On coefficient inequalities for certain subclasses of meromorphic bi-univalent functions

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Abstract. In the present paper, we investigate and define two subclasses of meromorphic bi-univalent function class Σ' which are defined on the domain $\mathbb{U}^* = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Further, by using the well-known coefficients estimates of the Carathéodory functions (i.e functions with positive real part) we obtain the estimates on the coefficients $|b_0|$, $|b_1|$ and $|b_2 + b_0^3|$ for functions in these subclasses.

Keywords: analytic function, meromorphic function, univalent function, bi-univalent function, meromorphic bi-univalent function.

1. Introduction

Let the class $\mathcal{A} = \{f : \mathbb{U} \to \mathbb{C} : f$ is analytic in $\mathbb{U}$ and $f(0) = f'(0) - 1 = 0\}$ and its subclass $\mathcal{S} = \{f : \mathbb{U} \to \mathbb{C} : f \in \mathcal{A}$ and also univalent in $\mathbb{U}\}$ where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and such functions $f \in \mathcal{A}$ have the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

In 1972, Ozaki and Nunokawa [14] proved the following Lemma (univalence criterion). In fact, this result is appeared in the paper by Aksentév [1] (also see the paper by Aksentév and Avhadiev [2]).

Lemma 1.1. *If for* $f(z) \in \mathcal{A}$

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathbb{U}),$$

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then $f(z)$ *is univalent in* $\mathbb{U}$ *and hence* $f(z) \in \mathcal{S}$.

Also, a functions $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{T}(\mu)$, $(0 < \mu \leq 1)$ if

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu \quad (z \in \mathbb{U})$$

and $\mathcal{T}(1) = \mathcal{T}$. Clearly, $\mathcal{T}(\mu) \subset \mathcal{T} \subset \mathcal{S}$. Further (see Kuroki et al. [10]), for $f(z) \in \mathcal{T}(\mu)$ see that:

$$\Re \left(\frac{z^2 f'(z)}{(f(z))^2} \right) > 1 - \mu \quad (z \in \mathbb{U}).$$

In particular, for initial coefficient estimates of bi-univalent function classes $\mathcal{T}_\Sigma(\mu)$ and $\mathcal{T}_\Sigma^\alpha$, see the paper by Naik and Patil [12].

In 1967, Lewin [11] introduced and studied the bi-univalent function class Σ. After which some researchers (viz. [3, 13]) found the initial coefficient estimates for the functions in Σ. Later, Srivastava et al. [17] revived it for the subclasses of Σ. Recently, the concept of bi-univalent functions is extend to meromorphic bi-univalent functions.

Let $\mathcal{S}'$ denote the class of meromorphic univalent functions g of the form:

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \tag{1.2}$$

defined on the domain $\mathbb{U}^* = \{z : z \in \mathbb{C},\ 1 < |z| < \infty\}$. Clearly, $g \in \mathcal{S}'$ has an inverse say g^{-1}, defined by:

$$g^{-1}(g(z)) = z, \quad (z \in \mathbb{U}^*)$$

and

$$g(g^{-1}(w)) = w, \quad (0 < M < |w| < \infty),$$

which has a series expansion of the form:

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{c_n}{w^n}, \quad (0 < M < |w| < \infty).$$

Some simple computations using equation (1.2) shows that:

$$g^{-1}(w) = h(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \cdots . \tag{1.3}$$

Let $\Sigma' = \{g \in \mathcal{S}' : \text{both } g \text{ and } g^{-1} \text{ are meromorphic univalent in } \mathbb{U}^*\}$ denote the class of all meromorphic bi-univalent functions in $\mathbb{U}^*$. Recently the coefficient estimate on functions of various subclasses of Σ' were obtained by some researchers viz. Halim et al. [6], Hamidi et al. [7, 8], Panigrahi [15], Janani and Murugusundaramoorthy [9], Bulut [4], etc. In the present investigation, we define two new subclasses of the function class Σ' and obtain the estimate on $|b_0|$, $|b_1|$ and $|b_2 + b_0^3|$ for the functions in these new subclasses.

We need to recall the Carathéodory lemma in the following form to prove our main results (see [5], [16]).

