Limiting direction of Julia sets and infinite radial order of solutions to complex linear differential equations

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Abstract. In this paper we find that for infinite order entire functions, the ray where it takes infinite radial order is a common limiting directions of Julia sets of their derivatives and their primitives. Applying this result to the solutions of some complex differential equations, we obtain the lower bound of the measure of sets of common limiting directions of Julia sets of the derivatives and integral primitives of any non-trivial solution of these equations, which give alternative proofs of previous results.
Keywords: radial order, limiting direction, Julia set, complex differential equation.

1. Introduction and main results

In this paper, we assume the reader is familiar with standard notations and basic results of Nevanlinna theory in the complex plane $\mathbb{C}$ and in an angle domain; see [5, 9, 19]. We use $\sigma(g)$ and $\mu(g)$ to denote the order and lower order of meromorphic function g in the complex plane respectively; see [19, p.10] for the definitions. Following [5], we give the notations of Nevanlinna theory in the angle. Set

$$\Omega(\alpha,\beta)=\{z\in\mathbb{C}:\alpha<\arg z<\beta\},\ \Omega(\alpha,\beta,r)=\{z:z\in\Omega(\alpha,\beta),|z|<r\}$$

and denote by $\overline{\Omega}(\alpha,\beta)$ the closure of $\Omega(\alpha,\beta)$. Let $g(z)$ be meromorphic on the closed angle $\overline{\Omega}(\alpha,\beta)$, where $\beta-\alpha\in(0,2\pi]$. Define

$$A_{\alpha,\beta}(r,g)=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\{\log^+|g(te^{i\alpha})|+\log^+|g(te^{i\beta})|\}\frac{dt}{t};$$
$$B_{\alpha,\beta}(r,g)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|g(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta;$$
$$C_{\alpha,\beta}(r,g)=2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^\omega}-\frac{|b_n|^\omega}{r^{2\omega}}\right)\sin\omega(\beta_n-\alpha),$$

where $\omega=\pi/(\beta-\alpha)$, and $b_n=|b_n|e^{i\beta_n}$ are poles of $g(z)$ in $\overline{\Omega}(\alpha,\beta)$ appearing according to their multiplicities. Thus, the Nevanlinna angular characteristic is defined as

$$S_{\alpha,\beta}(r,g) = A_{\alpha,\beta}(r,g) + B_{\alpha,\beta}(r,g) + C_{\alpha,\beta}(r,g).$$

Moreover, the order of $S_{\alpha,\beta}(r,g)$ is defined by

$$\rho_{\alpha,\beta}(g) = \limsup_{r\to\infty} \frac{\log S_{\alpha,\beta}(r,g)}{\log r}.$$

In addition, if $g(z)$ is analytic on the angle $\overline{\Omega}(\alpha,\beta)$, we define the order of g on $\Omega(\alpha,\beta)$ by

$$\sigma_{\alpha,\beta}(g) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,\Omega(\alpha,\beta),g)}{\log r},$$

where $M(r,\Omega(\alpha,\beta),g) = \sup_{\alpha\le\theta\le\beta} |g(re^{i\theta})|$. If $g(z)$ is analytic on $\mathbb{C}$, the order $\sigma(g)$ of g satisfies $\sigma(g) \ge \sigma_{\alpha,\beta}(g)$. Moreover, the sectorial order $\sigma_{\theta,\varepsilon}(g)$ and the radial order $\sigma_\theta(g)$ are defined by

$$\sigma_{\theta,\varepsilon}(g) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),g)}{\log r}, \quad \sigma_\theta(g) = \lim_{\varepsilon\to 0} \sigma_{\theta,\varepsilon}(g).$$

Similarly, the sectorial, respectively radial, exponent of convergence for zeros of $g(z)$ are defined by

$$\lambda_{\theta,\varepsilon}(g) = \limsup_{r\to\infty} \frac{\log^+ n(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),g=0)}{\log r}, \quad \lambda_\theta(g) = \lim_{\varepsilon\to 0} \lambda_{\theta,\varepsilon}(g),$$

where $n(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),g=0)$ stands for the number of zeros of $g(z)$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon,r)$ counting multiplicity.

