Fixed point results with Ω-distance by utilizing simulation functions

Anwar Bataihah*
Department of Mathematics
University of Jordan
Amman, 11942
Jordan
anwerbataihah@gmail.com
ano9160286@fgs.ju.edu.jo

Abdalla Tallafha
Department of Mathematics
School of Science
University of Jordan
Amman, 11942
Jordan
a.tallafha@ju.edu.jo

Wasfi Shatanawi
Department of Mathematics and General Courses
Prince Sultan University
Riyadh, 11586
Saudi Arabia
and
Department of Medical Research
China Medical University Hospital
China Medical University
Taichung, 40402
Taiwan
and
Department of M-Commerce and Multimedia Applications
Asia University
Taichung
Taiwan
wshatanawi@psu.edu.sa
wshatanawi@yahoo.com

Abstract. In this paper, we utilize the concept of simulation functions in sense of Khojasteh et al [10] and the notion of Ω-distance in the sense of Saadati et. al. [1] to introduce the notion of $(\Omega, \mathcal{Z})$-contraction and $(\Omega, \varphi, \mathcal{Z})$-contraction. We employ our contractions to formulate and prove many fixed point results for Ω-distance. Our results

*. Corresponding author

unify and improve many fixed point results in literature. Also, we give fixed point results of integral type as well as we support our result by introducing an example.
Keywords: fixed point theory, nonlinear contraction, simulation function, omega distance.

1. Introduction

The notion of Ω-distance in the sense of Saadati et al. [1] plays an important role in nonlinear analysis to extend and improve the Banach fixed point theorem to many directions. Saadati et al. [1] employed the notion of Ω-distance to prove many interesting results associated to the notion of G-metric spaces in the sense of Mustafa and Sims [2]. For some works in Ω-distance see [3]-[7] and all references cited their.

The definition of Ω-distance is given as follows:

Definition 1.1 ([1]). *Let (X, G) be a G-metric space. Then a function $\Omega : X \times X \times X \to [0, \infty)$ is called an Ω-distance on X if the following conditions satisfied:*

(*a*) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z) \, \forall \, x, y, z, a \in X$;

(*b*) *for any $x, y \in X, \Omega(x, y, .), \Omega(x, ., y) : X \to X$ are lower semi continuous;*

(*c*) *for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \epsilon$.*

Definition 1.2 ([1]). *Let (X, G) be a G-metric space and Ω be an Ω-distance on X. Then we say that X is Ω-bounded if there exists $M \geq 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, x \in X$.*

The following lemma plays a crucial role in the development of our results.

Lemma 1.1 ([1]). *Let X be a metric space with metric G and Ω be an Ω-distance on X. Let (x_n),(y_n) be sequences in X, (α_n),(β_n) be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following:*

(1) *If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \epsilon$ and hence $y = z$;*

(2) *If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for any $m > n \in \mathbb{N}$, then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$;*

(3) *If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $m, n, l \in \mathbb{N}$ with $n \leq m \leq l$, then (x_n) is a G-Cauchy sequence;*

(4) *If $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a G-Cauchy sequence.*

Khojasteh et al. [10] in 2015 introduced the concept of simulation mappings in which they used it to unify several fixed point results in the literature.

Definition 1.3 ([10]). *Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:*

($\zeta 1$) $\zeta(0, 0) = 0$;

($\zeta 2$) $\zeta(t, s) < s - t$ *for all $s, t > 0$;*

(ζ3) *If* t_n *and* s_n *are sequences in* $[0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$*, then* $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Hence forth, we denote by $\mathcal{Z}$ the set of all simulation functions.

Next, we list some examples of simulation functions, in the following ζ is defined from $[0,\infty)\times[0,\infty)$ to $\mathbb{R}$.

Example 1.1 ([10])**.** Let $h_1, h_2 : [0,\infty) \to [0,\infty)$ be two continuous functions such that $h_1(t) = h_2(t) = 0$ if and only if $t = 0$ and $h_2(t) < t \leq h_1(t)$ for all $t \in [0,\infty)$ and define $\zeta(t,s) = h_2(s) - h_1(t)$ for all $t, s \in [0,\infty)$. Then ζ is a simulation function.

Example 1.2 ([10])**.** Let $g : [0,\infty) \to [0,\infty)$ be a continuous function such that $g(t) = 0$ if and only if t=0 and define $\zeta(t,s) = s - g(s) - t$ for all $t, s \in [0,\infty)$. Then ζ is a simulation function.

