Chebyshev wavelet method (CWM) for the numerical solutions of fractional boundary value problems

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Abstract. This research article, is concerned with the numerical solutions of fractional boundary value problems by using Chebyshev Wavelet Method (CWM). Simulations based on (CWM), are better in terms of the numerical solutions of higher order boundary value problems. The results obtained by the proposed method are compared with the results of Optimal Homotopy Asymptotic Method (OHAM), Modified Optimal Homotopy Asymptotic Method (MOHAM), Variation Iteration Method (VIM) and exact solutions of the problems. By comparison, it is obvious that the current method improved the accuracy and is easy to implement. The numerical solutions of some examples are discussed to show the suitability of (CWM).

Keywords: fractional boundary value problem, Chebyshev wavelet method (CWM), iterative method, numerical solution.

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1. Introduction

Fractional calculus is the subject that dealing with the study of fractional order differential equations. Fractional order differential equations are the generalization of classical integer order differential equations. Gemant and Scot-Blair started the study of fractional calculus by proposing a fractional derivative models for viscoelasticity, anomalous stress and strain [1, 2]. Other applications of fractional calculus while modeling physical phenomena, such as nonlinear oscillation of earth quake [3], signal Processing [4], control theory [5] and fluid dynamics traffic [6] have made this area important for mathematicians and researchers.
The exact and numerical solutions are important to describe and analyze fractional order differential equations. Therefore a number of efficient techniques have been used to obtain the solution of fractional differential equation such as Adomian Decomposition Method (ADM) [7], the Variational Iteration Method (VIM) [8], Homotopy Analysis Method (HAM) [9], Homotopy Perturbation Method (HPM) [10, 11], Differential Transformation Method (DTM) [12], the Fractional Sub Equation Method (FSEM) [13], the First Integral Method [FIM] [14], Reproducing Kernel Hilbert Space Method (RKHSM) [15, 16], shifted Jacobi Polynomials Method [17], shifted Legendre polynomials [18, 19] and the Ex-Function Method (EFM) [20].

Recently, most of the researchers have shown great interest in wavelet theory [21, 22, 23, 24, 25, 26, 27, 28] The most relevant methods based on wavelets are Haar wavelet [25], Legendre wavelet [24] and Chebyshev wavelet [21, 23, 27]. Harmonic wavelet method [22]. CAS wavelet [26].

In the current work, we have used a numerical method based on Chebyshev Wavelets for the numerical solution of some fractional higher order differential equations. The numerical solution by (CWM) are compared with the results obtained by (OHAM), (MOHAM), (VIM) and exact solution of the problems. The numerical results have suggested that (CWM) has the higher degree of accuracy than other methods.

The organization of this paper is in the following manner. The definition and properties of fractional calculus will be given in Section 2 while in Section 3, we give some properties of the Chebyshev Wavelets. In section 4, we introduce the Chebyshev Wavelet Method (CWM). In Section 5, we give four numerical examples and finally in Section 6, we give the conclusion.

2. Preliminaries and definitions

In this section some of the important definitions and preliminary concepts are discussed for the continuation of the current work.

Definition 2.1. The Riemann fractional integral operator I of order μ on the usual Lebesgue space $L_1[a,b]$ is given by

$$(I^{\mu}g)(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\xi)^{\mu-1}g(\xi)d\xi, \mu > 0,\ (I^0 g)(t) = g(t),$$

This integral operator has the following properties

(a) $I^{\mu}I^{\eta} = I^{\mu+\eta}$,

(b) $I^{\mu}I^{\eta} = I^{\eta}I^{\mu}$,

(c) $I^{\mu}(t-a)^{\nu} = \dfrac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\mu+\nu}$

Where $\mu, \eta > 0, \nu > -1$.

Definition 2.2. The Riemann fractional derivative of order $\gamma > 0$ is defined as

$$(D^{\gamma}g)(t) = \Big(\frac{d}{dt}\Big)^n \Big(I^{(n-\gamma)}g)(t)\Big), n-1 < \gamma \leq n,$$

where n is an integer.

