Relationships between tropical eigenvectors and tropical fixed points of the group $GL(2, \mathbb{R})$

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Abstract. The eigenvalues, eigenvectors and fixed points of matrices have many applications in various branches of science and many mathematical disciplines. In this paper first we introduce the concept of tropical fixed points, then we calculate the tropical eigenvalues and tropical eigenvectors of $GL(2, \mathbb{R})$. Furthermore we give relationships between tropical eigenvectors and tropical fixed points of $GL(2, \mathbb{R})$.

Keywords: tropical eigenvalues, tropical eigenvectors, fixed points.

1. Introduction

The tropical semiring is the set $\mathbb{R} \cup \{\infty\}$ denoted by $\mathbb{T}$, with the two new operations $\oplus$ and $\odot$. The operation $\odot$ is defined as the classical $+$ and $\oplus$ is defined to be the minimum of two elements of $\mathbb{T}$. That is for all $a, b \in \mathbb{T}$,

$$a \oplus b = \min\{a, b\}, \qquad a \odot b = a + b.$$

Following examples explain the tropical operations.

Example 1.1.1. Let $3, 7 \in \mathbb{T}$, $3 \oplus 7 = \min\{3, 7\} = 3$, and $3 \odot 7 = 3 + 7 = 10$.

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Example 1.1.2. The elements $\infty, 0 \in \mathbb{T}$, are identities under the tropical operations $\oplus$ and $\odot$ respectively, as for any $a \in \mathbb{T}$,

$$a \oplus \infty = \min\{a, \infty\} = a = \min\{\infty, a\} = \infty \oplus a,$$
$$a \odot 0 = a + 0 = a = 0 + a = 0 \odot a.$$

It is easy to verify that many of classical axioms remains true in tropical linear algebra, see page 10 of [2].

Definition 1.1. [2] Let $X = [x_{ij}] \in \mathbb{R}^{n\times r}$, $Y = [y_{ij}] \in \mathbb{R}^{r\times m}$ be tropical matrices, the tropical product of X and Y is defined as $X \odot Y = [z_{ij}]$ where $z_{ij} = \bigoplus(x_{ik} \odot y_{kj})$, where $k = 1, 2, ..., r$.

Example 1.1. Let $X = \begin{bmatrix} 3 & 5 \\ 8 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 & 5 \\ 4 & 1 \end{bmatrix}$ be the tropical matrices in $\mathbb{R}^{2\times 2}$ then

$$\begin{aligned} X \odot Y &= \begin{bmatrix} 3 & 5 \\ 8 & 6 \end{bmatrix} \odot \begin{bmatrix} -1 & 5 \\ 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \odot -1 \oplus 5 \odot 4 & 3 \odot 5 \oplus 5 \odot 1 \\ 8 \odot -1 \oplus 6 \odot 4 & 8 \odot 5 \oplus 6 \odot 1 \end{bmatrix} = \begin{bmatrix} 3 + (-1) \oplus 5 + 4 & 3 + 5 \oplus 5 + 1 \\ 8 + (-1) \oplus 6 + 4 & 8 + 5 \oplus 6 + 1 \end{bmatrix} \\ &= \begin{bmatrix} \min(2, 9) & \min(8, 6) \\ \min(7, 10) & \min(13, 7) \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 7 & 7 \end{bmatrix} \end{aligned}$$

Definition 1.2. [2] Let $X = [x_{ij}] \in \mathbb{R}^{n\times r}$, be a tropical matrix, and $r \in \mathbb{R}$, the tropical scalar product is component wise just like the classical scalar product, that is $r \odot X = r \odot [x_{ij}] = [r \odot x_{ij}]$.

Example 1.2. Let $X = \begin{bmatrix} 7 & 5 & 1 \\ 8 & 6 & 5 \end{bmatrix}$ be a tropical matrix in $\mathbb{R}^{2\times 3}$ then

$$3 \odot X = 3 \odot \begin{bmatrix} 7 & 5 & 1 \\ 8 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 3 \odot 7 & 3 \odot 5 & 3 \odot 1 \\ 3 \odot 8 & 3 \odot 6 & 3 \odot 5 \end{bmatrix} = \begin{bmatrix} 10 & 8 & 4 \\ 11 & 9 & 8 \end{bmatrix}$$

Definition 1.3. [2] Let A be an $n\times n$-matrix with entries in the tropical semiring $(\mathbb{R}\cup\{\infty\},\oplus,\odot)$. An eigenvalue of A is a real number λ such that $A \odot v = \lambda \odot v$, for some $v \in \mathbb{R}^n$. We say that v is an eigenvector of the tropical matrix A .