2. Main result

We start our work by introducing the following definition:

Definition 2.1. *Let* (X, G) *be a* G*-metric space,* $\zeta \in \mathcal{Z}$ *and* Ω *be an* Ω*-distance on X. A self mapping* $f : X \to X$ *is said to be* $(\Omega, \mathcal{Z})$*-contraction with respect to* ζ *if f satisfies the following condition:*

$$\zeta(\Omega(fx, fy, fz), \Omega(x, y, z)) \geq 0 \quad for\ all \quad x, y, z \in X. \tag{2.1}$$

Lemma 2.1. *Let* (X, G) *be a* G*-metric space, and* Ω *be an* Ω*-distance on* X*. Let* $f : X \to X$ *be an* $(\Omega, \mathcal{Z})$*-contraction with respect to* $\zeta \in \mathcal{Z}$*. If* f *has a fixed point (say)* $u \in X$*, then it is unique.*

Proof. Assume that there is $v \in X$ such that $fv = v$. As f is $(\Omega, \mathcal{Z})$-contraction with respect to $\zeta \in \mathcal{Z}$, then by substituting $x = y = u$ and $z = v$ in 2.1 and taking into account (ζ2), we have

$$\begin{aligned} 0 &\leq \zeta(\Omega(fu, fu, fv), \Omega(u, u, v)) \\ &= \zeta(\Omega(u, u, v), \Omega(u, u, v)) \\ &< \Omega(u, u, v) - \Omega(u, u, v) = 0, \end{aligned}$$

a contradiction. Hence u is unique. □

Let (X, G) be a G-metric space, $x_0 \in X$ and $f : X \to X$ be a self mapping. Then the sequence (x_n) where $x_n = fx_{n-1}$ $n \in \mathbb{N}$ is called a picard sequence generated by f with initial point x_0.

Lemma 2.2. *Let* (X, G) *be a* G*-metric space,* $\zeta \in \mathcal{Z}$ *and* Ω *be an* Ω*-distance on X. If* $f : X \to X$ *is an* $(\Omega, \mathcal{Z})$*-contraction with respect to* ζ*, then*

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0. \tag{2.2}$$

for any initial point $x_0 \in X$ *where* (x_n) *is the picard sequence generated by* f *at* x_0.

Proof. Let $x_0 \in X$ be any point and (x_n) be the picard sequence generated by f at x_0. From 2.1 and $(\zeta 2)$, we have

$$\begin{aligned} 0 &\leq \zeta(\Omega(fx_{n-1}, fx_n, fx_n), \Omega(x_{n-1}, x_n, x_n)) \\ &= \zeta(\Omega(x_n, x_{n+1}, x_{n+1}), \Omega(x_{n-1}, x_n, x_n)) \\ &< \Omega(x_{n-1}, x_n, x_n) - \Omega(x_n, x_{n+1}, x_{n+1}). \end{aligned}$$

Thus, $(\Omega(x_n, x_{n+1}, x_{n+1}) : n \in \mathbb{N})$ is a non increasing sequence in $[0, \infty)$ and so there is $L \geq 0$ such that $\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = L$. Suppose to the contrary $L > 0$, then by 2.1 and $(\zeta 3)$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(\Omega(x_n, x_{n+1}, x_{n+1}), \Omega(x_{n-1}, x_n, x_n)) < 0,$$

a contradiction and so $\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0$. By the same way we can show that $\lim_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0$. □

Theorem 2.1. *Let (X, G) be a complete G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X. Suppose that $f : X \to X$ is $(\Omega, \mathcal{Z})$-contraction with respect to ζ that satisfies the following condition*

$$\text{for all } u \in X \text{ if } fu \neq u, \text{ then } \inf\{\Omega(x, fx, u) : x \in X\} > 0. \tag{2.3}$$

Then f has a unique fixed point $x \in X$.

Proof. Let $x_0 \in X$ and consider the picard sequence (x_n) in X generated by f at x_0.

We claim that $\lim_{n,m\to\infty} \Omega(x_n, x_m, x_m) = 0$ for $m, n \in \mathbb{N}$ with $m > n$.