However the Riemann fractional derivative has certain disadvantages and therefore Caputo introduced a modified differential operator.

Definition 2.3. The Caputo definition of fractional differential operator is given by

$$(D^{\mu}g)(t) = \frac{1}{\Gamma(n-\mu)}\int_0^t (t-\xi)^{n-\mu-1}g^{(n)}(\xi)d\xi,\ n-1 < \mu < n.$$

Where $t > 0, n$ is an integer.

3. Properties of the Chebyshev wavelets

Wavelets consist of family of functions generated from the dilation a and translation b of a single function $\psi(x)$ called the mother wavelet. When the dilation a and translation parameter b change continuously, we get the following continuous family of Wavelet [27]

$$\psi_{a,b}(x) = |a|^{\frac{1}{2}}\psi(\frac{x-b}{a}), a, b \in R, a \neq 0.$$

If we restrict the parameters a and b to discrete values as

$$a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0,$$

where n, k are positive integers.

Then the following family of discrete wavelets is obtained

$$\psi_{k,n}(x) = |a|^{\frac{k}{2}}\psi(a_0^k x - nb_0), k, n \in Z.$$

Especially when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthogonal basis.

The second kind of Chebyshev wavelets is constituted of four parameters, $\psi_{n,m}(x) = \psi(k, n, m, x)$, where $n = 1, 2, ..., 2^{k-1}, k$ is any nonnegative integer, m is the degree of the second Chebyshev polynomial. The Chebyshev wavelets are defined on the interval $0 \le x < 1$ as

$$\psi_{n,m} = \begin{cases} 2^{\frac{k}{2}} \tilde{T}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases}$$

Where $\tilde{T}_m(x) = \sqrt{\frac{2}{\pi}} T_m(x), m = 0, 1, 2..., M - 1.$

Here $T_m(x)$ are second Chebyshev polynomials of degree m with respect to the weight function $w(x) = \sqrt{1 - x^2}$ on the interval $[-1, 1]$, and satisfying the following recursive formula

$$T_0(x) = 1, T_1(x) = 2x, T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), m = 1, 2, 3, \ldots$$

Lemma 3.1. *If the Chebyshev Wavelet expansion of a continuous function $f(x)$ converges uniformly, then the Chebyshev Wavelet expansion converges to the function $f(x)$.*

Proof. See [29]. □

Theorem 3.2. *A function $f(x) \in L_2[0, 1]$, with bounded second derivative, say $|f''(x)| \le N$, can be expanded as an infinite sum of Chebyshev wavelets, and the series converges uniformly to $f(x)$, that is,*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x).$$

Proof. See [29]. □

4. Chebyshev Wavelet Method (CWM)

In the present paper we consider the fractional boundary value of the form

$$D^\alpha y(x) = g(x) + f(y), a < x \le b, 9 < \alpha \le 10, \tag{4.1}$$

with the boundary conditions given by $y(a) = \alpha_0, y^{(2)}(a) = \alpha_1, y^{(4)}(a) = \alpha_2, y^{(6)}(a) = \alpha_3, y^{(8)}(a) = \alpha_4, y(b) = \beta_0, y^{(2)}(b) = \beta_1, y^{(4)}(b) = \beta_2, y^{(6)}(b) = \beta_3, y^{(8)}(b) = \beta_4$, where $g(x)$ is a source function, $f(y)$ is linear or nonlinear continuous function and α_i and β_i are real valued constants.

The solution to equation (4.1) can be extended by Chebyshev Wavelets series as

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x) \tag{4.2}$$

The series in equation (4.2) is truncated to finite number of terms that is

$$y_{k,M}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \tag{4.3}$$

This shows that there are $2^{k-1}M$ conditions to determine $2^{k-1}M$ coefficients which are $c_{i,j}$.

Since, we have eighth boundary conditions; therefore eight conditions are obtained by these boundary conditions.

