i,n}, v_h - \zeta_h^{i,n}\right) \geq \left(f^{i,n} + \lambda w^i, v_h - \zeta_h^{i,n}\right),\ v_h \in \mathbb{V}_h, \\ \zeta_h^{i,n} \leq r_h\left(\rho + \inf_{\mu \neq i} \zeta_h^{\mu,n}\right),\ v \leq r_h\left(\rho + \inf_{\mu \neq i} \zeta_h^{\mu,n}\right). \end{cases}$$

Theorem 2. *Under the d.m.p and the preceding hypotheses and notation, the mapping $\mathbb{T}_h$ is a contraction in $\mathbb{H}^+$ with a contraction constant $\rho = \frac{1}{\beta \Delta t + 1}$. Therefore, $\mathbb{T}_h$ admits a unique fixed point which coincides with the solution of system* (3.5).

As in the continuous situation, one can define the following discrete iterative scheme.

3.3 A discrete algorithm

Starting from $U_h^0 = U_{0h} = \left(u_{0h}^1, ..., u_{0h}^J\right)$ solution of the following equation:

$$(3.7)\quad b^i\left(u_{0h}^i, v\right) = \left(f^i + \lambda u_{0h}^i, v\right),\ \forall v_h \in \mathbb{V}_h.$$

we define the sequences

$$(3.8)\quad u_h^{i,n} = \mathbb{T}_h u_h^{i,n-1},\ n = 1, ..., \mathcal{N}-1,$$

where $u_h^{i,n}$ is solution to (3.3).

Using the above result, we are able to establish the following geometric convergence of sequence U_h^n.

Proposition 2. *Under the d.m.p and Theorem 2, we have*

$$\max_{1\leq i\leq J}\left\|u_h^{i,n}-u_h^{i,\infty}\right\|_\infty \leq \left(\frac{1}{\beta\Delta t+1}\right)^n \max_{1\leq i\leq J}\left\|u_h^{i,0}-u_h^{i,\infty}\right\|_\infty . \tag{3.9}$$

where $u_h^{i,\infty}$ is the asymptotic solution of the discrete system of QVIs: find a vector $U_h^\infty=\left(u_h^{1,\infty},...,u_h^{J,\infty}\right)\in(\mathbb{V}_h)^J$ such that

$$\begin{cases} b^i\left(u_h^{i,\infty},\ v_h-u_h^{i,\infty}\right)\geq\left(f^{i,n}+\lambda u_h^{i,\infty},\ v_h-u_h^{i,\infty}\right), \\ u_h^{i,\infty}\leq r_h\left(\rho+\inf\limits_{\mu\neq i}u_h^{\mu,\infty}\right),\ v_h\leq r_h\left(\rho+\inf\limits_{\mu\neq i}u_h^{\mu,\infty}\right). \end{cases} \tag{3.10}$$

4. L^∞-asymptotic behavior

This section is devoted to estimating the error in the L^∞-norm between $U_h(T,.)$ the discrete solution calculated at the moment $T=n\,\Delta t$ and U^∞ the asymptotic solution of the continuous system of QVIs (2.14). To this end, we first recall some known L^∞-error estimates results, introduce an auxiliary discrete sequence and prove a fundamental Theorem.

Theorem 3 ([15, 16]). *Let u_0^i (respectively, $u_{0,h}^i$), be the solution of problem (2.11), (respectively (3.7)). Then, there exists a constant C independent of h, $\triangle t$ and n such that*

$$\max_{1\leq i\leq J}\left\|u_{0,h}^i-u_0^i\right\|_\infty\leq Ch^2\left|\log h\right|^{\frac{3}{2}} . \tag{4.1}$$

We introduce the following auxiliary discrete sequences

$$\tilde{u}_h^{i,n}=\mathbb{T}_h u^{i,n-1},\ n=1,...,\mathcal{N}-1, \tag{4.2}$$

with $u_{0,h}^i$ is defined in (3.7) and for any $n=1,...,\mathcal{N}-1$, $\tilde{u}_h^{i,n}$ is a solution to following discrete system of variational inequality (V.I.):

$$\begin{cases} b^i\left(\tilde{u}_h^{i,n},\ v-\tilde{u}_h^{i,n}\right)\geq\left(f^{i,n}+\lambda u^{i,n-1},\ v-\tilde{u}_h^{i,n}\right),\ v\in H_0^1(\Omega), \\ \tilde{u}_h^{i,n}\leq r_h\left(\rho+\inf\limits_{\mu\neq i}u^{\mu,n-1}\right),\ v\leq r_h\left(\rho+\inf\limits_{\mu\neq i}u^{\mu,n-1}\right), \end{cases} \tag{4.3}$$

$U^n=\left(u^{1,n},...,u^{J,n}\right)$ is the solution of the continuous problem (2.6).

Remark 2. We notice that $\tilde{u}_h^{i,n}$ represents the standard finite element approximation of $u^{i,n}$.