For this purpose assume to the contrary that $\lim_{n\to\infty} \Omega(x_n, x_m, x_m) \neq 0$. Hence, there is $\epsilon > 0$ and two subsequences (x_{n_k}) and (x_{m_k}) of (x_n) such that (x_{m_k}) is chosen as the smallest index for which

$$\Omega(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon, \ k < n_k < m_k. \tag{2.4}$$

This implies that

$$\Omega(x_{n_k}, x_{m_k-1}, x_{m_k-1}) < \epsilon. \tag{2.5}$$

Now, by using 2.4,2.5 and part (a) of the definition of Ω, we have

$$\begin{aligned} \epsilon &\leq \Omega(x_{n_k}, x_{m_k}, x_{m_k}) \\ &\leq \Omega(x_{n_k}, x_{m_k-1}, x_{m_k-1}) + \Omega(x_{m_k-1}, x_{m_k}, x_{m_k}) \\ &< \epsilon + \Omega(x_{m_k-1}, x_{m_k}, x_{m_k}). \end{aligned}$$

Passing the limit as $n \to \infty$ and taking into account 2.2, we get

$$\lim_{n\to\infty} \Omega(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon.$$

Also,

$$\begin{aligned}\epsilon &\leq \Omega(x_{n_k}, x_{m_k}, x_{m_k})\\ &\leq \Omega(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) + \Omega(x_{m_k+1}, x_{m_k}, x_{m_k})\end{aligned}$$

and

$$\begin{aligned}&\Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1})\\ &\leq \Omega(x_{n_k+1}, x_{n_k}, x_{n_k}) + \Omega(x_n, x_{m_k}, x_{m_k}) + \Omega(x_{m_k}, x_{m_k+1}, x_{m_k+1}).\end{aligned}$$

Passing the limit as $n \to \infty$ in the above two inequalities and taking into account 2.2, we get

$$\lim_{n\to\infty} \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) = \epsilon.$$

Now, by letting $s_n = \Omega(x_{n_k}, x_{m_k}, x_{m_k})$ and $t_n = \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1})$ then $(\zeta 3)$ and 2.1 yield that

$$0 \leq \limsup_{n\to\infty} \zeta(\Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}), \Omega(x_{n_k}, x_{m_k}, x_{m_k})) < 0$$

which is a contradiction. Therefore $\lim_{n,m\to\infty} \Omega(x_n, x_m, x_m) = 0,\ m > n$. By the same argument we can show that $\lim_{n,m\to\infty} \Omega(x_n, x_n, x_m) = 0,\ m > n$.

For $l > m > n$ we have $\Omega(x_n, x_m, x_l) \leq \Omega(x_n, x_m, x_m) + \Omega(x_m, x_m, x_l)$.

By taking the limit as $n, m, l \to \infty$, we get $\lim_{n,m,l\to\infty} \Omega(x_n, x_m, x_l) = 0$. Thus by Lemma 1.1 (x_n) is a G-Cauchy sequence. So there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

By the lower semi-continuity of Ω, we get

$$\Omega(x_n, x_m, u) \leq \liminf_{p\to\infty} \Omega(x_n, x_m, x_p) \leq \epsilon,\ \forall m \geq n.$$

Now, suppose that $fu \neq u$, then we get

$$\begin{aligned}0 &< \inf\{\Omega(x, fx, u) : x \in X\}\\ &\leq \inf\{\Omega(x_n, x_{n+1}, u) : n \in \mathbb{N}\}\\ &\leq \epsilon,\end{aligned}$$

for every $\epsilon > 0$ which is a contradiction. Therefore $fu = u$. The uniqueness of u follows from Lemma 2.1. □

We introduce the following example to support our main result.

Example 2.1. Let $X = [0, 1]$ and let $G : X \times X \times X \to [0, \infty)$, $\Omega : X \times X \times X \to [0, \infty)$, $f : X \to X$ and $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined as follow:
$G(x, y, z) = |x - y| + |y - z| + |x - z|$, $\Omega(x, y, z) = |x - y| + |x - z|$, $fx = ax$ and $\zeta(t, s) = bs - t$ where $0 \leq a \leq b < 1$. Then

(1) (X, G) is a complete G-metric space and Ω is an Ω-distance on X;

(2) $\zeta \in \mathcal{Z}$ and f is $(\Omega, \mathcal{Z})$-contraction with respect to ζ

(3) for every $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Proof. We show (2) and (3)

(2) Clearly $\zeta \in \mathcal{Z}$.

