More on almost countably compact spaces

Zuhier Altawallbeh
Department of Mathematics
Tafila Technical University
Tafila, 66110
Jordan
zuhier1980@gmail.com

Abstract. We study a recent general space of countably compact space called almost countably compact. A topological space X is almost countably compact space if for every countable open cover $\{U_n : n \in \mathbb{N}\}$ of X, there is a finite subfamily $\{U_{n_i}\}_{i=1}^m$, where $m \in \mathbb{N}$ such that $X = \bigcup_{i=1}^m Cl(U_{n_i})$. In particular, we investigate this new class of spaces and some other properties in the view of regular cover notion and semiregularization topology.
Keywords: regularly open sets, regularly closed sets, nearly countably compact, semiregularization topology.

1 Introduction

Compactness has come to be one of the most important concepts in advanced mathematics. In the 19th century, many mathematical properties were used that would be later seen as consequences of compactness. This notion that arises from topology and metric spaces is very useful in analysis and so in applied mathematics. A generalization of compact spaces, the countable compact spaces arise in different study fields. For metrizable spaces, the countable compactness, sequential compactness, limit point compactness and compactness are all equivalent.

Not only compactness, but among various covering properties of topological spaces a lot of attention has been made to those covers involving regularly open sets and regularly closed sets. In 1959, weakly Lindelöf spaces were introduced by Frolik [5]. After that and as an analogous work on Lindelöfness; nearly compact spaces were defined in by Singal and Mathur in 1969 [8] as a generalization of compact spaces. By the definition, a topological space is nearly compact if for every open cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there is a finite subfamily $\{U_{\alpha_i}\}_{i=1}^m$ where $m \in \mathbb{N}$ and $X = \bigcup_{i=1}^m Int(Cl(U_{\alpha_i}))$. Further, nearly Lindelöf spaces are defined in by Balasubramanianin 1982 [2]. In the other way around, some generalizations as almost countbly compact and nearly countably compact of countably compact spaces are presented by different authors as Song and Zhao 2012 [11], and Altawallbeh and Al-Momani in their paper [1]. In addition to the mentioned references, the reader may take a look at [3] and [10]. By the definition, a topological space X is said to be nearly countably compact space if for every countable (regularly) open cover $\{U_n : n \in \mathbb{N}\}$ of X, there is a finite subfamily $\{U_{n_i}\}_{i=1}^{m}$, where $m \in \mathbb{N}$ such that $X = \bigcup_{i=1}^{m} Int(Cl(U_{n_i}))$ $(X = \bigcup_{i=1}^{m} U_{n_i})$. It is clear that every nearly countably compact space X is almost countably compact.

We know that a compact space is countably compact. In this paper, we investigate one more general space, called almost countably compact. In particular, we study some properties of almost countably compact spaces by using regularly open (closed) sets and semiregularization topology. Porter and Thomson [7] have shown the importance of semiregularization topologies in the study of H-closed and minimal Hausdorff spaces.

Throughout this paper, a space X stands for a topological space (X, τ). The interior and closure of a subset A in a space X are denoted by $Int(A)$ and $Cl(A)$, respectively. Regularly open sets are defined by Stone [12] in 1937 and investigated with more interesting results about the topic. A subset A is said to be regularly open if and only if $A = Int(Cl(A))$. It is obvious that every regularly open set is an open set. The complement of a regularly open set is called regularly closed set. In addition, we denote the set of all regularly open and closed sets in a space X by $RO(X)$ and $RC(X)$, respectively. In 1985 [6], Mršević, Rielly, and Vamanamurthy have been studied the topology of a space X whose base is the set of all regularly open sets in the space (X, τ) which is called semiregularization topology and denoted by τ^*. If $\tau = \tau^*$, then the space X is called semiregular. Furthermore, Cameron [4] has called a topological property ρ semiregular provided that the space (X, τ) has the property ρ. Moreover, a space X is called extremally disconnected if the closure of every open set in it is open.

The following lemma is well known and it is easily can be proved. In its results, it contains some preliminaries matching the regularly open and regularly closed sets of any topology and its semiregularization in such useful facts that can be appear in the text later.