Therefore, adapting [13], we have the following

Proposition 3. *There exists a constant C independent of h, $\triangle t$ and n such that*

$$\max_{1\leq i\leq J}\left\|\tilde{u}_h^{i,n}-u^{i,n}\right\|_\infty\leq Ch^2\left|\log h\right|^2 . \tag{4.4}$$

Next, by using the above result, we introduce the following:

Lemma 1 ([8]).

$$\max_{1\le i\le J}\left\|u_h^{i,n}-u^{i,n}\right\|_\infty \le \sum_{p=0}^{n}\max_{1\le i\le J}\left\|\tilde{u}_h^{i,p}-u^{i,p}\right\|_\infty . \tag{4.5}$$

Remark 3. Lemma 1 given above plays a crucial role in proving the following Theorem.

Theorem 4. *There exists a constant C independent of h, $\triangle t$ and n such that*

$$\max_{1\le i\le J}\left\|u_h^{i,\infty}-u^{i,\infty}\right\|_\infty \le Ch^2\left|\log h\right|^3 . \tag{4.6}$$

Proof. By combining estimates (2.13), (3.9), and (4.5), we get

$$\begin{aligned}
&\left\|u_h^{i,\infty}-u^{i,\infty}\right\|_\infty = \left\|u_h^{i,\infty}-u^{i,n}+u^{i,n}-u^{i,\infty}\right\|_\infty \\
&\le \left\|u_h^{i,\infty}-u^{i,n}\right\|_\infty + \left\|u^{i,n}-u^{i,\infty}\right\|_\infty \\
&\le \left\|u_h^{i,\infty}-u_h^{i,n}+u_h^{i,n}-u^{i,n}\right\|_\infty + \left\|u^{i,n}-u_h^{i,n}+u_h^{i,n}-u^{i,\infty}\right\|_\infty \\
&\le \left\|u_h^{i,\infty}-u_h^{i,n}\right\|_\infty + \left\|u_h^{i,n}-u^{i,n}\right\|_\infty + \left\|u^{i,n}-u_h^{i,n}\right\|_\infty + \left\|u_h^{i,n}-u^{i,\infty}\right\|_\infty \\
&\le \left\|u_h^{i,\infty}-u_h^{i,n}\right\|_\infty + 2\left\|u_h^{i,n}-u^{i,n}\right\|_\infty + \left\|u_h^{i,n}-u^{i,n}+u^{i,n}-u^{i,\infty}\right\|_\infty \\
&\le \left\|u_h^{i,\infty}-u_h^{i,n}\right\|_\infty + 3\left\|u_h^{i,n}-u^{i,n}\right\|_\infty + \left\|u^{i,n}-u^{i,\infty}\right\|_\infty .
\end{aligned}$$

Applying the previous results of Propositions 1, 2, Theorem 3 and Lemma 1, we get

$$\begin{aligned}
&\left\|u_h^{i,\infty}-u^{i,\infty}\right\|_\infty \le \left\|u_h^{i,\infty}-u_h^{i,n}\right\|_\infty + 3\left\|u_h^{i,n}-u^{i,n}\right\|_\infty + \left\|u^{i,n}-u^{i,\infty}\right\|_\infty \\
&\le \left(\frac{1}{\beta\Delta t+1}\right)^n \left\|u^{i,\infty}-u_0^i\right\|_\infty + \left(\frac{1}{\beta\Delta t+1}\right)^n \left\|u_h^{i,\infty}-u_{h0}^i\right\|_\infty + 3\sum_{p=0}^{n}\left\|\bar{u}_h^{i,p}-u^{i,p}\right\|_\infty \\
&\le \left(\frac{1}{\beta\Delta t+1}\right)^n \left\|u^{i,\infty}-u_0^i\right\|_\infty + \left(\frac{1}{\beta\Delta t+1}\right)^n \left\|u_h^{i,\infty}-u_{h0}^i\right\|_\infty \\
&+ 3\left(\left\|\bar{u}_h^{i,0}-u^{i,0}\right\|_\infty + \sum_{p=1}^{n}\left\|\bar{u}_h^{i,p}-u^{i,p}\right\|_\infty\right) \\
&\le \left(\frac{1}{\beta\Delta t+1}\right)^n \left\|u^{i,\infty}-u_0^i\right\|_\infty + \left(\frac{1}{\beta\Delta t+1}\right)^n \left\|u_h^{i,\infty}-u_{h0}^i\right\|_\infty \\
&+ Ch^2\left|\log h\right|^{\frac{3}{2}} + nCh^2\left|\log h\right|^2 .
\end{aligned}$$

Finally, taking $h^2 = (\frac{1}{\beta\Delta t+1})^n$, we obtain

$$\max_{1\leq i\leq J}\left\|u_h^{i,\infty} - u^{i,\infty}\right\|_\infty \leq Ch^2\left|\log h\right|^3,$$

which completes the proof. □

Remark 4. It should be noted that the same result was obtained in [8].

Now guided by Propositions 2, Theorem 4, we are in a position to prove the main result.