To see that f is $(\Omega, \mathcal{Z})$-contraction with respect to ζ let $x, y, z \in X$. Then

$$\begin{aligned}\zeta(\Omega(fx, fy, fz), \Omega(x, y, z)) &= b\Omega(x, y, z) - \Omega(fx, fy, fz)\\ &= b(|x - y| + |x - z|) - (|ax - ay| + |ax - az|)\\ &= b(|x - y| + |x - z|) - a(|x - y| + |x - z|)\\ &= (b - a)(|x - y| + |x - z|)\\ &\geq 0\end{aligned}$$

(3) If $fu \neq u$, then $u \neq 0$. Therefore

$$\begin{aligned}\inf\{\Omega(x, fx, u) : x \in X\} &= \inf\{\Omega(x, \frac{1}{5}x, u) : x \in X\}\\ &= \inf\{|x - ax| + |x - u| : x \in X\}\\ &= \inf\{(1 - a)|x| + |x - u| : x \in X\}\\ &= (1 - a)u > 0.\end{aligned}$$

Thus all hypotheses of Theorem 2.1 hold true. Hence f has a unique fixed point in X. Here the unique fixed point of f is 0. □

Now, we derive some interesting results based on our main result. To facilitate our work we define the following:

$$\Phi = \{\phi : [0, \infty) \to [0, \infty) : \phi \textit{ is continuous function}\}$$

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \psi \textit{ is lower semi continuous function}\},$$

where $\phi^{-1}(\{0\}) = \psi^{-1}(\{0\}) = \{0\}$ for all $\phi \in \Phi$ and $\psi \in \Psi$.

Corollary 2.1. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X and $f : X \to X$ be a self mapping. Assume that there are $\phi_1, \phi_2 \in \Phi$ where $\phi_1(t) < t \leq \phi_2(t)$ $\forall t > 0$ such that f satisfies the following condition:*

$$\phi_2\Omega(fx, fy, fz) \leq \phi_1\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.6}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. Define $\zeta_A : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_A(t, s) = \phi_1(s) - \phi_2(t)$. Clearly $\zeta_A \in \mathcal{Z}$ and f is $(\Omega, \mathcal{Z})$-contraction with respect to ζ_A. Hence the result follows from Theorem 2.1 □

As a consequence result from Corollary 2.1, we have the following results:

Corollary 2.2. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X and $f : X \to X$ be a self mapping. Assume that there is $\phi \in \Phi$ where $\phi(t) < t \ \forall t > 0$ such that f satisfies the following condition:*

$$\Omega(fx, fy, fz) \leq \phi\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.7}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Corollary 2.3. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X and $f : X \to X$ be a self mapping. Assume that there is $\lambda \in [0, 1)$ such that f satisfies the following condition:*

$$\Omega(fx, fy, fz) \leq \lambda\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.8}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Corollary 2.4. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X and Let $f : X \to X$ be a self mapping. Assume that there is $\psi \in \Psi$ such that f satisfies the following condition:*

$$\Omega(fx, fy, fz) \leq \Omega(x, y, z) - \psi\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.9}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. Define $\zeta_B : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_B(t, s) = s - \psi(s) - t$. Clearly $\zeta_B \in \mathcal{Z}$ and f is $(\Omega, \mathcal{Z})$-contraction with respect to ζ_B. Hence the result follows from Theorem 2.1 □

As a consequence result from Corollary 2.4 we have the following result:

Corollary 2.5. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X and $f : X \to X$ be a self mapping. Assume that there are $\phi \in \Phi$ and $\psi \in \Psi$ where $\phi(t) < t \ \forall t > 0$ such that f satisfies the following conditions:*

$$\Omega(fx, fy, fz) \leq \phi\Omega(x, y, z) - \psi\Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.10}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Definition 2.2. *The function $\varphi : [0, \infty) \to [0, \infty)$ is called a c-comparison function if the following properties are satisfied:*

(1) *φ is monotone increasing;*

(2) $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ *for all $t \geq 0$.*

It is clear that if φ is a c-comparison function then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.

Before, we present our second main results we introduce the following definition in order to facilitate our arguments.

