Estimation for the parameter of a class of diffusion processes

Chao Wei
School of Mathematics and Statistics
Anyang Normal University
Anyang 455000
China
chaowei0806@aliyun.com

Abstract. This paper is concerned with the parameter estimation problem for a stationary ergodic diffusion process with drift coefficient $a(X_t, \theta)$ and diffusion coefficient $b(X_t)$ under the case of continuous-time observations. Firstly, we find a closed interval on which the likelihood function is continuous and does not attain the maximum at two endpoints of this interval. Secondly, we prove that the maximum likelihood estimator exists in the interval when the sample size is large enough. Finally, the strong consistency of the estimator and the asymptotic normality of the error of estimation are proved. All of the results are obtained by applying the maximal inequality for martingales, Borel-Cantelli lemma and uniform ergodic theorem.

Keywords: maximum likelihood estimation, ergodic diffusion processes, strong consistency, asymptotic normality.

1. Introduction

Diffusion processes have been widely used for model building in demographic theory, safety science, computer science and management science. Moreover, diffusion processes are the essential stochastic modeling tools in the modern financial theories and applications. Recently, diffusion models are applied to describe the dynamics of a financial asset, such as Black-Scholes-Merton option pricing formula (see e.g. [6, 21]), and Vasicek and Cox-Ingersoll-Ross pricing formulas for the zero coupon bond (see e.g. [25, 7, 8]). Statistical inference for diffusion processes is very important to the stochastic theories as well as to the applications in model building. In the stochastic models describing the actual systems, part or all of the parameters are always unknown, but the observed values are known. Therefore, parameter estimation becomes an important problem needed to be solved depending on observed values. Parameters in the stochastic models have to be estimated either from continuous-time observation if continuous observation is possible, or from discrete sampled data set for the process if the process is not observed continuously. As far as we know, it is of great importance for the special stochastic model to estimate the parameters for the purpose of obtaining a proper structure of the model no matter which method is used.

In earlier works, some methods such as maximum likelihood estimation, Bayes estimation and least-squares estimation have been employed to solve the parameter estimation problem for the diffusion processes and the asymptotic properties of the estimators have been discussed as well. In previous literatures, numerical approximation schemes have been employed together with the estimation methods to estimate the parameters based on discrete observations. For example, by applying Euler method to discrete the original continuous-time processes and employing least squares estimation (see e.g. [12, 11, 19]), generalized method of moments (see e.g. [10]), and maximum likelihood estimation (see e.g. [1, 14]), or by using Itô sum and Riemann sum to approximate the integrals in the continuous-time likelihood function and applying maximum likelihood estimation (see e.g. [28]), martingale estimation (see e.g. [4]), and estimation based on eigenfunctions (see e.g. [5]). When the process is observed partially, both the parameter estimation method and the state estimation method such as extended Kalman filter are employed to solve the estimation problem (see e.g. [13, 23, 18]). However, numerical approximation schemes have some disadvantages. For example, the discretized processes may not converge to the original continuous-time processes and the estimators obtained may not consistent when the time between observations is bounded away from zero. As a consequence, it is of great importance to estimate parameters for the original continuous-time diffusion processes based on continuous observations. In previous literatures, some methods have been employed to solve the estimation problem for continuous-time diffusion processes described by stochastic differential equations. For example, estimation of the drift parameter for a linear stochastic differential equation (see e.g. [22]), maximum likelihood estimation in the scalar parameter case and vector parameter case for a nonlinear stochastic differential equation (see e.g. [24, 2]), and other methods such as Moment estimation (see e.g. [17]), M-estimation method (see e.g. [27]), and minimum distance method (see e.g. [15, 9]). Moreover, Kutoyants([16]) and Wei ([26]) used likelihood ratio process and maximum likelihood estimation respectively to investigate the parameter estimation in probability for ergodic diffusion processes.