Lemma 1.1. *For any topological space $(X, \tau)$and its' semiregularization (X, τ^*), we have:*

1. $RO(X, \tau) = RO(X, \tau^*)$.
2. $RC(X, \tau) = RC(X, \tau^*)$.
3. $Int_{\tau}(F) = Int_{\tau^*}(F)$ *for any* $F \in RC(X, \tau)$.
4. $Cl_{\tau}(O) = Cl_{\tau^*}(O)$ *for any* $O \in \tau$.

2 Almost countably comapct spaces

In this section, we study a recent class of generalized spaces of the countably compact spaces which is called the class of almost countably compact spaces. In particular, we prove that almost countably compact property is a semiregular property and a regularly closed subset of an almost countably compact space is an almost countably compact.

Definition 2.1. *A topological space X is said to be almost countably compact, if for every countable open cover $\{U_n : n \in \mathbb{N}\}$ of X, there is a finite subfamily $\{U_{n_i}\}_{i=1}^m$, where $m \in \mathbb{N}$ such that $X = \bigcup_{i=1}^m Cl(U_{n_i})$.*

It is clear that every countably compact space X is almost countably compact but the converse needs not be true as it is shown in the follwoing example.

Example 1. Let $X = [0, \Omega] \times [0, \omega]/\{(\Omega, \omega)\}$ be the deleted Tychonoff plank space where Ω is the first infinite ordinal and ω is the first uncountable ordinal. Since the set $\{(n, \omega) : n \in \mathbb{N}\}$ is an infinite discrete closed subset of X, we deduce that X is not countably compact. On the other hand, it is easy to see that X is almost countably compact because there is the subset $\Omega \times [0, \omega]$ which is a dense countably compact subspace of X.

Proposition 0.1. *A topological space (X, τ) is almost countably compact space if and only if its' semiregularization (X, τ^*) is almost countably compact.*

Proof. Assume that (X, τ) is almost countably compact. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of (X, τ^*). Since $\tau^* \subseteq \tau$, the cover $\mathcal{U}$ is an open cover of (X, τ). From our assumption, the space (X, τ) is almost countably compact, and so there is a finite subfamily $\{U_{n_i}\}_{i=1}^m$, for some natural number m, of $\mathcal{U}$ such that

$$X = \bigcup_{i=1}^{m} Cl_\tau(U_{n_i}).$$

By using Lemma 1.1 (4), we get

$$X = \bigcup_{i=1}^{m} Cl_{\tau^*}(U_{n_i}).$$

This proves that (X, τ^*) is almost countably compact.

Conversely, we assume that (X, τ^*) is almost countably compact. Let $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ be a countable open cover of (X, τ). From the definition of the topology τ^*, the family $\{Int_\tau(Cl_\tau(V_n)) : n \in \mathbb{N}\}$ is an open cover of the space (X, τ^*). Since (X, τ^*) is almost countably compact, there is a finite subfamily $\{Int_\tau(Cl_\tau(V_{n_i}))\}_{i=1}^m$ such that

$$X = \bigcup_{i=1}^{m} Cl_{\tau^*}(Int_\tau(Cl_\tau(V_{n_i}))).$$

By using Lemma 1.1 (4), we get

$$X = \bigcup_{i=1}^{m} Cl_\tau(Int_\tau(Cl_\tau(V_{n_i}))).$$

Since $Cl_\tau(Int_\tau(Cl_\tau(V_{n_i}))) \subseteq Cl_\tau(V_{n_i})$ for every set V_{n_i}, we have

$$X = \bigcup_{i=1}^{m} Cl_\tau(V_{n_i}).$$

This proves that (X, τ) is almost countably compact space and the proof is complete. □

Corollary 2.1. *Almost countably compact property is a semiregular property.*

From Example 1, observe that the closed subset $\{(n, \omega) : n \in \mathbb{N}\}$ of a Tychonoff almost countably compact space X is not almost countably compact. Thus, the closed subset of an almost countably compact space need not be almost countably compact. Anyway, the following proposition shows that there is a positive result regarding that.