Lemma 1.2. *If $p(z) \in \mathcal{P}$, the class of all functions analytic in $\mathbb{U}^*$, for which*

$$\Re(p(z)) > 0,$$

then $|p_n| \leq 2$ for each $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$, where

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots, \quad (z \in \mathbb{U}^*).$$

2. Coefficient estimates

Definition 2.1. A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $\mathcal{T}_{\Sigma'}(\mu)$ if the following conditions are satisfied:

$$\Re\left(\frac{z^2 g'(z)}{(g(z))^2}\right) > 1 - \mu, \quad (z \in \mathbb{U}^*;\ 0 < \mu \leq 1)$$

and

$$\Re\left(\frac{w^2 h'(w)}{(h(w))^2}\right) > 1 - \mu, \quad (w \in \mathbb{U}^*;\ 0 < \mu \leq 1),$$

where the function h is an inverse of g given by (1.3).

Theorem 2.2. *Let the function $g(z) \in \Sigma'$ given by (1.2) be in the class $\mathcal{T}_{\Sigma'}(\mu)$, where $0 < \mu \leq 1$. Then,*

$$|b_0| \leq \begin{cases} \mu; & \left(0 < \mu \leq \frac{2}{3}\right) \\ \sqrt{\frac{2\mu}{3}}; & \left(\frac{2}{3} \leq \mu \leq 1\right), \end{cases} \tag{2.1}$$

$$|b_1| \leq \frac{2\mu}{3}, \tag{2.2}$$

$$|b_2 + b_0^3| \leq \frac{\mu}{2}. \tag{2.3}$$

Proof. Let the function $g(z) \in \mathcal{T}_{\Sigma'}(\mu)$. See that clearly, the conditions given in the definition of meromorphic bi-univalent function class $\mathcal{T}_{\Sigma'}(\mu)$ can be written as:

$$\frac{z^2 g'(z)}{(g(z))^2} = (1 - \mu) + \mu\, s(z) \tag{2.4}$$

and

$$\frac{w^2 h'(w)}{(h(w))^2} = (1-\mu) + \mu\, t(w), \tag{2.5}$$

where $s(z), t(w) \in \mathcal{P}$ have the form:

$$s(z) = 1 + \frac{s_1}{z} + \frac{s_2}{z^2} + \frac{s_3}{z^3} + \cdots, \ (z \in \mathbb{U}^*) \tag{2.6}$$

and

$$t(w) = 1 + \frac{t_1}{w} + \frac{t_2}{w^2} + \frac{t_3}{w^3} + \cdots, \ (w \in \mathbb{U}^*)\,. \tag{2.7}$$

Hence we have:

$$(1-\mu) + \mu\, s(z) = 1 + \frac{\mu s_1}{z} + \frac{\mu s_2}{z^2} + \frac{\mu s_3}{z^3} + \cdots$$

and

$$(1-\mu) + \mu\, t(w) = 1 + \frac{\mu t_1}{w} + \frac{\mu t_2}{w^2} + \frac{\mu t_3}{w^3} + \cdots.$$

Also, using (1.2) and (1.3) we obtain:

$$\frac{z^2 g'(z)}{(g(z))^2} = 1 - \frac{2b_0}{z} + \frac{3\left(b_0^2 - b_1\right)}{z^2} + \frac{8b_0b_1 - 4b_2 - 4b_0^3}{z^3} + \cdots$$

and

$$\frac{w^2 h'(w)}{(h(w))^2} = 1 + \frac{2b_0}{w} + \frac{3\left(b_0^2 + b_1\right)}{w^2} + \frac{12b_0b_1 + 4b_2 + 4b_0^3}{w^3} + \cdots.$$

Now, equating the coefficients in (2.4) and (2.5) we get:

$$-2b_0 = \mu s_1, \tag{2.8}$$

$$3\left(b_0^2 - b_1\right) = \mu s_2, \tag{2.9}$$

$$8b_0b_1 - 4b_2 - 4b_0^3 = \mu s_3, \tag{2.10}$$

$$2b_0 = \mu t_1, \tag{2.11}$$

$$3\left(b_0^2 + b_1\right) = \mu t_2, \tag{2.12}$$

$$12b_0b_1 + 4b_2 + 4b_0^3 = \mu t_3. \tag{2.13}$$

Clearly, equation (2.8) and (2.11) in light of Lemma 1.2 gives:

$$|b_0| \leq \mu. \tag{2.14}$$

Also by adding (2.9) in (2.12), we obtain:

$$6b_0^2 = \mu\left(s_2 + t_2\right)$$

which, by using Lemma 1.2 gives:

$$\left|b_0^2\right| \le \frac{2\mu}{3}. \tag{2.15}$$

Equation (2.14) and (2.15) together yields:

$$|b_0| \le \min\left\{\mu, \sqrt{\frac{2\mu}{3}}\right\},$$

which, for $0 < \mu \le 1$ gives the desired result (2.1).