Although the parameter estimation has been studied by some authors, the almost sure convergence of the parameter estimator has not been discussed. In this paper, the parameter estimation problem for a class of stationary ergodic diffusion processes is investigated by applying maximum likelihood estimation under the case of continuous-time observations. The idea of solving the estimation problem for ergodic diffusion processes in this article is different from that in Kutoyants([16]) and Wei ([26]). In ([16]) and ([26]), only the weak convergence of the estimator has been considered, but in this paper, the strong convergence, namely the almost sure convergence of the estimator is considered. For the purpose of proving the existence of the maximum likelihood estimator, we find a compact set on which the likelihood function is continuous and does not attain the maximum at two endpoints of this compact set when the sample size is large enough. Hence, the likelihood function has a local maximum in this compact set and the existence of the maximum likelihood estimator is proved. The strong consistency of the parameter and the asymptotic normality of the error of estimation are proved by applying maximal inequality for martingale, Borel-Cantelli lemma, the dominated convergence theorem and the uniform ergodic theorem.

This paper is organized as follows. In Section 2, some assumptions are provided and the likelihood function is given based on the Girsanov theorem. The main results are given in Section 3 where the existence and strong consistency of the maximum likelihood estimator are proved and the limit distribution of the error of estimation is obtained. An example is given to verify the effectiveness of the estimator in Section 4. The conclusion is given in Section 5.

2. Problem formulation and preliminaries

In this paper, the one-dimensional stationary ergodic diffusion processes described by the following class of stochastic differential equation will be studied:

$$\begin{cases} dX_t = a(X_t,\theta)dt + b(X_t)dW_t \\ X_0 = x_0, \end{cases} \tag{1}$$

where $\theta \in \Theta$ a open subset of $\mathbb{R}$ is the unknown one-dimensional parameter.

Suppose (1) satisfies the conditions that ensure the existence and uniqueness of the solution, (see e.g. [20]). The process is observed over $[0,T]$. x_0 is distributed according to the stationary distribution of the process. The drift and diffusion coefficients are supposed to be known and do not depend on the time t. $(W_t, t \geq 0)$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Now the Girsanov theorem is introduced below.

Lemma 1 ([3])**.** *Let $Y(t)$ be an Itô process of the form*

$$dY(t) = a(t,\omega)dt + dB(t); t \leq T,$$

where $T \leq \infty$ is a given constant and $B(t)$ is Brownian motion. Put

$$M_t = \exp(-\int_0^t a(s,\omega)dB_s - \frac{1}{2}\int_0^t a^2(s,\omega)ds); t \leq T.$$

Assume that $a(s,\omega)$ satisfies Novikov's condition

$$\mathbb{E}[\exp(\frac{1}{2}\int_0^T a^2(s,\omega)ds)] < \infty,$$

where $\mathbb{E}$ is the expectation with respect to P. Define the measure Q on $(\Omega, \mathcal{F}_T)$ by

$$dQ(\omega) = M_T dP(\omega).$$

Then $Y(t)$ is a Brownian motion with respect to the probability law Q, for $t \leq T$.

From now on we shall work under the assumptions below.

Assumption 1. *$\mathbb{P}_{\theta_1} \neq \mathbb{P}_\theta$ for $\theta_1 \neq \theta$ in Θ where θ denotes the true parameter.*

Assumption 2. *$|a(x,s)| \leq M(x)$ and $|a'(x,s)| \leq Q(x)$ for all $s \in I(\theta)$ where $I(\theta)$ is a closed interval containing θ and a' denotes the differential with respect to θ. Moreover, $\mathbb{E}_\theta[\frac{M(X_0)}{b(X_0)}]^2 < \infty$ and $\mathbb{E}_\theta[\frac{Q(X_0)}{b(X_0)}]^2 < \infty$.*

Assumption 3.

$$\mathbb{E}_\theta[\int_0^T (\frac{a(X_t,\theta)}{b(X_t)})^2 dt] < \infty,$$
$$\mathbb{E}_\theta[\int_0^T (\frac{a'(X_t,\theta)}{b(X_t)})^2 dt] < \infty$$

and

$$\mathbb{E}_\theta[\int_0^T (\frac{a''(X_t,\theta)}{b(X_t)})^2 dt] < \infty,$$

which ensure the existence of the stochastic integrals $\int_0^T \frac{a(X_t,\theta)}{b(X_t)} dW_t$, $\int_0^T \frac{a'(X_t,\theta)}{b(X_t)} dW_t$ and $\int_0^T \frac{a''(X_t,\theta)}{b(X_t)} dW_t$.