The conditions are

$$\begin{cases} y_{k,m} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_0, \\ \frac{d^2}{dx^2} y_{k,M}(a) = \frac{d^2}{dx^2} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_1, \\ \frac{d^4}{dx^4} y_{k,M}(a) = \frac{d^4}{dx^4} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_2, \\ \frac{d^6}{dx^6} y_{k,M}(a) = \frac{d^6}{dx^6} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_3, \\ \frac{d^8}{dx^8} y_{k,M}(a) = \frac{d^8}{dx^8} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(a) = \alpha_4, \\ y_{k,M}(b) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_0, \\ \frac{d^2}{dx^2} y_{k,M}(b) = \frac{d^2}{dx^2} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_1, \\ \frac{d^4}{dx^4} y_{k,M}(b) = \frac{d^4}{dx^4} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_2, \\ \frac{d^6}{dx^6} y_{k,M}(b) = \frac{d^6}{dx^6} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_3, \\ \frac{d^8}{dx^8} y_{k,M}(b) = \frac{d^8}{dx^8} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(b) = \beta_4. \end{cases} \tag{4.4}$$

The remaining $2^{k-1}M - 10$ conditions can be obtained by substituting equation (4.4) in equation (4.3), we get

$$\frac{d^\alpha}{dx^\alpha}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\psi_{n,m}(x) = f(x) + \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\psi_{n,m}(x) \tag{4.5}$$

Assume that equation (4.5) is exact at $2^{k-1}M - 10$ points, which we call it x_i, then

$$\frac{d^\alpha}{dx^\alpha}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\psi_{n,m}(x_i) = f(x_i) + \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\psi_{n,m}(x_i) \tag{4.6}$$

The x_i are obtain by using the following formula $x_i = \frac{i-0.5}{2^{k-1}M}$, $i = 1, 2, 3, ..., 2^{k-1}M - 10$. The combination of equations (4.5) and (4.6) form the linear system of $2^{k-1}M$ linear equations. The solution of this linear system of equations determine the unknown coefficients $c_{i,j}$.

The same procedure can be repeated for other fractional boundary value problems of any order.

5. Method implementation

Example 5.1. Consider the following fractional order nonlinear boundary value problem

$$\frac{d^\alpha}{dx^\alpha}y(x) = \frac{d^3}{dx^3}y(x) + 2e^x y^2,\ 0 \le x \le 1, 9 < \alpha \le 10.$$

Subject to the boundary conditions $y(0) = 1, y(1) = \frac{1}{e}, y^{(2)}(1) = \frac{1}{e}, y^{(4)}(0) = 1, y^{(4)}(1) = \frac{1}{e}, y^{(6)}(0) = 1, y^{(6)}(1) = \frac{1}{e}, y^{(8)}(0) = 1, y^{(8)}(1) = \frac{1}{e}$. The exact solution of this problem is $y(x) = e^{-x}$.

Table 1 shows the comparison of the absolute error between exact solutions and approximate solutions for $\alpha = 10$, when $M = 20$ and $k = 1$. Here y_{exact} and y_{10} represent the exact solution and approximate solution of the problem at $\alpha = 10$. The numerical results given by the present method are also compared with (OHAM) solutions. From the table it is obvious that the results of the current method are far better than (OHAM) method.

Table 2 displays the approximate solutions $y_{9.25}, y_{9.5}, y_{9.75}$ and $y_{9.95}$ for different values of $\alpha = 9.25, 9.50, 9.75$ and 9.95 respectively. Error($y_{9.25}$), Error($y_{9.5}$), Error($y_{9.75}$) and Error(y_{10}) are the errors obtain for different values of $\alpha =$ 9.25, 9.5, 9.75 and 9.95 respectively, comparing with the exact solutions of the given problem. The error associated with different fractional order differential equations shows that the error in each fractional order decreases as the order of the fractional order differential equation approaches to integer order. This phenomena shows the consistency and reliability of the fractional order solutions.