Definition 1.1 *Let $f(z)$ be a transcendental meromorphic function of order σ. The ray $\arg z = \theta$ is called a Borel direction of f if for any $\varepsilon > 0$, $\lambda_{\theta,\varepsilon}(f-a) = \sigma$ with at most two exceptional value $a \in \mathbb{C}\cup\{\infty\}$.*

Some basic knowledge of complex dynamics of meromorphic functions is also needed; see [3, 22]. We define $f^n, n \in \mathbb{N}$ denote the nth iterate of f. The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of the plane $\mathbb{C}$ where the iterates f^n of f form a normal family. The complement of $F(f)$ in $\mathbb{C}$ is called the Julia set $J(f)$ of f. It is well known that $F(f)$ is open and completely invariant under f, $J(f)$ is closed and non-empty. Given $\theta \in [0,2\pi)$, if $\Omega(\theta-\varepsilon,\theta+\varepsilon)\cap J(f)$ is unbounded for any $\varepsilon > 0$, then we call the ray $\arg z = \theta$ the radial distribution of $J(f)$. Define $meas\Delta(f)$ is the linear measure of $\theta \in [0,2\pi)$ such that $J(f)$ has the radial distribution with respect to $\arg z = \theta$.

Baker [2] first proved that the Julia set of a transcendental entire function can not lie in finite many rays emanating from the original point. Later, Qiao [11] proved the measure of limiting direction of Julia set of finite lower order transcendental entire function f satisfies $meas\Delta(f) \ge \min\{2\pi, \frac{\pi}{\mu(f)}\}$. Moreover, for a transcendental entire function of finite lower order, Qiao [12] also found out the lower bounded of the measure of the common limiting direction of Julia set of its derivatives and its primitive. There are also some other papers related to this aspect; see [10, 12, 13, 16].

For transcendental meromorphic function, Zheng [23] has given significant results. Recent years, many results about the measure of limiting directions of Julia sets of solutions of complex linear differential equations have been appeared; see [6, 7, 15, 20, 21].

For a transcendental entire function $f(z)$ of infinite order, it is easy to see that there exist some angular domains $\Omega(\alpha,\beta)$ such that $\sigma_{\alpha,\beta}(f)=\infty$. But, $\sigma(f)=\infty$ cannot guarantee $\sigma_{\alpha,\beta}(f)=\infty$ for any angular domain $\Omega(\alpha,\beta)$. For example, it's known that $\sigma(\exp\{\exp z\})=\infty$, while $\sigma_{\frac{\pi}{2},\frac{3\pi}{2}}(\exp\{\exp z\})=0$. We know that a main study content about the complex differential equations is the estimation of the order of their solutions. There has many results about the infinite order of the solutions to complex differential equations under various conditions. Naturally, a question that how wide are the angular domains $\Omega(\alpha,\beta)$ such that $\sigma_\theta(f)=\infty$ for any ray $\theta=\arg z\in\Omega(\alpha,\beta)$ is raised. For convenience of the following, set $I(f)=\{\theta\in[0,2\pi):\sigma_\theta(f)=\infty\}$ for infinite order entire function f. By the idea of study the measure of Julia set of solution of complex differential equations, Huang and Wang [8] obtained the lower bound of $I(f)$ of the non-trivial solutions of second order linear differential equations. In order to give subsequent results, we firstly give the following theorem about the high order differential equation.

Theorem 1.1. *Let $A_i(z)(i=0,1,\ldots,n-1)$ be entire functions of finite lower order such that A_0 is transcendental and $T(r,A_i)=o(T(r,A_0)),(i=1,2,\ldots,n-1)$ as $r\to\infty$. Then every non-trivial solution f of the equation*

$$f^{(n)}+A_{n-1}f^{(n-1)}+\ldots+A_0f=0 \tag{1.1}$$

satisfies $measI(f)\ge\min\{2\pi,\pi/\mu(A_0)\}$. Moreover, if $\sigma(A_i)<\sigma(A_0),(i=1,2,\cdots,n-1)$, then there exist a closed interval $I_0\subseteq I(f)$ with $meas(I_0)\ge\min\{2\pi,\pi/\mu(A_0)\}$.