Example 1.3. Consider $\begin{pmatrix} 2 & 4 \\ 13 & 7 \end{pmatrix} \in \mathbb{R}^{2\times 2}$,

$$\begin{pmatrix} 2 & 4 \\ 13 & 7 \end{pmatrix} \odot \begin{pmatrix} 2 \\ 13 \end{pmatrix} = \begin{pmatrix} 4 \\ 15 \end{pmatrix} = 2 \odot \begin{pmatrix} 2 \\ 13 \end{pmatrix}$$

The scalar 2 is tropical eigenvalue of $\begin{pmatrix} 2 & 4 \\ 13 & 7 \end{pmatrix}$, while $\begin{pmatrix} 2 \\ 13 \end{pmatrix}$ is corresponding tropical eigenvector.

Definition 1.4 ([3])**.** A graph G consists of two finite sets, $V(G)$ the set of vertices and $E(G)$ the set of edges. The edges connect the different vertices in a graph. A graph is said to be strongly connected if every vertex is reachable from every other vertex.

Theorem 1.1 ([2])**.** Let A be a tropical $n \times n$-matrix whose graph $G(A)$ is strongly connected. Then A has precisely one eigenvalue λ. That eigenvalue equals the minimal normalized length of any directed cycle in $G(A)$.

The power algorithm ([3])**.** Let A be a tropical matrix of order $n \times n$, with tropical eigenvalue λ and v be the corresponding tropical eigenvector then we calculate λ and v as:

(1) Choose $x(0) \in \mathbb{T}^n$ such that $x(0)$ contains at least one finite entry.

(2) Compute $x(k+1) = A \odot x(k)$, until a positive integer k is reached such that $x(k+p) = q \odot x(k)$ for some $p \in \mathbb{N}$ and $q \in \mathbb{R}$.

(3) Calculate $\lambda = \frac{q}{p}$.

(4) Calculate $v = \min(\lambda^{\odot(p-j)} \odot x(k+j-1))$ for $j = 1, 2, ..., p$.

Remarks. If $p = 1$ then $v = x(k)$, if $p = 2$ then $v = x(k+1) \oplus \lambda \odot x(k)$. Many authors have studied the eigenvalues, eigenvectors and fixed points of matrix groups, see for example [1,4].

2. Main results

Definition 2.1. Let $A \in GL(2, \mathbb{R})$, a vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{T}^2$ is called tropical fixed point of A if $A \odot X = X$.

Theorem 2.1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$, if $2a \le b + c$ and $a \le d$ then a is the tropical eigenvalue and $\begin{pmatrix} a \\ c \end{pmatrix}$ is the corresponding tropical eigenvector of A.

Proof. Let $x(0) = \begin{pmatrix} 0 \\ \infty \end{pmatrix}$, we calculate $x(k+1) = A \odot x(k)$, until we get $x(k+p) = q \odot x(k)$, where q is a real number and p is a natural number.

Now,

$$x(1) = A \odot x(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 0 \\ \infty \end{pmatrix} = \begin{pmatrix} a \odot 0 \oplus b \odot \infty \\ c \odot 0 \oplus d \odot \infty \end{pmatrix} = \begin{pmatrix} \min(a, \infty) \\ \min(c, \infty) \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$

$$x(2) = A \odot x(1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \min(2a, b+c) \\ \min(c+a, c+d) \end{pmatrix}$$

since $2a \leq b + c$ and $a \leq d$ therefore

$$x(2) = \begin{pmatrix} 2a \\ c + a \end{pmatrix} = a \odot \begin{pmatrix} a \\ c \end{pmatrix},$$

this implies $x(2) = x(1+1) = a \odot x(1)$, here $k = 1, q = a, p = 1$, so tropical eigenvalue $= \frac{q}{p} = \frac{a}{1} = a$ and tropical eigenvector$=\begin{pmatrix} a \\ c \end{pmatrix}$.

