On modular flats and pushouts of matroids

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Abstract. In this paper, a sufficient condition for a submatroid of a loopless matroid to be a modular flat is given. Moreover, it is shown that if the injective pushout of two loopless matroids relative to a common submatroid exists, then the join of the given matroids exists and is isomorphic to the indicated pushout.
Keywords: matroid, flat, modular flat, injective pushout.

1. Background

We follow the terminology of White [17] and Lawvere and Schanuel [18]. In particular, the *ground set* of a matroid M, the *rank* of M and the *closure* of a subset $A \subseteq E(M)$ are denoted by $E(M)$, $r(M)$, $\bar{A}$, respectively. A *loopless* matroid is a matroid which has no single element set with rank zero. Let A and B be flats of M. Then (A, B) is a *modular pair of flats* if $r(A) + r(B) = r(A \cup B) + r(A \cap B)$. If F is a flat of M such that (F, A) is a modular pair for all flats A, then F is a *modular flat* of M.

By a *join* of two matroids M and N relative to a common submatroid S, we mean a matroid on the point set consisting of the disjoint union of $M - S$, $N - S$ and S, the flats of which are all subsets F such that $F \cap M$ is a flat of M and $F \cap N$ is a flat of N.

By an *injective pushout* of two matroids M and N relative to a common submatroid S, we mean a colimit for the diagram in Figure 1 where i_M and i_N are non-rank-decreasing injective strong maps. We will show the existence of the injective pushout guarantees the existence of the join. In fact, the join is isomorphic to the injective pushout.

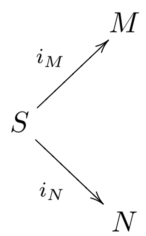


Fig. 1 Injective pushout of M and N relative to S.

For a complete background on the previous notions and the following ones, the reader is referred to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

2. Joins of matroids

We begin this section by recalling the following result which is needed to prove Theorem 1:

Lemma 1 ([16]). *Let F be a flat of a matroid M.Then F is a modular flat if and only if $r(F) + r(A) = r(M)$ for all complements A of F.*

Theorem 1. *Let M be a loopless matroid with a submatroid S and suppose that $r(S) + r(X) \leq r(M)$ for all flats X of M disjoint from S. Then S is a modular flat.*

Proof. If $\bar{S} \neq S$, then there exists a point $c \in \bar{S} - S$. Let X be a subset of M satisfying $X \cap \bar{S} = \{c\}$ and $X \cup \bar{S} = M$. Then by the semimodularity of the rank and as $r(S) + r(X) = r(\bar{S}) + r(X)$,

$$r(S) + r(X) \geq r(X \cup \bar{S}) + r(X \cap \bar{S}) = r(M) + r(c) > r(M), \tag{1}$$

and $S \cap X = \varnothing$, which is a contradiction to the assumption. Hence S is a flat. By the semimodularity of the rank for every complement X of S,

$$r(S) + r(X) \geq r(S \cup X) + r(S \cap X) = r(\overline{S \cup X}) = r(M),$$

and then by assumption $r(S) + r(X) = r(M)$. Hence S is modular by Lemma 1. □

Next, we recall the following two results from [16]:

Lemma 2. *Suppose that T is a modular flat of $M$and every non-loop element of $\overline{T} - T$ is parallel to some element of T.Then T is fully embedded in M.*

Lemma 3. *Let M be a matroid on a set E and suppose that, for some subset T of E, the matroid $M/T = M1 \oplus M2$. If T is a modular flat of the simple matroid associated with $M \setminus (E(M_2)$, then*

$$M = P_{M|T}(M \setminus (E(M_2), M \setminus (E(M_1)).$$

Next, we look at some sufficient conditions for a join to be exist. The proofs of the first two theorems follow from Lemma 2 and Lemma 3 combined with Theorem 1.

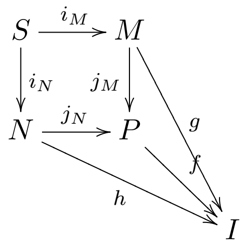
Theorem 2. *Let M_1 be a loopless matroid, M_2 be a matroid and T be the intersection of the ground sets of M_1 and M_2. If $\overline{T}^{M_1}$ satisfies the rank property in Theorem 1 and every loop element of $\overline{T}^{M_1} - T$ is parallel to some element of T, then the join of M_1 and M_2 relative to $M_1|T$ exists, termed the generalized parallel connection $P_{M_1|T}(M_1, M_2)$.*

Theorem 3. *Let M be a matroid on a set E and suppose that, for some subset T of E, the matroid $M/T = M_1 \oplus M_2$, $\widetilde{T}$ be the simple matroid associated with $M_1|T$ and $\widetilde{M\backslash E(M_2)}$ the simple matroid associated with $M\backslash E(M_2)$. If $r(\widetilde{T}) + r(X) \leq r(\widetilde{M\backslash E(M_2)})$ for all flats X of $\widetilde{M\backslash E(M_2)}$ disjoint from $\widetilde{T}$, then $P_{M_1|T}(M\backslash E(M_2), M\backslash E(M_1))$ exists, termed the matroid M.*

Injective pushouts of matroids M and N relative to a common submatroid S have been known to exist for S equal to the empty set in which case it is the direct sum; and for S equal to a point in which case it is the parallel connection. Let S be the rank zero matroid with the points consisting of the disjoint union of $M-S$, $N-S$ and S. Then the identity maps from M and N into S are strong, so that by the unique existence of the colimit map $P \longrightarrow S$, the points of an injective pushout when it exists can be identified with the point set consisting of the disjoint union of $M-S$, $N-S$ and S so that it is a combinatorial geometry. Now we are ready to prove our main theorem which is an extremal matroid result, that the existence of the injective pushout guarantees the existence of the join.

Theorem 4. *If P is an injective pushout of matroids M and N relative to a common submatroid S, then the join of M and N relative to S exists and is isomorphic to P.*

Proof. By assumption there are strong maps $j_M : M \longrightarrow P$ and $j_N : N \longrightarrow P$ such that $j_M i_M = j_N i_N$. Also if I is a matroid and $g : M \longrightarrow I$ and $h : N \longrightarrow I$ are strong maps for which $gi_M = hi_N$, then there exists a unique strong map $f : P \longrightarrow I$ which make the diagram in Figure 2 commutative. By the paragraph preceding this theorem, j_M and j_N are injective and the point set of P is consisting of the disjoint union of $M - S$, $N - S$ and S. Let $K \subseteq P$ and assume $j_M^{-1}(K \cap M)$ and $j_N^{-1}(K \cap N)$ are flats of M and N, respectively. We need only show K is a flat of P since then P is the join of submatroids isomorphic to M and N relative to a common submatroid isomorphic to S. Let I be the matroid with a single loop y and $(P - K)$ parallel elements. Define a strong map $g : M \longrightarrow I$ by $g(z) = z$ when $z \in M - j_M^{-1}(K)$, and $g(z) = y$ when $z \in j_M^{-1}(K)$. Define a strong map $h : N \longrightarrow I$ similarly. For the strong map $f : P \longrightarrow I$, which makes the diagram in Figure 2 commutative, we find that $f(z) = z$ when $z \in P - K$ and $f(z) = y$ when $z \in K$. It follows that $K = f^{-1}(y)$ is a flat, which was to be proved. □



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