Table 1: Numerical results for Example 5.1 for $\alpha = 10$

x_i	y_{exact}	y_{10} (CWM)	Error (CWM)	Error (OHAM)
0.0	1.000000000000000000	0.999999999999999999	1.00E-21	1.10E-11
0.1	0.904837418035959573	0.904837418035960081	5.07E-16	8.39E-7
0.2	0.81873075307798185	0.818730753077982824	9.65E-16	1.59E-6
0.3	0.74081822068171786	0.740818220681719194	1.32E-15	2.19E-6
0.4	0.670320046035639300	0.670320046035640860	1.55E-15	2.58E-6
0.5	0.606530659712633423	0.606530659712635061	1.63E-15	2.71E-6
0.6	0.548811636094026432	0.54881163609402798	1.56E-15	2.58E-6
0.7	0.496585303791409514	0.496585303791410837	1.32E-15	2.20E-6
0.8	0.449328964117221591	0.449328964117222551	9.59E-16	1.60E-6
0.9	0.406569659740599111	0.406569659740599616	5.04E-16	8.41E-7
1.0	0.367879441171442321	0.367879441171442321	4.00E-16	4.85E-11

Table 2: Numerical results for Example 5.1 for different fractional order

x_i	$y_{9.25}$	Error ($y_{9.25}$)	$y_{9.5}$	Error ($y_{9.5}$)	$y_{9.75}$	Error ($y_{9.75}$)	$y_{9.95}$	Error ($y_{9.95}$)
0.0	1.000000	0.0	0.9999999	2.0E-20	0.99999999	3.0E-20	1.00000000	0.0
0.1	0.904839	2.05E-6	0.9048383	8.97E-7	0.90483767	2.59E-7	0.90483744	2.80E-8
0.2	0.818734	3.91E-6	0.8187324	1.70E-6	0.81873124	4.94E-7	0.81873080	5.34E-8
0.3	0.740823	5.39E-6	0.7408205	2.35E-6	0.74081890	6.80E-7	0.74081829	7.36E-8
0.4	0.670326	6.34E-6	0.6703228	2.76E-6	0.67032084	8.01E-7	0.67032013	8.68E-8
0.5	0.606537	6.67E-6	0.6065337	2.91E-6	0.60653150	8.43E-7	0.60653075	9.15E-8
0.6	0.548817	6.35E-6	0.5488144	2.77E-6	0.54881243	8.03E-7	0.54881172	8.72E-8
0.7	0.496590	5.40E-6	0.4965876	2.35E-6	0.49658530	6.84E-7	0.49658537	7.43E-8
0.8	0.449333	3.92E-6	0.4493306	1.17E-6	0.44932946	4.97E-7	0.44932896	5.41E-8
0.9	0.406571	2.06E-6	0.4065705	9.01E-7	0.40656992	2.61E-7	0.40656965	2.84E-8
1.0	0.367879	1.0E-20	0.3678794	4.0E-20	0.36787944	1.0E-20	0.36787944	5.0E-20

Example 5.2. Consider the following fractional order nonlinear boundary value problem

$$\frac{d^\alpha}{dx^\alpha}y(x) = y(x) - 15e^x - 10xe^x,\ 0 \leq x \leq 1, 4 < \alpha \leq 5.$$

Subject to the boundary conditions

$$y(0) = 0, y(1) = 0, y'(0) = 1, y'(1) = -e, y''(0) = 0.$$

The analytic solution of this problem is $y(x) = x(1-x)e^x$.

Table 3 illustrates the comparison of the absolute error between the present method and other numerical methods such as (MOHPM), (OHAM) and (VIM).

Figure 1: The solution graph, of example 5.1 for different fractional order $\alpha = 10, 9.95, 9.75, 9.5, 9.25$