Example 2.1. Let $A = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = 3$, and $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is the tropical eigenvector.

Verification. $A \odot v = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} \odot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} = 3 \odot \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. This implies $A \odot v = \lambda \odot v$.

Corollary 2.1. If in Theorem 2.1 $a = 0$, then tropical eigenvalue of A is zero, and $\begin{pmatrix} 0 \\ c \end{pmatrix}$ is the tropical eigenvector moreover any tropical scalar multiple of this vector is the tropical fixed point of A.

Proof. Here $a = 0$, $0 \leq b + c$ and $0 \leq d$, so matrix A becomes $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$, then by Theorem 2.1 tropical eigenvalue is zero and tropical eigenvector is $\begin{pmatrix} 0 \\ c \end{pmatrix}$. Now we show that $X = r \odot v = \begin{pmatrix} r \\ r + c \end{pmatrix}$ is the fixed point of A,

$$A \odot X = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} r \\ r + c \end{pmatrix} = \begin{pmatrix} \min(0 + r, b + r + c) \\ \min(r + c, r + c + d) \end{pmatrix} = \begin{pmatrix} r \\ r + c \end{pmatrix} = X.$$

Hence the required result.

Theorem 2.2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$, if $\frac{b+c}{2} \leq a$ and $\frac{b+c}{2} \leq d$ then $\frac{b+c}{2}$ is the tropical eigenvalue and $\begin{pmatrix} b + c \\ \frac{b+c}{2} + c \end{pmatrix}$ is the corresponding tropical eigenvector of A.

Proof. Let $x(0) = \begin{pmatrix} 0 \\ \infty \end{pmatrix}$, we calculate $x(k+1) = A \odot x(k)$, until we get $x(k+p) = q \odot x(k)$, where q is a real number and p is a natural number. Now

$$x(1) = A \odot x(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 0 \\ \infty \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$x(2) = A \odot x(1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \min(2a, b+c) \\ \min(c+a, c+d) \end{pmatrix},$$

here $b + c \leq 2a$, for $\min(c + a, c + d)$ two cases arise:

Case I. If $c + a \leq c + d$ this implies $a \leq d$ then we have $x(2) = \begin{pmatrix} b+c \\ a+c \end{pmatrix}$,

$$x(3) = A \odot x(2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c \\ a+c \end{pmatrix} = \begin{pmatrix} \min(a+b+c, a+b+c) \\ \min(b+2c, a+c+d) \end{pmatrix},$$

if $a + c + d < b + 2c$ this implies $a + d < b + c$ this means $2a < b + c$ (since $a \leq d$), which is not true therefore $b + 2c \leq a + c + d$, and we have

$$x(3) = \begin{pmatrix} a+b+c \\ b+2c \end{pmatrix}$$

$$x(3) = (b+c) \odot \begin{pmatrix} a \\ c \end{pmatrix},$$

this implies $x(3) = x(1 + 2) = (b + c) \odot x(1)$, here $k = 1, q = b + c, p = 2$, so

$$\text{tropical eigenvalue} = \lambda = \frac{q}{p} = \frac{b+c}{2}$$

and

$$\text{tropical eigenvector} = x(2) \oplus \lambda \odot x(1) = \begin{pmatrix} b+c \\ a+c \end{pmatrix} \oplus \begin{pmatrix} a + \frac{b+c}{2} \\ c + \frac{b+c}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \min(b+c, a+\frac{b+c}{2}) \\ \min(a+c, c+\frac{b+c}{2}) \end{pmatrix},$$

if $a + \frac{b+c}{2} < b + c$ this means $a < \frac{b+c}{2}$, which is not true. Therefore $b + c \leq a + \frac{b+c}{2}$. If $a + c < c + \frac{b+c}{2}$ then again we get $a < \frac{b+c}{2}$ so $c + \frac{b+c}{2} \leq a + c$. Hence $v = \begin{pmatrix} b+c \\ c+\frac{b+c}{2} \end{pmatrix}$.