Definition 2.3. *Let (X, G) be a G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X. A self mapping $f : X \to X$ is said to be $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ if there is is a c-comparison function φ such that f satisfies the following condition:*

$$\zeta(2\Omega(fx, f^2x, fy), \varphi\Omega(x, fx, x) + \varphi\Omega(y, fy, y)) \geq 0 \ \forall x, y \in X. \tag{2.11}$$

Lemma 2.3. *Let (X, G) be a G-metric space,$\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X. Let $f : X \to X$ be an $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ. If f has a fixed point (say) $u \in X$, then it is unique.*

Proof. First we show that for all $w \in X$ if $fw = w$, then $\Omega(w, w, w) = 0$. Assume that $\Omega(w, w, w) > 0$. From 2.11 and $\zeta 2$, we have

$$\begin{aligned}
0 &\leq \zeta(2\Omega(fw, f^2w, fw), \varphi\Omega(w, fw, w) + \varphi\Omega(w, fw, w)) \\
&= \zeta(2\Omega(w, w, w), 2\varphi\Omega(w, w, w)) \\
&< 2\varphi\Omega(w, w, w) - 2\Omega(w, w, w), \\
&< 2\Omega(w, w, w) - 2\Omega(w, w, w), \\
&= 0
\end{aligned}$$

a contradiction. Hence $\Omega(w, w, w) = 0$.

Now, assume that there is $v \in X$ such that $fv = v$ and $\Omega(u, v, v) > 0$. Since f is $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ, then by substituting $x = u$ and $y = v$ in 2.1 and taking into account ($\zeta 2$), we have

$$\begin{aligned}
0 &\leq \zeta(2\Omega(fu, f^2u, fv), \varphi\Omega(u, fu, u) + \varphi\Omega(v, fv, v)) \\
&= \zeta(2\Omega(u, u, v), \varphi\Omega(u, u, u) + \varphi\Omega(v, v, v)) \\
&< \varphi\Omega(u, u, u) + \varphi\Omega(v, v, v) - 2\Omega(u, u, v) \\
&< \Omega(u, u, u) + \Omega(v, v, v) - 2\Omega(u, u, v).
\end{aligned}$$

Hence $2\Omega(u, u, v) < \Omega(u, u, u) + \Omega(v, v, v) = 0 + 0 = 0$ a contradiction. Hence $\Omega(u, u, v) = 0$. Thus by the definition of Ω-distance we have $G(u, v, v) = 0$ and so $u = v$. □

Theorem 2.2. *Let (X, G) be a complete G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X such that X is Ω-bounded. Suppose that there is is a c-comparison function φ such that $f : X \to X$ is a $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ that satisfies the following condition*

$$\forall\ u \in X \ if \ fu \neq u, \ then \ \inf\{\Omega(x, fx, u) : x \in X\} > 0. \tag{2.12}$$

Then f has a unique fixed point in X.

Let $x_0 \in X$ and consider the picard sequence (x_n) in X generated by f at x_0.

Consider $s \geq 0$. Then by 2.11, we have for all $n \in \mathbb{N}$

$$\begin{aligned}
0 &\leq \zeta(2\Omega(fx_{n-1}, f^2x_{n-1}, fx_{n+s-1}), \varphi\Omega(x_{n-1}, fx_{n-1}, x_{n-1}) \\
&\quad + \varphi\Omega(x_{n+s-1}, fx_{n+s-1}, x_{n+s-1})) \\
&= \zeta(2\Omega(x_n, x_{n+1}, x_{n+s}), \varphi\Omega(x_{n-1}, x_n, x_{n-1}) + \varphi\Omega(x_{n+s-1}, x_{n+s}, x_{n+s-1})). \\
&< \varphi\Omega(x_{n-1}, x_n, x_{n-1}) + \varphi\Omega(x_{n+s-1}, x_{n+s}, x_{n+s-1}) - 2\Omega(x_n, x_{n+1}, x_{n+s}).
\end{aligned}$$

Thus,

$$\Omega(x_n, x_{n+1}, x_{n+s}) < \frac{1}{2}[\varphi\Omega(x_{n-1}, x_n, x_{n-1}) + \varphi\Omega(x_{n+s-1}, x_{n+s}, x_{n+s-1})]. \tag{2.13}$$

Now,

$$\begin{aligned}
0 &\leq \zeta(2\Omega(fx_{n-2}, f^2x_{n-2}, fx_{n-2}), \varphi\Omega(x_{n-2}, fx_{n-2}, x_{n-2}) + \varphi\Omega(x_{n-2}, fx_{n-2}, x_{n-2})) \\
&= \zeta(2\Omega(x_{n-1}, x_n, x_{n-1}), 2\varphi\Omega(x_{n-2}, x_{n-1}, x_{n-2})) \\
&< 2\varphi\Omega(x_{n-2}, x_{n-1}, x_{n-2}) - 2\Omega(x_{n-1}, x_n, x_{n-1}).
\end{aligned}$$

So, $\Omega(x_{n-1}, x_n, x_{n-1}) < \varphi\Omega(x_{n-2}, x_{n-1}, x_{n-2})$.