Now, by subtracting (2.9) from (2.12), we get:

$$6b_1 = \mu\left(t_2 - s_2\right) \tag{2.16}$$

which, by using Lemma 1.2 gives:

$$|b_1| \le \frac{2\mu}{3}.$$

This is the desired result (2.2).

Finally, for the last inequality subtracting (2.10) from (2.13), we get:

$$4b_0b_1 + 8b_2 + 8b_0^3 = \mu\left(t_3 - s_3\right). \tag{2.17}$$

Also, by adding (2.10) in (2.13), we get:

$$20b_0b_1 = \mu\left(s_3 + t_3\right). \tag{2.18}$$

Eliminating b_0b_1 from (2.17) and (2.18), we obtain:

$$40\left(b_2 + b_0^3\right) = \mu\left(4t_3 - 6s_3\right)$$

which, in light of Lemma 1.2, yields the desired inequality (2.3).

This completes the proof of Theorem 2.2. □

Definition 2.3. A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $\mathcal{T}_{\Sigma'}^{\alpha}$ if the following conditions are satisfied:

$$\left|arg\left(\frac{z^2g'(z)}{(g(z))^2}\right)\right| < \frac{\alpha\pi}{2} \quad \left(z \in \mathbb{U}^*; 0 < \alpha \le 1\right)$$

and

$$\left|arg\left(\frac{w^2h'(w)}{(h(w))^2}\right)\right| < \frac{\alpha\pi}{2} \quad \left(w \in \mathbb{U}^*; 0 < \alpha \le 1\right),$$

where the function h is an inverse of g given by (1.3).

Theorem 2.4. *Let the function* $g(z) \in \Sigma'$ *given by* (1.2) *be in the class* $\mathcal{T}_{\Sigma'}^{\alpha}$, *where* $0 < \alpha \leq 1$. *Then,*

$$|b_0| \leq \sqrt{\frac{2}{3}}\alpha, \tag{2.19}$$

$$|b_1| \leq \frac{2}{3}\alpha^2, \tag{2.20}$$

$$\left|b_2 + b_0^3\right| \leq \frac{\alpha\left(2\alpha^2+1\right)}{6}. \tag{2.21}$$

Proof. Since $g(z) \in \mathcal{T}_{\Sigma'}^{\alpha}$; for $s(z), t(w) \in \mathcal{P}$ the conditions given in the definition of the function class $\mathcal{T}_{\Sigma'}^{\alpha}$ can be written as:

$$\frac{z^2 g'(z)}{(g(z))^2} = [s(z)]^{\alpha} \tag{2.22}$$

and

$$\frac{w^2 h'(w)}{(h(w))^2} = [t(w)]^{\alpha}, \tag{2.23}$$

where $s(z)$ and $t(w)$ have the form as given in (2.6) and (2.7), respectively.

Clearly, we have:

$$[s(z)]^{\alpha} = 1 + \frac{\alpha s_1}{z} + \frac{\frac{1}{2}\alpha(\alpha-1)s_1^2 + \alpha s_2}{z^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)s_1^3 + \alpha(\alpha-1)s_1 s_2 + \alpha s_3}{z^3} + \cdots$$

and

$$[t(w)]^{\alpha} = 1 + \frac{\alpha t_1}{w} + \frac{\frac{1}{2}\alpha(\alpha-1)t_1^2 + \alpha t_2}{w^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)t_1^3 + \alpha(\alpha-1)t_1 t_2 + \alpha t_3}{w^3} + \cdots.$$

Also, just as in proof of Theorem 2.2 we have:

$$\frac{z^2 g'(z)}{(g(z))^2} = 1 - \frac{2b_0}{z} + \frac{3\left(b_0^2 - b_1\right)}{z^2} + \frac{8b_0 b_1 - 4b_2 - 4b_0^3}{z^3} + \cdots$$

and

$$\frac{w^2 h'(w)}{(h(w))^2} = 1 + \frac{2b_0}{w} + \frac{3\left(b_0^2 + b_1\right)}{w^2} + \frac{12b_0 b_1 + 4b_2 + 4b_0^3}{w^3} + \cdots.$$