Case II. If $c + d \leq c + a$ this implies $d \leq a$ then we have $x(2) = \begin{pmatrix} b+c \\ c+d \end{pmatrix}$,

$$x(3) = A \odot x(2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c \\ c+d \end{pmatrix} = \begin{pmatrix} \min(a+b+c, b+c+d) \\ \min(b+c+c, c+d+d) \end{pmatrix},$$

if $a + b + c < b + c + d$ this implies $a < d$, which is not true therefore $b + c + d \leq a + b + c$, if $c + d + d < b + c + c$ then we have $2d < b + c$, which is not true so $b + c + c \leq c + d + d$ and we get $x(3) = \begin{pmatrix} b+c+d \\ b+2c \end{pmatrix}$,

$$x(4) = A \odot x(3) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c+d \\ c+2c \end{pmatrix} = \begin{pmatrix} \min(a+b+c+d, 2b+2c) \\ \min(b+2c+d, b+2c+d) \end{pmatrix},$$

if $a+b+c+d < 2b+2c$ this implies $a+d < b+c$, this means $2d < b+c$ (since $d \le a$) which is not true so $2b+2c \le a+b+c+d$, we have $x(4) = \begin{pmatrix} 2b+2c \\ b+2c+d \end{pmatrix}$,

$$x(5) = A \odot x(4) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 2b+2c \\ b+2c+d \end{pmatrix} = \begin{pmatrix} \min(a+2b+2c, 2b+2c+d) \\ \min(2b+3c, b+2c+2d) \end{pmatrix}$$

if $a+2b+2c < 2b+2c+d$ this means $a < d$, which is not true (since $d \le a$) therefore $2b+2c+d \le a+2b+2c$, if $b+2c+d < 2b+3c$ this implies $d < b+c$ which is not true, so $2b+3c \le b+2c+d$, hence we get

$$x(5) = \begin{pmatrix} 2b+2c+d \\ 2b+3c \end{pmatrix} = (b+c) \odot \begin{pmatrix} b+c+d \\ b+2c \end{pmatrix}$$

this implies $x(5) = x(3+2) = (b+c) \odot x(3)$, here $k = 3, q = b+c, p = 2$, so

$$\text{tropical eigenvalue} = \lambda = \frac{q}{p} = \frac{b+c}{2}$$

and

$$\begin{aligned} \text{tropical eigenvector} = x(2) \oplus \lambda \odot x(1) &= \begin{pmatrix} b+c \\ c+d \end{pmatrix} \oplus \begin{pmatrix} a + \frac{b+c}{2} \\ c + \frac{b+c}{2} \end{pmatrix} \\ &= \begin{pmatrix} \min(b+c, a+\frac{b+c}{2}) \\ \min(a+c, c+\frac{b+c}{2}) \end{pmatrix}, \end{aligned}$$

if $a+\frac{b+c}{2} < b+c$ this means $a < \frac{b+c}{2}$, which is not true, therefore $b+c \le a+\frac{b+c}{2}$. If $a+c < c+\frac{b+c}{2}$ then again we get $a < \frac{b+c}{2}$ so $c+\frac{b+c}{2} \le a+c$. Hence $v = \begin{pmatrix} b+c \\ c+\frac{b+c}{2} \end{pmatrix}$.

Example 2.2.1 (case I). Let $A = \begin{pmatrix} \frac{5}{3} & \frac{-1}{2} \\ \frac{2}{3} & 2 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = \frac{b+c}{2} = \frac{1}{12}$, and $v = \left(\begin{pmatrix} b+c \\ c+\frac{b+c}{2} \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{6} \\ \frac{3}{4} \end{pmatrix}$ is the tropical eigenvector.

Verification.

$$A \odot v = \begin{pmatrix} \frac{5}{3} & \frac{-1}{2} \\ \frac{2}{3} & 2 \end{pmatrix} \odot \begin{pmatrix} \frac{1}{6} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{5}{6} \end{pmatrix},$$

and

$$\lambda \odot v = \frac{1}{12} \odot \begin{pmatrix} \frac{1}{6} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{5}{6} \end{pmatrix}.$$

This implies $A \odot v = \lambda \odot v$.

Example 2.2.2 (case II). Let $A = \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = \frac{b+c}{2} = 1$, and $v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is the tropical eigenvector.

