A note on unitarily invariant norm inequalities for accretive-dissipative operator matrices

Junjian Yang
School of Mathematical Sciences
Guizhou Normal University
Guiyang
P. R. China
and
Hainan Key Laboratory for Computational Science and Application
P. R. China
junjianyang1981@163.com

Abstract. In this paper, we present a unitarily invariant norm inequality for accretive-dissipative operator matrices, which is similar to an inequality obtained by Zhang in [J. Math. Anal. Appl. 412 (2014) 564-569]. Examples are provided to show that neither Zhang's inequality nor our inequality is uniformly better than the other.
Keywords: unitarily invariant norms, accretive-dissipative operators, inequalities.

1. Introduction

In this note, we use the same notation as in [11, 14]. For convenience, recall that, as usual, let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For $\mathbf{H} := \mathcal{H} \oplus \mathcal{H}$ and $T \in \mathcal{B}(\mathbf{H})$, the operator T can be represented as a 2×2 operator matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ with $T_{jk} \in \mathcal{B}(\mathcal{H})$, $j, k = 1, 2$.

For any $T \in \mathcal{B}(\mathbf{H})$, we can write

$$T = A + iB, \tag{1.1}$$

in which $A = \frac{T+T^*}{2}$ and $B = \frac{T-T^*}{2i}$ are Hermitian operators. This is the Cartesian decomposition of T. In this paper, we always represent the decomposition of (1.1) as follows,

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} + i \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{pmatrix}, \tag{1.2}$$

where $T_{jk}, A_{jk}, B_{jk} \in \mathcal{B}(\mathcal{H})$, $j, k = 1, 2$. Then $A_{12} = A_{21}^*$, $B_{12} = B_{21}^*$.

If T is a compact operator, we denote by $s_1(T) \geq s_2(T) \geq \cdots$ the eigenvalues of $(T^*T)^{\frac{1}{2}}$, which are called the singular values of T. Thus, whenever we talk about singular values, the operators are necessarily compact. We denote by

$W(A)$ the numerical range of A. A norm $\|\cdot\|_u$ on $\mathcal{B}(\mathcal{H})$ is unitarily invariant if $\|T\|_u = \|UTV\|_u$ for all unitaries $U, V \in \mathcal{B}(\mathcal{H})$. Every unitarily invariant norm is defined on an ideal in $\mathcal{B}(\mathcal{H})$. It will be implicity understood that the operator T is in this ideal when we talk of $\|T\|_u$. Recall that T with $T = A+iB$ is accretive-dissipative if both A and B are positive. For the study of accretive-dissipative matrices in matrix theory and numerical linear algebra, the readers can refer to [2, 3, 7, 8]. Recent works devoted to studying the accretive-dissipative operators or matrices are in [6, 9, 10].

Zhang [14, Theorem 2] obtained the following unitarily invariant norm inequality.

Theorem 1. *Let $T \in \mathcal{B}(\mathbf{H})$ be accretive-dissipative and partitioned as in (1.2). Then*

$$\|T\|_u \quad \leq \quad 2\|T_{11} + T_{22}\|_u \tag{1.3}$$

for any unitarily invariant norm $\|\cdot\|_u$.

However, there is a gap in the proof of Zhang [14, Theorem 2]. Since in the proof of Theorem 2 in [14] the author proves that the last equality

$$2\|A_{11} + B_{11} + i(A_{22} + B_{22})\|_u \quad = \quad 2\|T_{11} + T_{22}\|_u$$

holds, actually it is as follows:

$$\begin{aligned} 2\|A_{11} + B_{11} + i(A_{22} + B_{22})\|_u &\leq 2\|A_{11} + B_{11} + A_{22} + B_{22}\|_u \\ &\leq 2\sqrt{2}\|A_{11} + A_{22} + i(B_{11} + B_{22})\|_u \\ &= 2\sqrt{2}\|A_{11} + iB_{11} + A_{22} + iB_{22}\|_u \\ &= 2\sqrt{2}\|T_{11} + T_{22}\|_u. \end{aligned}$$

The purpose of this paper is to discuss unitarily invariant norm inequalities for the accretive-dissipative operator matrix (1.1), which are similar to the inequality (1.3) . Our main result is the following theorem.

Theorem 2. *Let $T \in \mathcal{B}(\mathbf{H})$ be accretive-dissipative and partitioned as in (1.2). Then*

$$\|T\|_u \leq \sqrt{2}[\|T_{11} + T_{22}\|_u + 2\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}] \tag{1.4}$$

for any unitarily invariant norm $\|\cdot\|_u$. Furthermore, if $0 \notin W(B_{12} + C_{12})$, then

$$\|T\|_u \leq \sqrt{2}[\|T_{11} + T_{22}\|_u + \|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}]. \tag{1.5}$$

2. Main results

Before proving the main theorem of this paper, we need a few auxiliary results.