Comparing the proof of [8, Theorem 1.3] and that of [7, Theorem 1.1], we can see that the method of finding the lower bound of $measI(f)$ and $meas\Delta(f)$ of the solutions of complex differential equations is similar in some sense. Therefore, there may has special relations between the infinite radial order and limiting direction of Julia set for infinite order entire function f. Indeed, we find the following relationship.

Theorem 1.2 *Suppose that f is an entire function of infinite order, then for any $\theta\in I(f)$, the ray $\arg z=\theta$ is a common limiting direction of Julia sets of $f^{(n)}$, where $f^{(n)}$ denotes the n-th derivative or the n-th integral primitive of f, for $n\ge0$ or $n<0$, respectively.*

Then by the above Theorems 1.1 and 1.2, we obtain the following result, which has been proved by Chen and Wang [15]. Our method is different from that in [15].

Corollary 1.3 *Suppose f is any non-trivial solution of* (1.1) *in Theorem 1.1, then the measure of set of common limiting direction of Julia sets of $f^{(n)}$ satisfies $meas(\bigcap_{n\in\mathbf{z}}\Delta(f^{(n)}))\geq\min\{2\pi,\pi/\mu(A_0)\}$.*

In [15], the authors also proved that every solution of a second order linear differential equation, the coefficient of which has finite deficient value, is of infinite lower order, see the following.

Theorem 1.4 *Suppose that A_0 is a transcendental entire function and $T(r,A_0)\sim\log M(r,A_0)$ as $r\to\infty$ outside a set of finite logarithmic measure, $A_1(z)$ is a finite order entire function and has a finite deficient value a, i.e. $\delta(a,A_1)>0$, then every non-trivial solution f of*

$$f''+A_1(z)f'+A_0(z)f=0 \tag{1.2}$$

satisfies $\mu(f)=\infty$.

Together the proof of above theorem with the proof of [8, Theorem 1.3], we can obtain the lower bound of $measI(f)$ for the non-trivial solution of equation (1.2).

Theorem 1.5 *Suppose that f is a non-trivial solution of* (1.2)*, where $A_0(z)$ and $A_1(z)$ satisfy the conditions in Theorem 1.4, then*

$$measI(f)\geq\min\left\{2\pi,\frac{4}{\mu(A_1)}\arcsin\sqrt{\frac{\delta(a,A_1)}{2}}\right\}.$$

Combining the above theorem and Theorem 1.2, we can obtain the following result easily, which is proved in [15] by another method.

Theorem 1.6 *Under the hypotheses of Theorem 1.4, the measure of set of common limiting direction of Julia sets of $f^{(n)}$ satisfies*

$$meas(\bigcap_{n\in\mathbf{z}}\Delta(f^{(n)}))\geq\min\left\{2\pi,\frac{4}{\mu(A_1)}\arcsin\sqrt{\frac{\delta(a,A_1)}{2}}\right\}.$$

2. Preliminary lemmas

We call W is a hyperbolic domain if $\overline{\mathbb{C}}\backslash W$ contains at least three points, where $\overline{\mathbb{C}}$ is the extended complex plane. For an $a\in\mathbb{C}\backslash W$, define

$$C_W(a)=\inf\{\lambda_W(z)|z-a|:\forall z\in W\},$$

where $\lambda_W(z)$ is the hyperbolic density on W. It's well known that, if every component of W is simply connected, then $C_W(a)\geq 1/2$; see [23].

Lemma 2.1 ([23, Lemma 2.2]). *Let $f(z)$ be analytic in $\Omega(r_0, \theta_1, \theta_2)$, U be a hyperbolic domain, and $f : \Omega(r_0, \theta_1, \theta_2) \to U$. If there exists a point $a \in \partial U \backslash \{\infty\}$ such that $C_U(a) > 0$, then there exists a constant $d > 0$ such that, for sufficiently small $\varepsilon > 0$, we have*

$$|f(z)| = O(|z|^d), \quad z \to \infty, \; z \in \Omega(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon).$$

The next lemma shows some estimates for the logarithmic derivative of functions being analytic in an angle. Before this, we recall the definition of R-set; for reference, see [9]. Set $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$. If $\sum_{n=1}^{\infty} r_n < \infty$ and $z_n \to \infty$, then $\bigcup_{n=1}^{\infty} B(z_n, r_n)$ is called an R-set. Clearly, the set $\{|z| : z \in \bigcup_{n=1}^{\infty} B(z_n, r_n)\}$ is of finite linear measure.