Remark 1. Assumption 1 means that the value the likelihood function takes at the true parameter is not equal to the value at other parameters. Assumptions 2 and 3 play a key role in applying the Borel-Cantelli lemma and uniform ergodic theorem.

Let P_θ^T be the probability measure generated by the process $\{X_t, 0 \leq t \leq T\}$ and P_W^T be the probability measure induced by the standard Wiener process. Then, by applying the Girsanov theorem, the log likelihood function is described as follows:

$$\ell_T(\theta) = \log \frac{dP_\theta^T}{dP_W^T} = \int_0^T \frac{a(X_t,\theta)}{b^2(X_t)} dX_t - \frac{1}{2} \int_0^T \frac{a^2(X_t,\theta)}{b^2(X_t)} dt. \tag{2}$$

Assume that $\ell_T(\theta)$ is continuous and differentiable with respect to θ.

In the next section two problems will be solved. One is that the existence and strong consistency of the maximum likelihood estimator at which the likelihood function attains a local maximum will be proved, the other one is that the limit distribution of the error of estimation will be discussed.

3. Main results and proofs

In the following theorem, the existence and strong consistency of the maximum likelihood estimator are proved by applying the maximal inequality for martingale, Borel-Cantelli lemma and the uniform ergodic theorem. In this case, the likelihood function attains a maximum on a compact set, which means that the likelihood function has a local maximum. The process of finding the compact set in which the likelihood function has a maximum plays a key role in the proof of this theorem.

Theorem 1. *Under Assumptions 1–3, there exists a solution of the equation* $\ell'_T(\theta) = 0$ *which is strongly consistent for* θ *as* $T \to \infty$.

Proof. Suppose θ denotes the true value of the parameter. According to the expression of the likelihood function, for any $\sigma > 0$ such that $\theta \pm \sigma \in \Theta$, it is easy to check that

$$\begin{aligned}
&\ell_T(\theta \pm \sigma) - \ell_T(\theta) \\
&= \int_0^T \frac{(a(X_t, \theta \pm \sigma) - a(X_t, \theta))}{b^2(X_t)} dX_t - \frac{1}{2} \int_0^T \frac{(a^2(X_t, \theta \pm \sigma) - a^2(X_t, \theta))}{b^2(X_t)} dt \\
&= \int_0^T \frac{(a(X_t, \theta \pm \sigma) - a(X_t, \theta))}{b(X_t)} dW_t - \frac{1}{2} \int_0^T \frac{(a(X_t, \theta \pm \sigma) - a(X_t, \theta))^2}{b^2(X_t)} dt.
\end{aligned}$$

First of all, we will prove that $\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)} dW_t \overset{a.s.}{\to} 0$ as $T \to \infty$ where $s \in I(\theta)$.

By applying maximal inequality for martingale and the stationarity of the process, it follows that

$$\begin{aligned}
\mathbb{P}_\theta(\sup_{0<T\le T_0} |\int_0^T \frac{a(X_t, s)}{b(X_t)} dW_t| > \varepsilon) &\le \frac{\mathbb{E}_\theta(\int_0^{T_0} \frac{a(X_t,s)}{b(X_t)} dW_t)^2}{\varepsilon^2} \\
&= \frac{T_0 \mathbb{E}_\theta(\frac{a(X_0,s)}{b(X_0)})^2}{\varepsilon^2}.
\end{aligned}$$