Theorem 5. *There exists a constant C independent of h, $\triangle t$ and n such that*

$$\left\|U_h\left(T,\ .\right) - U^\infty\left(.\right)\right\|_\infty \leq C\left(h^2\left|\log h\right|^3 + \left(\frac{1}{\beta\Delta t+1}\right)^N\right). \tag{4.7}$$

Proof. We have

$$u_h^{i,n}\left(x\right) = u_h^i\left(t,\ x\right) \text{ for all } t \in \left](n-1).\Delta t,\ n.\Delta t\right[,$$

thus

$$u_h^{i,N}\left(x\right) = u_h^i\left(T,\ x\right)$$

So,

$$\begin{aligned}\left\|u_h^i\left(T,\ x\right) - u^{i,\infty}\left(x\right)\right\|_\infty &= \left\|u_h^{i,N}\left(x\right) - u^{i,\infty}\left(x\right)\right\|_\infty \\ &\leq \left\|u_h^{i,N} - u_h^{i,\infty}\right\|_\infty + \left\|u_h^{i,\infty} - u^{i,\infty}\right\|_\infty.\end{aligned}$$

Applying the previous results of Propositions 2 and Theorem 4, we get

$$\begin{aligned}\left\|u_h^i\left(T,\ .\right) - u^{i,\infty}\right\|_\infty &\leq \left(\frac{1}{\beta\Delta t+1}\right)^N\left\|u_h^{i,\infty} - u_{h0}^i\right\|_\infty + Ch^2\left|\log h\right|^3 \\ &\leq C\left(h^2\left|\log h\right|^3 + \left(\frac{1}{\beta\Delta t+1}\right)^N\right),\end{aligned}$$

which completes the proof. □

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References

[1] A. Bensoussan and J. L. Lions, *Impulse control and quasi-variational inequalities*, Gauthier Villars, Paris, 1982.

[2] G. L. Blankenship and J. L. Menaldi, *Optimal stochastic scheduling of power generation system with scheduling delays and large cost differentials*, SIAM J. Control Optim., 22 (1984), 121–132.

[3] M. A. Bencheikh Le Hocine, S. Boulaaras and M. Haiour, *An optimal L^∞-error estimate for an approximation of a parabolic variational inequality*, Numer. Funct. Anal. Optim., 37 (2016), 1-18.

[4] M. A. Bencheikh Le Hocine, M. Haiour, *L^∞-error analysis for parabolic quasi-variational inequalities related to impulse control problems*, Comput. Math. Model., 28 (2017), 89-108.

[5] S. Boulaaras, M. A. Bencheikh Le Hocine and M. Haiour, *The finite element approximation in a system of parabolic quasi-variational inequalities related to management of energy production with mixed boundary condition*, Comput. Math. Model., 25 (2014), 530–543.

[6] M. Boulbrachene, *On the finite element approximation of variational inequalities with noncoercive operators*, Numer. Funct. Anal. Optim., 36 (2015), 1107-1121.

[7] M. Boulbrachene, *Pointwise error estimate for a noncoercive system of quasi-variational inequalities related to the management of energy production*, J. Ineq.Pure. Appl. Math., 3 (2002), 1-9.

[8] M. Boulbrachene, M. Haiour, *The finite element approximation of Hamilton Jacobi Bellman equations*, Comp. Math. with. Appl., 41 (2001), 993–1007.

[9] S. Boulaaras, M. Haiour, *A new proof for the existence and uniqueness of the discrete evolutionary HJB equations*, Appl. Math. Comput, 262 (2015), 42-55.

[10] S. Boulaaras, M. Haiour, *The finite element approximation of evolutionary Hamilton–Jacobi–Bellman equations with nonlinear source terms*, Indagationes Mathematicae, 24 (2013), 161-173.

[11] S. Boulaaras, M. Haiour, *The finite element approximation in parabolic quasi-variational inequalities related to impulse control problem with mixed boundary conditions*, Journal of Taibah University for Science, 7 (2013), 105-113.

[12] S. Boulaaras, M. Haiour, *L^∞-asymptotic behavior for a finite element approximation in parabolic quasi-variational inequalities related to impulse control problem*, Appl.Math. Comput., 217 (2011), 6443-6450.

[13] P. Cortey-Dumont, *Sur l'analyse numerique des equations de Hamilton-Jacobi-Bellman*, Math. Meth. in Appl. Sci., 9 (1987), 198-209.

[14] P. G. Ciarlet and P. A. Raviart, *Maximum principle and uniform convergence for the finite element method*, Comp. Meth. in Appl. Mech. and Eng., 2 (1973), l7-31.

[15] P. G. Ciarlet and J.L. Lions, *Editors, Handbook of Numerical Analysis Vol. II, Finite Element Methods, (Part l)*, North-Holland, 1991.

[16] J. Nitsche, L^∞*-convergence of finite element approximations*, In Mathematical Aspects of Finite Element Methods, Lect. Notes Math., 606 (1977), 261-274.

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