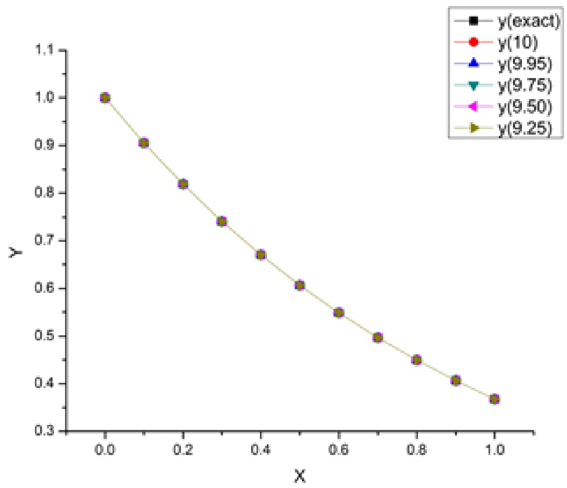


Table 3: Numerical results for Example 5.2 for $\alpha = 5$

x_i	y_{exact}	y_5 (CWM)	Error (CWM)	Error (MOHAM)	Error (OHAM)	Error (VIM)
0.0	0.000000000000000000	0.000000000000000002	2.53E-20	0.0000	0.00000	0.0000
0.1	0.099465382626808286	0.099465382626808625	3.39E-16	5.4E-14	9.00E-11	1.0E-9
0.2	0.195424441305627173	0.195424441305628649	1.47E-15	3.3E-13	4.00E-10	2.0E-9
0.3	0.283470349590960651	0.283470349590963518	2.86E-15	1.0E-13	5.00E-10	1.0E-9
0.4	0.358037927433904876	0.358037927433908857	3.98E-15	1.9E-12	2.00E-11	2.0E-9
0.5	0.412180317675032036	0.412180317675036519	4.48E-15	2.7E-12	1.00E-9	3.1E-8
0.6	0.437308512093722153	0.437308512093726389	4.23E-15	3.0E-12	2.00E-9	3.7E-8
0.7	0.422888068568800069	0.422888068568803370	3.30E-15	2.1E-12	2.00E-9	4.1E-8
0.8	0.356086548558794816	0.356086548558796758	1.94E-15	3.7E-12	1.00E-9	3.1E-8
0.9	0.221364280004125469	0.221364280004126090	6.20E-16	-3.2E-11	4.0E-10	1.4E-8
1.0	0.000000000000000000	0.000000000000000004	4.58E-20	-1.6E-10	0.00000	0.0000

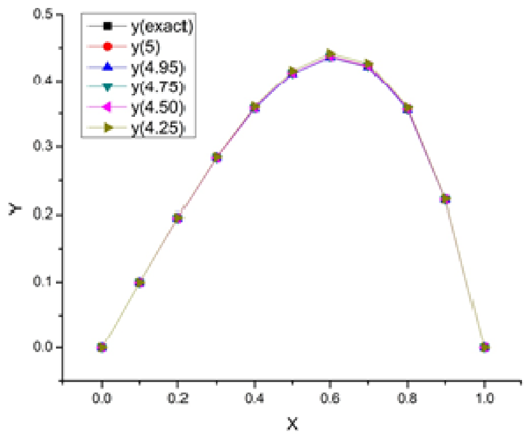
The results reveal that the present method has higher accuracy than any other method given the table. Here we have used $M = 17$ and $k = 1$, to obtain these results given in Table 3. y_{exact} and y_5 are the exact and approximate solution of example 5.2 at $\alpha = 5$. Error y_5, Error (MOHPM), Error (OHAM) and Error (VIM) are the corresponding errors in the present method, (MOHPM), (OHAM) and (VIM) respectively.