Corollary 2.2. If in above theorem $b+c=0$, then tropical eigenvalue of A is zero, and $\begin{pmatrix}0\\c\end{pmatrix}$ is the tropical eigenvector, moreover any tropical scalar multiple of this vector is the tropical fixed point of A.

Proof. Here $b+c=0$ implies $c=-b$, also $b+c \leq a$ and $b+c \leq d$, so matrix A becomes $\begin{pmatrix}a & b\\-b & d\end{pmatrix}$, then by Theorem 2.2 tropical eigenvalue $b+c$ is 0 and tropical eigenvector $\begin{pmatrix}0\\-b\end{pmatrix}$. Now, we show that $X = r \odot v = \begin{pmatrix}r\\r-b\end{pmatrix}$ is the fixed point of A,

$$A \odot X = \begin{pmatrix}a & b\\-b & d\end{pmatrix} \odot \begin{pmatrix}r\\r-b\end{pmatrix} = \begin{pmatrix}\min(a+r, b+r-b)\\\min(r-b, r-b+d)\end{pmatrix} = \begin{pmatrix}r\\r-b\end{pmatrix} = X.$$

Hence the required result.

Theorem 2.3. Let $A = \begin{pmatrix}a & b\\c & d\end{pmatrix} \in GL(2,\mathbb{R})$, if $2d \leq b+c$ and $d \leq a$ then d is the tropical eigenvalue and $\begin{pmatrix}b+c+(n-2)d\\c+(n-1)d\end{pmatrix}$ where n is a natural number, is the corresponding tropical eigenvector of A (here $n=k$, where k is from Power Algorithm).

Proof. Let $x(0) = \begin{pmatrix}0\\\infty\end{pmatrix}$, we calculate $x(k+1) = A \odot x(k)$, until we get $x(k+p) = q \odot x(k)$, where q is a real number and p is a natural number. Now

$$x(1) = A \odot x(0) = \begin{pmatrix}a & b\\c & d\end{pmatrix} \odot \begin{pmatrix}0\\\infty\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix},$$

$$x(2) = A \odot x(1) = \begin{pmatrix}a & b\\c & d\end{pmatrix} \odot \begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}\min(2a, b+c)\\\min(c+a, c+d)\end{pmatrix},$$

here $c+a \leq c+d$, for $\min(2a, b+c)$ two cases arise.
Case I. If $b+c \leq 2a$ this implies

$$x(2) = \begin{pmatrix}b+c\\c+d\end{pmatrix},$$

$$x(3) = A \odot x(2) = \begin{pmatrix}a & b\\c & d\end{pmatrix} \odot \begin{pmatrix}b+c\\c+d\end{pmatrix} = \begin{pmatrix}\min(a+b+c, b+c+d)\\\min(b+2c, c+2d)\end{pmatrix},$$

if $a+b+c < b+c+d$ this implies $a < d$, which is not true (since $d \leq a$), so $b+c+d \leq a+b+c$, if $b+2c < c+2d$ this means $b+c < 2d$, which is not true (since $2d \leq b+c$), therefore $c+2d \leq b+2c$, and we get

$$x(3) = \begin{pmatrix}b+c+d\\c+2d\end{pmatrix},$$

$$x(3) = d \odot \begin{pmatrix} b+c \\ c+d \end{pmatrix},$$

this implies $x(3) = x(2+1) = d \odot x(2)$, here $k = 2, q = d, p = 1$, so

$$\text{tropical eigenvalue} = \lambda = \frac{q}{p} = \frac{d}{1} = d$$

and

$$\text{tropical eigenvector} = \begin{pmatrix} b+c \\ c+d \end{pmatrix}.$$

Case II. If $2a \leq b+c$ then

$$x(2) = \begin{pmatrix} 2a \\ c+d \end{pmatrix},$$
$$x(3) = A \odot x(2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 2a \\ c+d \end{pmatrix} = \begin{pmatrix} \min(a+2a, b+c+d) \\ \min(c+2a, c+2d) \end{pmatrix}$$

here if we take $3a < b+c+d$ this implies

$$x(3) = \begin{pmatrix} 3a \\ c+2d \end{pmatrix},$$
$$x(4) = A \odot x(3) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} 3a \\ c+2d \end{pmatrix} = \begin{pmatrix} \min(a+3a, b+c+2d) \\ \min(c+3a, c+3d) \end{pmatrix},$$