Lemma 2.2 ([7, Lemma 2.2]). *Let $z = re^{i\psi}, r_0 + 1 < r$ and $\alpha \leq \psi \leq \beta$, where $0 < \beta - \alpha \leq 2\pi$. Suppose that $n(\geq 2)$ is an integer, and that $g(z)$ is analytic in $\Omega(r_0, \alpha, \beta)$ with $\rho_{\alpha,\beta}(g) < \infty$. Choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon_j \in (0, (\beta_j - \alpha_j)/2)(j = 1, 2, \dots, n-1)$ outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \dots, n-1,$$

there exists $K>0$ and $M>0$ only depending on g, $\varepsilon_1, \dots, \varepsilon_{n-1}$ and $\Omega(\alpha_{n-1}, \beta_{n-1})$, and not depending on z, such that

$$\left|\frac{g'(z)}{g(z)}\right| \leq Kr^M (\sin k(\psi - \alpha))^{-2}$$

and

$$\left|\frac{g^{(n)}(z)}{g(z)}\right| \leq Kr^M \left(\sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_{\varepsilon_j}(\psi - \alpha_j)\right)^{-2},$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R-set D, where $k = \pi/(\beta - \alpha)$ and $k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j)(j = 1, 2, \dots, n-1)$.

Lemma 2.3 ([18, 22]). *Let $f(z)$ be a transcendental meromorphic function with lower order $\mu(f) < \infty$ and order $0 < \sigma(f) \leq \infty$. Then, for any positive number λ with $\mu(f) \leq \lambda \leq \sigma(f)$ and any set H of finite measure, there exists a sequence $\{r_n\}$ satisfies:*

(1) $r_n \notin H, \lim_{n\to\infty} r_n/n = \infty$;

(2) $\liminf_{n\to\infty} \log T(r_n, f)/\log r_n \geq \lambda$;

(3) $T(r, f) < (1 + o(1))(2t/r_n)^{\lambda} T(r_n/2, f), t \in [r_n/n, nr_n]$;

(4) $t^{-\lambda-\varepsilon_n} T(t, f) \leq 2^{\lambda+1} r_n^{-\lambda-\varepsilon_n} T(r_n, f), 1 \leq t \leq nr_n, \varepsilon_n = (\log n)^{-2}$.

Such $\{r_n\}$ is called a sequence of Pólya peaks of order λ outside H. The following lemma, which related to Pólya peaks, is called the spread relation; see [1].

Lemma 2.4 ([1]). *Let $f(z)$ be a transcendental meromorphic function with positive order and finite lower order, and has a deficient value $a \in \overline{\mathbb{C}}$. Then, for any sequence of Pólya peaks $\{r_n\}$ of order $\lambda > 0$, $\mu(f) \leq \lambda \leq \sigma(f)$, and any positive function$\Upsilon(r) \to 0$ as $r_n \to \infty$, we have*

$$\liminf_{r_n \to \infty} meas D_\Upsilon(r_n, a) \geq \min\left\{2\pi, \frac{4}{\lambda} \arcsin\sqrt{\frac{\delta(a,f)}{2}}\right\},$$

where

$$D_\Upsilon(r,a) = \left\{\theta \in [0, 2\pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \Upsilon(r) T(r,f)\right\}, \ a \in \mathbb{C}$$

and

$$D_\Upsilon(r,\infty) = \left\{\theta \in [0, 2\pi) : \log^+ |f(re^{i\theta})| > \Upsilon(r) T(r,f)\right\}.$$

For the Borel directions of entire functions with infinite order, Sun [14] obtained the following lemma.