Lemma 3 ([12]). *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then for any complex number z,*

$$\prod_{j=1}^{k} s_j(A + zB) \quad \leq \quad \prod_{j=1}^{k} s_j(A + |z|B)$$

for all $k = 1, 2, \ldots$. As a consequence,

$$\sum_{j=1}^{k} s_j(A + zB) \quad \leq \quad \sum_{j=1}^{k} s_j(A + |z|B)$$

for all $k = 1, 2, \ldots$.

Lemma 4 ([4, Corollary 2.1]). *If $A, B, X \in \mathcal{B}(\mathcal{H})$ and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is positive, then we have the following decomposition*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \quad = \quad U \begin{pmatrix} \dfrac{A+B}{2} + \mathrm{Re}X & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{A+B}{2} - \mathrm{Re}X \end{pmatrix} V^*$$

for some unitary operator matrices $U, V \in \mathcal{B}(\mathbf{H})$.

Lemma 5 ([13, p. 42]). *The operator matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is positive if and only if both A and C are positive and there exists a contraction W such that $B = A^{\frac{1}{2}} W C^{\frac{1}{2}}$.*

Lemma 6 ([14, Lemma 2]). *Let $P_i, Q_i \in \mathcal{B}(\mathcal{H})$ be positive and let $C_i \in \mathcal{B}(\mathcal{H})$ be contractive, $i = 1, 2, \ldots, m$. Then*

$$\sum_{j=1}^{k} s_j \left(\sum_{i=1}^{m} P_i C_i Q_i \right) \leq \sum_{j=1}^{k} s_j \left((\sum_{i=1}^{m} P_i^2)^{\frac{1}{2}} \right) s_j \left((\sum_{i=1}^{m} Q_i^2)^{\frac{1}{2}} \right),$$

for all $k = 1, 2, \ldots$.

Lemma 7 ([1, Theorem 1.1]). *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then*

$$s_j(A + B) \quad \leq \quad \sqrt{2} s_j(A + iB) for all j = 1, 2, \ldots.$$

Remark 8. Reverse inequality of Lemma 7 was given in [5].

Lemma 9. *Let $T \in \mathcal{B}(\mathbf{H})$ be accretive-dissipative and partitioned as in* (1.2). *Then*

$$\|B_{12} + C_{12}\|_u \leq \sqrt{2} \|T_{11}\|_u^{\frac{1}{2}} \|T_{22}\|_u^{\frac{1}{2}}.$$

Proof. Compute

$$
\begin{aligned}
&\|B_{12}+C_{12}\|_u = \sum_{j=1}^{k}\alpha_j s_j(B_{12}+C_{12})\\
&= \sum_{j=1}^{\infty}\alpha_j s_j(B_{11}^{\frac{1}{2}}W_1B_{22}^{\frac{1}{2}}+C_{11}^{\frac{1}{2}}W_2C_{22}^{\frac{1}{2}}) && \text{(by Lemma 5)}\\
&\leq \sum_{j=1}^{\infty}\alpha_j s_j((B_{11}+C_{11})^{\frac{1}{2}})s_j((A_{22}+B_{22})^{\frac{1}{2}}) && \text{(by Lemma 6)}\\
&= \sum_{j=1}^{\infty}\alpha_j (s_j(B_{11}+C_{11}))^{\frac{1}{2}}(s_j(A_{22}+B_{22}))^{\frac{1}{2}}\\
&\leq \sum_{j=1}^{\infty}\alpha_j [\sqrt{2}s_j(B_{11}+iC_{11})]^{\frac{1}{2}}[\sqrt{2}s_j(B_{22}+iC_{22})]^{\frac{1}{2}} && \text{(by Lemma 7)}\\
&\leq \sqrt{2}\sum_{j=1}^{\infty}\alpha_j [s_j(T_{11})]^{\frac{1}{2}}[s_j(T_{22})]^{\frac{1}{2}}\\
&\leq \sqrt{2}(\sum_{j=1}^{\infty}\alpha_j s_j(T_{11}))^{\frac{1}{2}}(\sum_{j=1}^{\infty}\alpha_j s_j(T_{22}))^{\frac{1}{2}} && \text{(by Cauchy-Schwarz inequality)}\\
&= \sqrt{2}\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}.
\end{aligned}
$$

Thus,

$$\|B_{12}+C_{12}\|_u \leq \sqrt{2}\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}.$$

This completes the proof. □

Lemma 10 ([4, Corollary 2.6]). *If $A, B, X \in \mathcal{B}(\mathcal{H})$ and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is positive, then for $0 \notin W(X)$ we have*

$$\left\|\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\right\|_u \leq \|A+B\|_u + \|X\|_u$$

for any unitarily invariant norm.