Table 4: Numerical results for Example 5.2 for different fractional order

x_i	$y_{4.25}$	Error $(y_{4.25})$	$y_{4.5}$	Error $(y_{4.5})$	$y_{4.75}$	Error $(y_{4.75})$	$y_{4.95}$	Error $(y_{4.95})$
0.0	0.999999	1.6E-19	0.0000000	7.2E-21	0.00000000	8.5E-21	0.00000000	1.87E-9
0.1	0.099502	4.54E-6	0.0994459	1.94E-5	0.09945966	4.08E-5	0.09943453	3.08E-5
0.2	0.195751	2.50E-5	0.1953345	7.91E-5	0. 19534526	2.50E-4	0.19522367	2.00E-4
0.3	0.284518	5.81E-5	0.2833612	1.09E-4	0.28336120	6.29E-4	0.28347034	5.34E-4
0.4	0.360170	9.26E-5	0.3579946	4.32E-5	0.35799467	1.07E-3	0.35707949	9.58E-4
0.5	0.415450	1.16E-4	0.4123023	1.22E-4	0.41230239	1.43E-3	0.41084175	1.33E-3
0.6	0.441314	1.20E-4	0.4376282	3.19E-4	0.43762826	1.56E-3	0.43578474	152E-3
0.7	0.426794	1.01E-4	0.4233281	4.40E-4	0.42332813	1.37E-3	0.42148765	1.40E-3
0.8	0.358885	6.39E-5	0.3564752	3.88E-4	0.35647526	9.04E-4	0.35513104	9.55E-3
0.9	0.222431	2.17E-5	0.2213553	1.71E-4	0.22153538	3.19E-4	0.22101464	3.49E-4
1.0	0.000000	0.0000	0.0000000	3.7E-20	0.00000000	2.2E-19	0.00000000	1.8E-9

Table 4 represents the solution of fractional order differential equations for different values of $\alpha = 4.25, 4.50, 4.75$ and 4.95. The solutions $y_{4.25}, y_{4.5}, y_{4.75}$ and $y_{4.95}$ and Error $(y_{4.25})$, Error $(y_{4.5})$, Error $(y_{4.75})$ and Error $(y_{4.95})$ are the corresponding errors of fractional order differential equations at $\alpha = 4.25, 4.50, 4.75, 4.95$ respectively.

Figure 2: The exact solution is represented by y(exact), while $y_5, y_{4.95}, y4.75$ and $y_{4.5}$ show (CWM) solutions at $\alpha = 5, 4.95, 4.75$ and 4.5 respectively.



Example 5.3. Consider the following fractional order nonlinear boundary value problem

$$\frac{d^\alpha}{dx^\alpha} y(x) - y^2 e^{-x} = 0,\ \ 0 \le x \le 1, 4 < \alpha \le 5.$$

Subject to the boundary conditions

$$y(0) = 1, y(1) = e, y'(0) = 1, y'(1) = e, y''(0) = 1,$$

The exact solution of this problem is $y(x) = e^x$.

Table 5: Numerical results for Example 5.3 for $\alpha = 5$

x_i	y_{exact}	y_5 (CWM)	Error (CWM)	Error (MOHPM)	Error (OHAM)	Error (VIM)
0.0	1.0000000000000000000	1.0000000000000000001	1.00E-19	0.00000	0.000000	0.0000
0.1	1.1051709180756476248	1.1051709180756476237	1.10E-18	3.1E-15	1.9E-10	-8.0E-3
0.2	1.2214027581601698339	1.2214027581601698286	5.30E-18	1.9E-14	4.00E-9	-1.2E-3
0.3	1.3498588075760031040	1.3498588075760030936	1.04E-17	5.4E-14	5.00E-9	-5.0E-3
0.4	1.4918246976412703178	1.4918246976412703033	1.45E-17	1.0E-13	2.00E-9	3.0E-3
0.5	1.6487212707001281468	1.6487212707001281305	1.63E-17	1.4E-13	1.00E-9	8.0E-3
0.6	1.8221188003905089749	1.8221188003905089595	1.54E-17	1.6E-13	2.00E-8	6.0E-3
0.7	2.0137527074704765216	2.0137527074704765096	1.20E-17	1.5E-13	2.00E-8	1.0E-4
0.8	2.2255409284924676046	2.2255409284924675976	7.00E-18	9.9E-14	1.00E-9	9.0E-3
0.9	2.4596031111569496638	2.4596031111569496617	2.10E-18	1.1E-14	4.00E-9	-9.0E-3
1.0	2.7182818284590452354	2.7182818284590452354	0.000000	1.0E-13	0.00000	0.0000