since $d \leq a$ so proceeding as above we get some natural number n such that $b+c+(n-2)d \leq na$. Therefore, we have

$$x(n) = \begin{pmatrix} b+c+(n-2)d) \\ c+(n-1)d) \end{pmatrix}$$
$$x(n+1) = A \odot x(n) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot \begin{pmatrix} b+c+(n-2)d) \\ c+(n-1)d) \end{pmatrix}$$
$$= \begin{pmatrix} \min(a+b+c+(n-2)d, b+c+(n-1)d) \\ \min(b+c+c+(n-2)d, c+(n-1)d)+d \end{pmatrix},$$

if $a+b+c+(n-2)d < b+c+(n-1)d$ this implies $a+b+c+(n-2)d < b+c+(n-2)d+d$, this means $a < d$ which is not true (since $d \leq a$), therefore $b+c+(n-1)d \leq a+b+c+(n-2)d$, if $b+c+c+(n-2)d < c+(n-1)d+d$ this implies $b+c+(n-2)d < (n-2)d+2d$, this means $b+c < 2d$ which is not true, so $c+(n-1)d+d \leq b+c+c+(n-2)d$, hence we have

$$x(n+1) = \begin{pmatrix} b+c+(n-1)d) \\ c+(n-1)d+d \end{pmatrix} = d \odot \begin{pmatrix} b+c+(n-2)d) \\ c+(n-1)d \end{pmatrix},$$

this implies $x(n+1) = d \odot x(n)$, here $k = n, q = d, p = 1$, so

$$\text{tropical eigenvalue} = \lambda = \frac{q}{p} = \frac{d}{1} = d$$

and

$$\text{tropical eigenvector} = \begin{pmatrix} b+c+(n-2)d \\ c+(n-1)d \end{pmatrix}.$$

Example 2.3.1 (case I). Let $A = \begin{pmatrix} \frac{11}{2} & \frac{6}{5} \\ \frac{9}{2} & 2 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = d = 2$, and $v = \begin{pmatrix} b+c \\ c+d \end{pmatrix} = \begin{pmatrix} \frac{57}{10} \\ \frac{13}{2} \end{pmatrix}$ is the tropical eigenvector.

Example 2.3.2 (case II). Let $A = \begin{pmatrix} 5 & 9 \\ 7 & 3 \end{pmatrix}$, then tropical eigenvalue of A is $\lambda = d = 3$, here $b+c+(n-2)d \leq na$ implies $9+7+(n-2)3 \leq 5n$, solving this we get $5 \leq n$, let us take $n = 5$, therefore $v = \begin{pmatrix} b+c+3d \\ c+4d \end{pmatrix} = \begin{pmatrix} 25 \\ 19 \end{pmatrix}$ is the tropical eigenvector.

Corollary 2.3. If in Theorem 2.3 $d = 0$, then tropical eigenvalue of A is zero, and $\begin{pmatrix} b+c \\ c \end{pmatrix}$ is the tropical eigenvector, moreover any tropical scalar multiple of this vector is the tropical fixed point of A.

Proof. Here $d = 0$, $0 \leq b+c$ and $0 \leq a$, so matrix A becomes $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, then by Theorem 2.3 tropical eigenvalue is zero and tropical eigenvector is $\begin{pmatrix} b+c \\ c \end{pmatrix}$. Now we show that $X = r \odot v = \begin{pmatrix} r+b+c \\ r+c \end{pmatrix}$ is the fixed point of A,

$$\begin{aligned} A \odot X &= \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \odot \begin{pmatrix} r+b+c \\ r+c \end{pmatrix} \\ &= \begin{pmatrix} \min(a+r+b+c, b+r+c) \\ \min(r+b+2c, r+c) \end{pmatrix} = \begin{pmatrix} b+r+c \\ r+c \end{pmatrix} = X. \end{aligned}$$

Hence the required result.

3. Conclusion

In this paper we first calculate the tropical eigenvalues and tropical eigenvectors of the group $GL(2, \mathbb{R})$. Then we show that if the tropical eigenvalue is zero then the tropical fixed points of elements of $GL(2, \mathbb{R})$ are the tropical scalar multiple of the tropical eigenvectors.

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