Let

$$\mathcal{B}_n = \{\sup_{2^{n-1}<T<2^n} \sup_s |\int_0^T \frac{a(X_t, s)}{b(X_t)} dW_t| > 2^{\frac{n}{2}} n^\alpha\}, \tag{3}$$

where $n \ge 1$ and $\alpha > \frac{1}{2}$. Then

$$\begin{aligned}
\mathbb{P}_\theta(\mathcal{B}_n) &= \mathbb{P}_\theta(\sup_{0<T<2^{n-1}} \sup_s |\int_0^T \frac{a(X_t, s)}{b(X_t)} dW_t| > 2^{\frac{n}{2}} n^\alpha) \\
&\le \frac{2^{n-1} \mathbb{E}_\theta(\frac{a(X_0,s)}{b(X_0)})^2}{2^n n^{2\alpha}} = \frac{\mathbb{E}_\theta(\frac{a(X_0,s)}{b(X_0)})^2}{2} \frac{1}{n^{2\alpha}}.
\end{aligned}$$

It can be obtained that

$$\sum_{n=1}^\infty \mathbb{P}_\theta(\mathcal{B}_n) < \infty. \tag{4}$$

According to Borel-Cantelli lemma, it follows that

$$\mathbb{P}_\theta(\limsup_{n\to\infty}\mathcal{B}_n)=0. \tag{5}$$

Therefore,

$$\limsup_{T\to\infty}\sup_s \frac{|\int_0^T \frac{a(X_t,s)}{b(X_t)}dW_t|}{T^{\frac{1}{2}}(\ln T)^\alpha} \le 2^{\frac{1}{2}}(\frac{1}{\ln 2})^\alpha \quad a.s. \tag{6}$$

Hence, for large T,

$$\sup_s |\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)}dW_t| \le \frac{(\ln T)^\alpha}{T^{\frac{1}{2}}} 2^{\frac{1}{2}}(\frac{1}{\ln 2})^\alpha, \tag{7}$$

with probability one.

It is easy to check that

$$\sup_s |\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)}dW_t| \overset{a.s.}{\to} 0, \tag{8}$$

as $T\to\infty$.

Since

$$|\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)}dW_t| \le \sup_s |\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)}dW_t|, \tag{9}$$

it follows that

$$\frac{1}{T}\int_0^T \frac{a(X_t,s)}{b(X_t)}dW_t \overset{a.s.}{\to} 0. \tag{10}$$

As a result, one has

$$\frac{1}{T}\int_0^T \frac{(a(X_t,\theta\pm\sigma)-a(X_t,\theta))}{b(X_t)}dW_t \overset{a.s.}{\to} 0, \tag{11}$$

as $T\to\infty$.

According to the uniform ergodic theorem (see e.g. [24]), it can be checked that

$$\frac{1}{T}\int_0^T \frac{(a(X_t,\theta\pm\sigma)-a(X_t,\theta))^2}{b^2(X_t)}dt \overset{a.s.}{\to} \mathbb{E}_\theta[\frac{(a(X_0,\theta\pm\sigma)-a(X_0,\theta))^2}{b^2(X_0)}], \tag{12}$$

as $T\to\infty$.

We assume that $\mathbb{E}_\theta[\frac{(a(X_0,\theta\pm\sigma)-a(X_0,\theta))^2}{b^2(X_0)}]>0$.

Therefore,

$$\frac{1}{T}(\ell_T(\theta\pm\sigma)-\ell_T(\theta)) \overset{a.s.}{\to} -\frac{1}{2}\mathbb{E}_\theta[\frac{(a(X_0,\theta\pm\sigma)-a(X_0,\theta))^2}{b^2(X_0)}]<0, \tag{13}$$

as $T \to \infty$.

Hence, for almost every $\omega \in \Omega$, σ and θ, there exists T_0 such that for $T \geq T_0$,

$$\ell_T(\theta \pm \sigma) < \ell_T(\theta). \tag{14}$$

Since $\ell_T(\theta)$ is continuous on the interval $[\theta-\sigma, \theta+\sigma]$, there exists an element $\overline{\theta}_T \in (\theta-\sigma, \theta+\sigma)$ such that $\ell_T(\theta)$ reaches the maximum at this element, that is to say, $\ell'_T(\overline{\theta}_T) = 0$. As $|\overline{\theta}_T - \theta| < \sigma$, it leads to the relation

$$\overline{\theta}_T \overset{a.s.}{\to} \theta, \tag{15}$$

as $T \to \infty$.