If we apply the previous steps repeatedly, we get

$$\Omega(x_{n-1}, x_n, x_{n-1}) \leq \varphi^{n-1}\Omega(x_0, x_1, x_0).$$

Therefore $\varphi\Omega(x_{n-1}, x_n, x_{n-1}) \leq \varphi^n\ \Omega(x_0, x_1, x_0)$. Since X is Ω-bounded, there is $M \geq 0$, such that $\Omega(x, y, z) \leq M,\ \forall x, y, z, \in X$. Thus,

$$\varphi\Omega(x_{n-1}, x_n, x_{n-1}) \leq \varphi^n\,(M).$$

In analogous manner, we can show that

$$\varphi\Omega(x_{n+s-1}, x_{n+s}, x_{n+s-1}) \leq \varphi^n\,(M).$$

Thus, (2.13) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq \varphi^n(M). \tag{2.14}$$

Now, by using the definition of Ω-distance and (2.14), we have for all $l \geq m \geq n$

$$\begin{aligned}
\Omega(x_n, x_m, x_l) &\leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l) \\
&\leq \varphi^n(M) + \varphi^{n+1}(M) + \cdots + \varphi^{m-1}(M) \\
&= \sum_{k=n}^{m-1} \varphi^k(M) \\
&\leq \sum_{k=n}^{\infty} \varphi^k(M).
\end{aligned}$$

Since φ is c-comparison function, then the sequence $\left(\sum_{k=n}^{\infty} \varphi^k(M) : n \in \mathbb{N}\right)$ converges to 0. Thus for any $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\sum_{k=n}^{\infty} \varphi^k(M) < \epsilon \ \forall\ n \geq N$. Hence for $l \geq m \geq n \geq N$, we have

$$\Omega(x_n, x_m, x_l) \leq \sum_{k=n}^{m-1} \varphi^k(M) \leq \sum_{k=n}^{\infty} \varphi^k(M) < \epsilon \ \forall n \geq N.$$

By Lemma 1.1, (x_n) is a G-Cauchy sequence. Therefore there is $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Consider $\delta > 0$. Then there exists $r_0 \in \mathbb{N}$ such that $\Omega(x_n, x_m, x_l) \leq \delta \ \forall n, m, l \geq r_0$. Therefore, $\lim_{l\to\infty} \Omega(x_n, x_m, x_l) \leq \lim_{l\to\infty} \delta = \delta$.

By the lower semi continuity of Ω, we have

$$\Omega(x_n, x_m, u) \leq \liminf_{p\to\infty} \Omega(x_n, x_m, x_p) \leq \delta \, \forall\, m, n \geq r_0.$$

Consider $m = n+1$. Then $\Omega(x_n, x_{n+1}, u) \leq \liminf_{p\to\infty} \Omega(x_n, x_{n+1}, x_p) \leq \delta \ \forall n \geq r_0$.

If $fu \neq u$, then (2.12) implies that

$$\begin{aligned} 0 &< \inf\{\Omega(x, fx, u) : x \in X\} \\ &\leq \inf\{\Omega(x_n, x_{n+1}, u) : n \geq r_0\} \\ &\leq \delta, \end{aligned}$$

for each $\delta > 0$ which is a contradiction. Therefore $fu = u$. The uniqueness follows from Lemma 2.3. □

Corollary 2.6. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X where X is Ω bounded and $f : X \to X$ be a self mapping. Assume that there is a c-comparison function φ and an upper semi continuous function $\eta : [0, \infty) \to [0, \infty)$ where $\eta(t) < t \ \forall t > 0$ and $\eta(0) = 0$ such that f satisfies the following condition:*

$$2\Omega(fx, f^2x, fy) \leq \eta(\varphi\Omega(x, fx, x) + \varphi\Omega(y, fy, y)) \ \forall x, y \in X. \tag{2.15}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. Define $\zeta_{AA} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta_{AA}(t, s) = \eta(s) - t$. Clearly $\zeta_{AA} \in \mathcal{Z}$ and f is $(\Omega, \varphi, \mathcal{Z})$-contraction with respect to ζ_{AA}. Hence the result follows from Theorem 2.2 □

Now, we introduce and prove the following fixed point theorems of integra type.