Proof of Theorem 2. Compute

$$
\begin{aligned}
&\|B+iC\|_u \leq \|B+C\|_u && \text{(by Lemma 3)}\\
&\leq \left\|\frac{B_{11}+C_{11}+B_{22}+C_{22}}{2}+\mathrm{Re}(B_{12}+C_{12})\right\|_u\\
&+ \quad \left\|\frac{B_{11}+C_{11}+B_{22}+C_{22}}{2}-\mathrm{Re}(B_{12}+C_{12})\right\|_u
\end{aligned}
$$

(by Lemma 4 and triangle inequality)

$$\leq 2\left\|\frac{B_{11}+C_{11}+B_{22}+C_{22}}{2}\right\|_u + 2\left\|\mathrm{Re}(B_{12}+C_{12})\right\|_u \text{ (by triangle inequality)}$$

$$\leq \sqrt{2}\|B_{11}+B_{22}+i(C_{11}+C_{22})\|_u + 2\|\mathrm{Re}(B_{12}+C_{12})\|_u \qquad \text{(by Lemma 7)}$$

$$\leq \sqrt{2}\|T_{11}+T_{22}\|_u + 2\sqrt{2}\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}} \qquad \text{(by Lemma 9)}$$

$$\leq \sqrt{2}[\|T_{11}+T_{22}\|_u + 2\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}].$$

Thus,

$$\|B+iC\| \leq \sqrt{2}\left[\|T_{11}+T_{22}\|_u + 2\,\|T_{11}\|_u^{\frac{1}{2}}\,\|T_{22}\|_u^{\frac{1}{2}}\right].$$

Furthermore, if $0 \notin W(B_{12}+C_{12})$, then we have

$$\|B+iC\|_u \leq \|B+C\|_u \qquad \text{(by Lemma 3)}$$

$$\leq \|B_{11}+C_{11}+B_{22}+C_{22}\|_u + \|B_{12}+C_{12}\|_u \qquad \text{(by Lemma 10)}$$

$$\leq \sqrt{2}\|B_{11}+B_{22}+i(C_{11}+C_{22})\|_u + \|B_{12}+C_{12}\|_u \quad \text{(by Lemma 7)}$$

$$= \sqrt{2}\|T_{11}+T_{22}\|_u + \|B_{12}+C_{12}\|_u$$

$$\leq \sqrt{2}\|T_{11}+T_{22}\|_u + \sqrt{2}\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}} \qquad \text{(by Lemma 9)}$$

$$= \sqrt{2}[\|T_{11}+T_{22}\|_u + \|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}].$$

This completes the proof. □

The following examples show that neither (1.3) nor (1.4) is uniformly better than the other.

Example 1. Let

$$\begin{aligned} T &= B+iC \\ &= \begin{pmatrix} 0.001 & 0 \\ 0 & 2 \end{pmatrix} + i\begin{pmatrix} 0.001 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.001+0.001i & 0 \\ 0 & 2+1i \end{pmatrix}, \end{aligned}$$

then $T_{11} = 0.001 + 0.001i$, $T_{22} = 2 + i$.

For the right side of (1.3), $2\|T_{11}+T_{22}\|_u = 6.3283$. For the right side of (1.4), $\sqrt{2}[\|T_{11}+T_{22}\|_u+2\|T_{11}\|_u^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}] = 3.3232$. This shows that (1.4) is better than (1.3) in some cases.

Example 2. If

$$
\begin{aligned}
T &= B + iC \\
&= \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} + i \begin{pmatrix} 0.001 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0.001 \end{pmatrix} \\
&= \begin{pmatrix} 1+i*0.001 & -1 & 0 & 0 \\ -1 & 1+i*0.001 & 0 & 0 \\ 0 & 0 & 1+i*0.001 & 1 \\ 0 & 0 & 1 & 1+i*0.001 \end{pmatrix},
\end{aligned}
$$

then

$$T_{11} = \begin{pmatrix} 1+0.001i & -1 \\ -1 & 1+0.001i \end{pmatrix}$$

and

$$T_{22} = \begin{pmatrix} 1+0.001i & 1 \\ 1 & 1+0.001i \end{pmatrix}.$$

For the right side of (1.3), $2\sqrt{2}\|T_{11} + T_{22}\|_2 = 5.6583$. For the right side of (1.4), $\sqrt{2}[\|T_{11} + T_{22}\|_u + 2\|T_{11}\|_2^{\frac{1}{2}}\|T_{22}\|_u^{\frac{1}{2}}] = 8.4860$. This implies that (1.4) is weaker than (1.3) in some cases.

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