The proof is complete. □

In the following theorem, the limit distribution of the error of estimation is obtained by applying the maximal inequality for martingale, Borel-Cantelli lemma and uniform ergodic theorem.

Theorem 2. *Under Assumptions 1–3,* $\sqrt{T}(\overline{\theta}_T - \theta) \overset{d}{\to} N(0, \frac{1}{\mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2}), (T \to \infty)$.

Proof. Expanding $\ell'_T(\theta)$ about $\overline{\theta}_T$, it follows that

$$\ell'_T(\theta) = \ell'_T(\overline{\theta}_T) + \ell''_T(\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T))(\theta - \overline{\theta}_T), \tag{16}$$

where $0 < \lambda < 1$.

In view of Theorem 1, it is known that $\ell'_T(\overline{\theta}_T) = 0$, then

$$\ell'_T(\theta) = \ell''_T(\overline{\theta}_T + \lambda(\theta - \overline{\theta}_T))(\theta - \overline{\theta}_T). \tag{17}$$

Since θ is the true value of the parameter,

$$\begin{aligned}\ell'_T(\theta) &= \int_0^T \frac{a'(X_t,\theta)}{b^2(X_t)} dX_t - \int_0^T \frac{a(X_t,\theta)a'(X_t,\theta)}{b^2(X_t)} dt \\ &= \int_0^T \frac{a'(X_t,\theta)}{b(X_t)} dW_t,\end{aligned}$$

with the stationarity of the process, it can be checked that

$$\mathbb{E}_\theta[\ell'_T(\theta)] = 0, \tag{18}$$

and

$$\mathbb{E}_\theta[\ell'_T(\theta)]^2 = T\mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2. \tag{19}$$

From the central limit theorem for stochastic integrals, one has

$$T^{-\frac{1}{2}}\ell'_T(\theta) \overset{d}{\to} N(0, \mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2). \tag{20}$$

As

$$\frac{1}{T}\ell_T''(\theta) = \frac{1}{T}(\int_0^T \frac{a''(X_t,\theta)}{b^2(X_t)}dX_t - \int_0^T \frac{(a'^2(X_t,\theta)+a(X_t,\theta)a''(X_t,\theta))}{b^2(X_t)}dt) = \frac{1}{T}(\int_0^T \frac{a''(X_t,\theta)}{b(X_t)}dW_t - \int_0^T \frac{a'^2(X_t,\theta)}{b^2(X_t)}dt).$$

By applying the same method used in Theorem 1, it follows that

$$\frac{1}{T}\int_0^T \frac{a''(X_t,\theta)}{b(X_t)}dW_t \overset{a.s.}{\to} 0, \tag{21}$$

as $T\to\infty$.

By employing the uniform ergodic theorem, it can be obtained that

$$\frac{1}{T}\int_0^T \frac{a'^2(X_t,\theta)}{b^2(X_t)}dt \overset{a.s.}{\to} \mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2, \tag{22}$$

as $T\to\infty$.

From the above analysis, one has

$$\frac{1}{T}\ell_T''(\theta) \overset{a.s.}{\to} \mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2, \tag{23}$$

as $T\to\infty$.

Since

$$\begin{aligned}
&\frac{1}{T}(\ell_T''(\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T))-\ell_T''(\theta))\\
&=\frac{1}{T}(\int_0^T \frac{(a''(X_t,\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T))-a''(X_t,\theta))}{b^2(X_t)}dX_t\\
&+\int_0^T \frac{a'^2(X_t,\theta)-a'^2(X_t,\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T))+a(X_t,\theta)a''(X_t,\theta)}{b^2(X_t)}dt\\
&-\int_0^T \frac{a(X_t,\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T))a''(X_t,\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T))}{b^2(X_t)}dt)\\
&=\frac{1}{T}(\int_0^T \frac{(a''(X_t,\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T))-a''(X_t,\theta))}{b(X_t)}dW_t\\
&+\int_0^T \frac{a'^2(X_t,\theta)-a'^2(X_t,\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T))}{b^2(X_t)}dt\\
&+\int_0^T \frac{a''(X_t,\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T))(a(X_t,\theta)-a(X_t,\overline{\theta}_T+\lambda(\theta-\overline{\theta}_T)))}{b^2(X_t)}dt),
\end{aligned}$$