Lemma 2.5 ([14]). *Let g be an entire function of infinite order, then the ray* $\arg z = \theta$ *is a Borel direction of infinite order for g if and only if* $\arg z = \theta$ *is a Borel direction of infinite order for g'.*

The following lemma is a weaker version of Chuang's result.

Lemma 2.6 ([4]). *Let f be a meromorphic function of infinite order, then the ray* $\arg z = \theta$ *is one Borel direction of infinite order of f if and only if f satisfies the equality*

$$\limsup_{r \to \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r,f)}{\log r} = \infty,$$

for any $\varepsilon \in (0, \pi/2)$.

Lemma 2.7 ([21]). *Let $f(z)$ be a transcendental entire function. If $\sigma_\theta(f) = \sigma(f)$, then the ray* $\arg z = \theta$ *is a radial distribution of the Julia set of f.*

Lemma 2.8 ([17, Corollary 2.3.6]). *If $g(z)$ is an entire function with $0 < \sigma(g) < \infty$, then there exists an angular domain $\Omega(\theta_1, \theta_2)$ with $\theta_2 - \theta_1 \geq \pi/\sigma(g)$ such that*

$$\limsup_{r \to \infty} \frac{\log^+ \log^+ |g(re^{i\theta})|}{\log r} = \sigma(g),$$

for any $\theta \in (\theta_1, \theta_2)$.

3. Proof of Theorems

Proof of Theorem 1.1

Suppose that f is a non-trivial solution of equation (1.1) under the hypotheses of this theorem. From [7, p.479] we know that $\sigma(f) \geq \mu(f) = \infty$. We assume that $measI(f) < \nu := \min\{2\pi, \pi/\mu(A_0)\}$, so $\zeta := \nu - measI(f) > 0$. Clearly $S = (0, 2\pi)\backslash\overline{I(f)}$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i := (\alpha_i, \beta_i), i = 1, 2, \cdots, m$ satisfying $[\alpha_i, \beta_i] \subset S$ and $meas(S\backslash\bigcup_{i=1}^{m} I_i) < \frac{\zeta}{4}$. For the angular domain $\Omega(\alpha_i, \beta_i)$, it is easy to see

$$\Omega(\alpha_i, \beta_i) \cap I(f) = \emptyset.$$

This implies that for each $i = 1, 2, \cdots, m$, we have $\sigma_{\alpha_i,\beta_i}(f) < \infty$, and from the definition of $\rho_{\alpha_i,\beta_i}(f)$ and [22, Corollary 2.2.2], we have $\rho_{\alpha_i,\beta_i}(f) < \infty$. Therefore, by Lemma 2.7, for sufficiently small $\varepsilon > 0$, there exist two constants $M > 0$ and $K > 0$ such that

$$\left|\frac{f^{(s)}(z)}{f(z)}\right| \leq Kr^{M}, s = 1, 2, \cdots, n, \tag{3.1}$$

for all $z \in \bigcup_{i=1}^{m} \Omega(\alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$, outside an R-set H.

Applying Lemma 2.3 to $A_0(z)$, there exist a sequence of Pólya peak $\{r_n\}$ of order $\mu(A_0)$ such that $r_n \notin \{|z|, z \in H\}$, and for sufficiently large n,

$$meas\{D_{\Upsilon}(r_n, \infty)\} \geq \nu - \frac{\zeta}{4}, \tag{3.2}$$

where we take the function $\Upsilon(r)$ as

$$\Upsilon(r) = \max\left\{\sqrt{\frac{\log r}{T(r, A_0)}}, \sqrt{\frac{T(r, A_1)}{T(r, A_0)}}, \cdots, \sqrt{\frac{T(r, A_{n-1})}{T(r, A_0)}}\right\}.$$

Without loss of generality, we assume that (3.2) holds for all n, and simplified denote $D(r_n) = D_{\Upsilon}(r_n, \infty)$. Obviously,

$$\begin{aligned} meas(D(r_n) \cap S) &= meas(D(r_n)\backslash(I(f) \cap D(r_n))) \\ &\geq measD(r_n) - measI(f) > \frac{3\zeta}{4}. \end{aligned} \tag{3.3}$$