Now, equating the coefficients in (2.22) and (2.23) we get:

$$-2b_0 = \alpha s_1, \tag{2.24}$$

$$3\left(b_0^2 - b_1\right) = \frac{1}{2}\alpha(\alpha-1)s_1^2 + \alpha s_2, \tag{2.25}$$

$$8b_0b_1 - 4b_2 - 4b_0^3 = \frac{1}{6}\alpha(\alpha-1)(\alpha-2)s_1^3 + \alpha(\alpha-1)s_1s_2 + \alpha s_3, \tag{2.26}$$

$$2b_0 = \alpha t_1, \tag{2.27}$$

$$3\left(b_0^2 + b_1\right) = \frac{1}{2}\alpha(\alpha-1)t_1^2 + \alpha t_2, \tag{2.28}$$

$$12b_0b_1 + 4b_2 + 4b_0^3 = \frac{1}{6}\alpha(\alpha-1)(\alpha-2)t_1^3 + \alpha(\alpha-1)t_1t_2 + \alpha t_3. \tag{2.29}$$

Clearly, equation (2.24) and (2.27) in light of Lemma 1.2 gives:

$$|b_0| \le \alpha. \tag{2.30}$$

Also by adding (2.25) in (2.28), we obtain:

$$6b_0^2 = \frac{1}{2}\alpha(\alpha-1)\left(s_1^2 + t_1^2\right) + \alpha(s_2 + t_2)$$

which, by using Lemma 1.2 gives:

$$\left|b_0^2\right| \le \frac{2}{3}\alpha^2. \tag{2.31}$$

Obviously, from (2.30) and (2.31) we can write:

$$|b_0| \le \sqrt{\frac{2}{3}}\alpha \le \alpha; \quad (0 < \alpha \le 1).$$

This gives the desired result (2.19).

Now, by subtracting (2.25) from (2.28), we get:

$$6b_1 = \frac{1}{2}\alpha(\alpha-1)\left(t_1^2 - s_1^2\right) + \alpha(t_2 - s_2)$$

which, by using Lemma 1.2 gives:

$$|b_1| \le \frac{2}{3}\alpha^2.$$

This is the desired result (2.20).

Finally, subtracting (2.26) from (2.29), we get:

$$\begin{aligned} 24\left(b_0b_1 + 2b_2 + 2b_0^3\right) =& \alpha(\alpha-1)(\alpha-2)\left(t_1^3 - s_1^3\right) + \\ & 6\alpha(\alpha-1)(t_1t_2 - s_1s_2) + 6\alpha(t_3 - s_3). \end{aligned} \tag{2.32}$$

Also, by adding (2.26) in (2.29), we get:

$$120b_0b_1 = \alpha(\alpha-1)(\alpha-2)(s_1^3+t_1^3)+6\alpha(\alpha-1)(s_1s_2+t_1t_2)+6\alpha(s_3+t_3). \tag{2.33}$$

Eliminating b_0b_1 from (2.32) and (2.33), we obtain:

$$\begin{aligned}240\left(b_2+b_0^3\right) = \alpha\,(\alpha-1)\,(\alpha-2)\left(4t_1^3-6s_1^3\right)+\\ 6\alpha\,(\alpha-1)\,(4t_1t_2-6s_1s_2)+6\alpha\,(4t_3-6s_3)\end{aligned}$$

which, in light of Lemma 1.2, yields the desired inequality (2.21).

This completes the proof of Theorem 2.4. □

3. Conclusion

It is interesting that, for functions in both the subclasses $\mathcal{T}_{\Sigma'}(\mu)$ and $\mathcal{T}_{\Sigma'}^{\alpha}$, $(0<\mu,\alpha\le 1)$; all the coefficient inequalities are similar in the following sense:

$$\max_{g\in\Sigma'}|b_0| \le \sqrt{\frac{2}{3}},$$

$$\max_{g\in\Sigma'}|b_1| \le \frac{2}{3},$$

$$\max_{g\in\Sigma'}\left|b_2+b_0^3\right| \le \frac{1}{2}.$$

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