Theorem 2.3. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X where X is Ω bounded and $f : X \to X$ be a self mapping. Assume that there is a function $\gamma : [0, \infty) \to [0, \infty)$ where $\int_0^\epsilon \gamma(u)du$ exists and $\int_0^\epsilon \gamma(u)du > \epsilon$ $\forall \epsilon > 0$ such that f satisfies the following condition:*

$$\int_0^{\Omega(fx,fy,fz)} \gamma(u)du \leq \Omega(x, y, z) \ \forall x, y, z \in X. \tag{2.16}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. Defining $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ via $\zeta(t, s) = s - \int_0^t \gamma(u)du$. Clearly $\zeta \in \mathcal{Z}$ and f is $(\Omega, \mathcal{Z})$. Hence the results follow from Theorem 2.1. □

Theorem 2.4. *Let (X, G) be a complete G-metric space, Ω be an Ω-distance on X where X is Ω bounded and $f : X \to X$ be a self mapping. Assume that there is a c-comparison function φ and a function $\gamma : [0, \infty) \to [0, \infty)$ where $\int_0^\epsilon \gamma(u)du$ exists and $\int_0^\epsilon \gamma(u)du > \epsilon$ $\forall \epsilon > 0$ such that f satisfies the following condition:*

$$\int_0^{2\Omega(fx,f^2x,fy)} \gamma(u)du \leq \varphi\Omega(x, fx, x) + \varphi\Omega(y, fy, y) \ \forall x, y \in X. \tag{2.17}$$

Also, suppose that for all $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x, fx, u) : x \in X\} > 0$.

Then f has a unique fixed point in X.

Proof. The results follow from Theorem 2.2 by defining $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ via $\zeta(t, s) = s - \int_0^t \gamma(u)du$. and noting that $\zeta \in \mathcal{Z}$ and f is $(\Omega, \varphi, \mathcal{Z})$.

□

References

[1] R. Saadati, S. M. Vaezpour, P. Vetro and B. E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Mathematical and Computer Modeling, 52 (2010), 797-801.

[2] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces,* J. Nonlinear Convex Anal., 7 (2006), 289-297.

[3] K. Abodayeh, W. Shatanawi, A. Bataihah and A.H. Ansari, *Some fixed point and common fixed point results through Ω-distance under nonlinear conractions,* GU J. Sci., 30 (2017), 293-302.

[4] W. Shatanawi, G. Maniu, A. Bataihah and F. Bani Ahmad, *Common fixed points for mappings of cyclic form satisfying linear contractive conditions with Omega-distance,* U.P.B. Sci., series A, 79 (2017).

[5] W. Shatanawi, A. Bataihah and A. Pitea, *Fixed and common fixed point results for cyclic mappings of* Ω*-distance,* J. Nonlinear Sci. Appl., 9 (2016), 727-735.

[6] K. Abodayeh, W. Shatanawi and A. Bataihah, *Fixed point theorem through* Ω*-distance of Suzuki type conraction condition*, GU J. Sci, 29 (2016), 129-133.

[7] L. Gholizadeh, R. Saadati, W. Shatanawi, S.M. Vaezpour, *Contractive mapping in generalized, ordered metric spaces with application in integral equations*, Math. Probl. Eng.,Article ID 380784, 2011.

[8] W. Shatanawi, A. Pitea, *Fixed and coupled fixed point theorems of omega-distance for nonlinear contraction*, Fixed Point Theory and Applications, 2013, 2013:275.

[9] W. Shatanawi, A. Pitea, Ω*-Distance and coupled fixed point in G-metric spaces*, Fixed Point Theory and Applications, 208 (2013).

[10] F. Khojasteh, S. Shukla and S. Radenovic, *A new approach to the study of fixed point theory for simulation functions*, Filomat, 29:6 (2015).

Accepted: 20.01.2018