Then, for each n we have

$$\begin{aligned} meas((\bigcup_{i=1}^{m} I_i) \cap D(r_n)) &= meas(S \cap D(r_n)) - meas((S\backslash\bigcup_{i=1}^{m} I_i) \cap D(r_n)) \\ &> \frac{3\zeta}{4} - \frac{\zeta}{4} = \frac{\zeta}{2} > 0. \end{aligned} \tag{3.4}$$

This means there exist at least one open interval $I_{i_0}=(\alpha,\beta)$ of $I_i,(i=1,2,\cdots,m)$ such that for infinitely many j,

$$meas(D(r_j)\cap(\alpha,\beta)) > \frac{\zeta}{2m} > 0. \tag{3.5}$$

Set $G_j = D(r_j)\cap(\alpha+2\varepsilon,\beta-2\varepsilon)$, it follows from the definition of $D(r_j)$ in Lemma 2.4, $T(r,A_0)=m(r,A_0)$ and (3.5) that

$$\int_{G_j}\log^+|A_0(r_je^{i\theta})|d\theta \geq meas(G_j)\Upsilon(r_j)m(r_j,A_0) \geq \frac{\zeta}{4m}\Upsilon(r_j)m(r_j,A_0). \tag{3.6}$$

We rewrite (1.1) as

$$A_0 = -\left(\frac{f^{(n)}}{f} + A_{n-1}\frac{f^{(n-1)}}{f} + \cdots + A_1\frac{f'}{f}\right). \tag{3.7}$$

Substituting (3.1) into (3.7) yields

$$\begin{aligned}\int_{G_j}\log^+|A_0(r_je^{i\theta})|d\theta &\leq \int_{G_j}\left(\sum_{i=1}^{n-1}\log^+\left|A_i(r_je^{i\theta})\right|\right)d\theta + O(\log r_j)\\ &\leq \sum_{i=1}^{n-1}m(r_j,A_i) + O(\log r_j).\end{aligned} \tag{3.8}$$

This and (3.6) give out

$$\frac{\zeta}{4m}\Upsilon(r_j)m(r_j,A_0) \leq \sum_{i=1}^{n-1}m(r_j,A_i) + O(\log r_j) \tag{3.9}$$

which is impossible since A_0 is transcendental and $T(r,A_i) = o(T(r,A_0))(i = 1,2,\cdots,n-1)$ as $r\to\infty$. Thus, we deduce that $measI(f) \geq \min\{2\pi,\pi/\mu(A_0)\}$.

In the following, we consider the case $\sigma(A_i) < \sigma(A_0)(i = 1,2,\cdots,n-1)$. By Lemma 2.8, there exists an interval (a,b) with $b-a \geq \min\{2\pi,\frac{\pi}{\sigma(A_0)}\}$ such that, for $\theta\in(a,b)$,

$$\limsup_{r\to\infty}\frac{\log^+\log^+|A_0(re^{i\theta})|}{\log r} = \sigma(A_0). \tag{3.10}$$

We shall prove $[a,b]\subset I(f)$. Assume that $[a,b]\not\subset I(f)$, then $(a,b)\setminus I(f)$ is bounded and open, so there must have $(\alpha,\beta)\subset(a,b)$ such that $\sigma_{\alpha,\beta}(f) < +\infty$. Thus, (3.1) still holds for $z\in\Omega(\alpha+\varepsilon,\beta-\varepsilon)$ outside an R-set H for sufficiently small ε. Since $\{r=|z| : z = re^{i\theta}\in H\}$ is a set of finite linear measure, then the set of θ, where the ray $\theta = \arg z$ meets R-set H infinitely many times, is of measure zero. Thus, we can find $\theta_0\in(\alpha+\varepsilon,\beta-\varepsilon)\in(a,b)$ such that the ray $\arg z = \theta_0$ meets H finitely many times. Furthermore, combining (3.1) and (3.7) yields, for sufficiently large r,