by applying the uniform ergodic theorem, the dominated convergence theorem together with $\overline{\theta}_T \overset{a.s.}{\to} \theta$ and (21), it follows that

$$\frac{1}{T}(\ell''_T(\overline{\theta} + \lambda(\theta - \overline{\theta}_T)) - \ell''_T(\theta)) \overset{a.s.}{\to} 0, \tag{24}$$

as $T \to \infty$.

Therefore,

$$\frac{1}{T}(\ell''_T(\overline{\theta} + \lambda(\theta - \overline{\theta}_T))) \overset{a.s.}{\to} \mathbb{E}_\theta[\frac{a'(X_0, \theta)}{b(X_0)}]^2, \tag{25}$$

as $T \to \infty$.

From the above analysis, we have

$$\sqrt{T}(\overline{\theta}_T - \theta) \overset{d}{\to} N(0, \frac{1}{\mathbb{E}_\theta[\frac{a'(X_0,\theta)}{b(X_0)}]^2}), \tag{26}$$

as $T \to \infty$.

The proof is complete. □

4. Example

We consider the diffusion process described by the following stochastic differential equation:

$$\begin{cases} dX_t = & -\theta X_t[1 + \frac{1}{2}\sin(X_t)]dt + \sigma dW_t \\ X_0 \sim & u_\theta, \end{cases} \tag{27}$$

where $\theta > 0$,$\sigma > 0$, u_θ is the invariant measure.

It is easy to check that the likelihood function has the following expression

$$\ell_T(\theta) = \int_0^T \frac{-\theta X_t[1 + \frac{1}{2}\sin(X_t)]}{\sigma^2} dX_t - \frac{1}{2}\int_0^T \frac{\theta^2 X_t^2[1 + \frac{1}{2}\sin(X_t)]^2}{\sigma^2} dt. \tag{28}$$

Then, we obtain the estimator

$$\widehat{\theta_T} = \frac{-\int_0^T X_t[1 + \frac{1}{2}\sin(X_t)]dX_t}{\int_0^T X_t^2[1 + \frac{1}{2}\sin(X_t)]^2 dt}. \tag{29}$$

Thus, the estimation error is

$$\widehat{\theta_T} - \theta = \frac{-\sigma\int_0^T X_t[1 + \frac{1}{2}\sin(X_t)]dW_t}{\int_0^T X_t^2[1 + \frac{1}{2}\sin(X_t)]^2 dt}. \tag{30}$$

Since $X_t^2[1 + \frac{1}{2}\sin(X_t)]^2 \leq \frac{9}{4}X_t^2$ and $\mathbb{E}[X_0]^2 < \infty$, it is obviously that this process satisfy the Assumptions 1–3. Then, we have

$$\widehat{\theta_T} \overset{a.s.}{\to} \theta, \tag{31}$$

and

$$\sqrt{T}(\widehat{\theta_T} - \theta) \overset{d}{\to} N(0, \frac{\sigma^2}{\mathbb{E}[X_0^2[1 + \frac{1}{2}\sin(X_0)]^2]}). \tag{32}$$

5. Conclusion

In this paper, the existence and strong consistency of the maximum likelihood estimator and the asymptotic normality of the error of estimation have been proved with the help of the maximal inequality for martingale, Borel-Cantelli lemma and uniform ergodic theorem. In this paper, the likelihood function has attained a local maximum at the maximum likelihood estimator in a compact set. This paper has considered a class of processes driven by Brown Motion, one of the further research topics will study the parameter estimation for the processes driven by fractional Brown Motion or small Lévy noises.

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