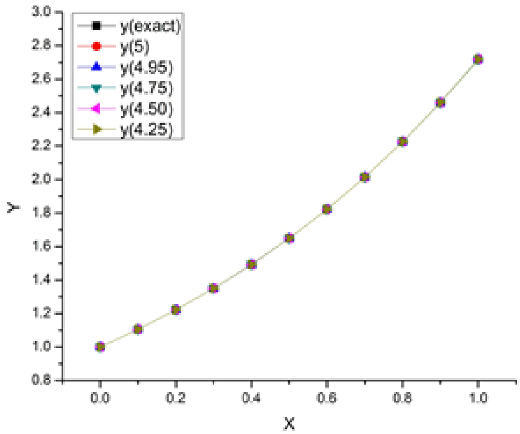
In Table 5, we have presented y_{exact} the exact solution and y_5 the approximate solution for example 5.3. Here we use $M = 17$ and $k = 1$ for the implementation of the current method. The accuracy is compared with other methods such as (MOHPM), (OHAM) and (VIM). This table reveals that the present method have the highest degree of accuracy than other methods. The exact and approximate solutions at $\alpha = 5$ are represented by y_{exact} and y_5 respectively. Error (y_5), Error (MOHPM), Error (OHAM) and Error (VIM) are the corresponding errors in the present method, (MOHPM), (OHAM) and (VIM) respectively.

Table 6: Numerical results for Example 5.3 for different fractional order

x_i	$y_{4.25}$	Error ($y_{4.25}$)	$y_{4.5}$	Error ($y_{4.5}$)	$y_{4.75}$	Error ($y_{4.75}$)	$y_{4.95}$	Error ($y_{4.95}$)
0.0	0.999999	1.6E-19	0.0000000	1.0E-19	0.00000000	1.0E-19	0.00000000	1.0E-19
0.1	1.105175	4.54E-6	1.1051747	3.84E-6	1.10517300	2.08E-6	1.10517133	4.20E-7
0.2	1.221427	2.50E-5	1.2214251	2.23E-5	1.22141525	1.24E-5	1.22140531	2.55E-6
0.3	1.349916	5.80E-5	1.3499128	1.09E-5	1.34988964	3.08E-5	1.34986520	6.39E-6
0.4	1.491917	9.26E-5	1.4919134	8.87E-5	1.49187625	5.15E-5	1.49183549	1.08E-5
0.5	1.648837	1.16E-4	1.6488356	1.14E-4	1.64872127	6.74E-5	1.64873551	1.42E-5
0.6	1.822239	1.20E-4	1.8222396	1.20E-4	1.82219088	7.20E-5	1.82213415	153E-5
0.7	2.013854	1.01E-4	2.0138561	1.03E-4	2.01381513	6.24E-5	2.01376610	1.33E-5
0.8	2.225604	6.39E-5	2.2256069	6.60E-5	2.22558116	4.02E-5	2.22554961	8.68E-6
0.9	2.459624	2.17E-5	2.4596257	2.26E-5	2.45961705	1.39E-5	2.45960661	3.02E-6
1.0	2.718281	0.0000	2.7182818	2.0E-19	2.71628182	1.0E-19	2.71828182	1.0E-19

Table 6 represents the solution of fractional order differential equations for different values of $\alpha = 4.25, 4.5, 4.75, 4.95$. The solutions $y_{4.25}, y_{4.5}, y_{4.75}$ and $y_{4.95}$ and Error $(y_{4.25})$, Error $(y_{4.5})$, Error $(y_{4.75})$ and Error $(y_{4.95})$ the corresponding errors of fractional order differential equations at $\alpha = 4.25, 4.5, 4.75, 4.95$ respectively.

Figure 3: The exact solution y_{exact}, while $y_5, y_{4.95}, y_{4.75}, y_{4.5}$ and $y_{4.25}$ show (CWM) solutions $\alpha = 5, 4.95, 4.75, 4.5$ and 4.25 respectively.



6. Conclusion

In this research paper, we have attempted to find the numerical solutions of fractional order boundary value problems by using Chebyshev Wavelet method. Three problems of different fractional order α such that $9 < \alpha \leq 10$ and $4 < \alpha \leq 5$ were considered for numerical treatment. The numerical simulation has shown that (CWM) has better accuracy than other methods which are under discussion.

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