$$\begin{aligned}
\log^+ |A_0(re^{i\theta_0})| &\le \sum_{i=1}^{n} \log^+ \left| \frac{f^{(i)}(re^{i\theta_0})}{f(re^{i\theta_0})} \right| + \sum_{i=1}^{n-1} \log^+ \left| A_i(re^{i\theta_0}) \right| + O(1) \\
&= \sum_{i=1}^{n-1} \log^+ \left| A_i(re^{i\theta_0}) \right| + O(\log r) \\
&\le r^{\sigma(A_0)-\varepsilon} \qquad (3.11)
\end{aligned}$$

which contradicts with (3.10). Hence, we prove that $[a, b] \in I(f)$. Thus, we complete the proof.

Proof of Theorem 1.2

Following Lemma 2.7, we can get that, for any $\theta \in I(f)$, the ray $\arg z = \theta$ must be a limiting direction of Julia set of f. Moreover, by Lemma 2.6, for any $\theta \in I(f)$, the ray $\arg z = \theta$ must be one infinite order Borel direction of f. Then, by Lemma 2.5, the ray $\arg z = \theta$ is also a Borel direction of infinite order for f' and F, setting which a primitive function of f. Applying Lemma 2.6 to f' and F, we obtain that the radial order of f' and F at $\arg z = \theta$ is infinity. Finally, by Lemma 2.7, we deduce that the ray $\arg z = \theta$ is not only a limiting direction of Julia sets of f', but also a limiting direction of Julia sets of F.

Repeating the above arguments infinitely many times, we can obtain that $\arg z = \theta$ is a common limiting direction of Julia set of $f^{(n)}$, where $f^{(n)}$ denotes the n-th derivative or the n-th integral primitive of f for $n \ge 0$ or $n < 0$, respectively.

Proof of theorem 1.5

We shall prove the conclusion by reduction to absurdity. We firstly assume that $measI(f) < \nu := \min\{2\pi, \frac{4}{\mu(A_1)} \arcsin \sqrt{\frac{\delta(a,A_1)}{2}}\}$, then $\zeta := \nu - measI(f) > 0$. For given $0 < c < 1$, set $I_c(r) = \{\theta \in [0, 2\pi) : \log |A_0(re^{i\theta})| < c \log M(r, A_0)\}$. The definition of proximity function yields that

$$\begin{aligned}
T(r, A_0) = m(r, A_0) \le &\left(1 - \frac{measI_c(r)}{2\pi}\right) \log M(r, A_0) \\
&+ c\left(\frac{measI_c(r)}{2\pi}\right) \log M(r, A_0).
\end{aligned}$$

Since $T(r, A_0) \sim \log M(r, A_0)$ outside a set F of finite linear measure, we have $measI_c(r) \to 0$ as $r \notin F \to \infty$. By Lemma 2.4, we can take an increasing and unbounded sequence $\{r_k\}$ such that $measD(r_k) \ge \nu - \frac{\zeta}{4}$, where $D(r) = \{\theta \in [0, 2\pi) : \log |A_1(re^{i\theta}) - a| < 1\}$, all $r_j \notin \{|z| : z \in H\} \cup F$ with H being an $R-$ set. Clearly, $|A_1(r_k e^{i\theta})| \le e + |a|$ for $\theta \in D(r_k)$. Similarly as in the proof of Theorem

1.1, there always exists an open interval $I_{i_0} = (\alpha, \beta)$ of $I_i, (i = 1, 2, \cdots, m)$ such that for infinitely many j,

$$meas(D(r_j) \cap (\alpha, \beta)) > \frac{\zeta}{2m} > 0, \tag{3.12}$$

and (3.1) still holds in $\Omega(r, \alpha + \varepsilon, \beta - \varepsilon)$. Hence, substituting (3.1) into

$$|A_0(z)| \le \left| \frac{f''(z)}{f(z)} \right| + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right| \tag{3.13}$$

yields

$$M(r_k, A_0)^c \le |A_0(r_k e^{i\theta})| \le (|a| + e + 1) K r_k^M,$$

for $\theta \in (D(r_k) \cap I_0) \setminus I_c(r_k)$. It is impossible since A_0 is transcendental. Then we complete the proof.

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