Proposition 0.2. *Every regularly closed subset of an almost countably compact space is almost countably compact.*

Proof. Let X be an almost countably compact space and let H be a regularly closed subset of X. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable open cover of H. So, for each n, there is an open subset V_n of X such that $V_n \bigcap H = U_n$. Now, we get

$$X = (\bigcup_{n\in\mathbb{N}} V_n) \bigcup (X/H).$$

It is clear that X/H is an open subset of X. Since X is almost countably compact, we deduce that there is a finite subfamily $\{U_{n_1}, U_{n2}, ..., U_{n_m}\}$ of the cover $\{U_n : n \in \mathbb{N}\}$ such that

$$X = (\bigcup_{i=1}^{m} Cl(V_{n_i})) \bigcup (Cl(X/H)).$$

Thus,

$$Int(H) \subseteq \bigcup_{i=1}^{m} Cl(V_{n_i}) = Cl(\bigcup_{i=1}^{m} V_{n_i}).$$

Since H is a regularly closed subset of X, we have $H = Cl(Int(H)) \subseteq Cl(\bigcup_{i=1}^{m} V_{n_i})$. That means

$$H = H \bigcap (Cl(\bigcup_{i=1}^{m} V_{n_i})) = Cl_H(H \bigcap (\bigcup_{i=1}^{m} V_{n_i})) = \bigcup_{i=1}^{m} Cl_H(V_{n_i} \bigcap H) = \bigcup_{i=1}^{m} Cl_H(U_{n_i}).$$

That means H is almost countably compact space and the proof is completed. □

3 Almost regular countably comapct spaces

We define one more general spaces called almost regular countably compact spaces by using the notion of regular covers. Besides that, we prove that almost regular countably compact property is a semiregular property with some other related results are presented.

Definition 3.1. *An open cover* $\{U_\alpha : \alpha \in \Delta\}$ *of a space* X *is called regular cover if, for every* $\alpha \in \Delta$ *there exists a nonempty regularly closed subset* C_α *in* X *such that* $C_\alpha \subseteq U_\alpha$ *and* $X = \bigcup_{\alpha\in\Delta} Int(C_\alpha)$*.*

In this paper and to make it clearer to the reader, we call the cover of regularly open sets a regularly open cover which is different from the regular cover.

Definition 3.2. *[9] A space* X *is called almost regular if any regularly closed set* F *and any singleton* $\{x\}$ *disjoint from* F*, then there exist two open disjoint sets* U *and* V *such that* $F \subseteq U$ *and* $\{x\} \subseteq V$*.*

Definition 3.3. *A space* X *is called almost regular countably compact if and only if for every countable regular cover* $\{U_n : n \in \mathbb{N}\}$ *of* X*, there is a finite subfamily* $\{U_{n_1}, U_{n}2, ..., U_{n_m}\}$ *where* $m \in \mathbb{N}$ *of* $\{U_n : n \in \mathbb{N}\}$ *such that* $X = \bigcup_{i=1}^{m} Cl(U_{n_i})$*.*

It is obvious that every almost countably compact space is almost regular countably compact.

Proposition 0.3. *A topological space* (X, τ) *is almost regular countably compact space if and only if* (X, τ^*) *is almost regular countably compact space.*

Proof. Assume that (X, τ) is almost regular countably compact. Let $\{U_n : n \in \mathbb{N}\}$ be a countable regular cover of (X, τ^*). From definition of the regular cover, there is a nonempty regularly closed set C_n in (X, τ^*) such that $C_n \subseteq U_n$ and $X = \bigcup_{n\in\mathbb{N}} Int_{\tau^*}(C_n)$. From Lemma 1.1, and the fact that $\tau^* \subseteq \tau$, we get $\{U_n : n \in \mathbb{N}\}$ is a countable regular cover of (X, τ). From the assumption, (X, τ) is almost regular countably compact, there is a finite subfamily $\{U_{n_1}, U_{n}2, ..., U_{n_m}\}$ of the cover $\{U_n : n \in \mathbb{N}\}$ such that such that

$$X = \bigcup_{i=1}^{m} Cl_\tau(U_{n_i}).$$

Using 4 in Lemma 1.1, we get

$$X = \bigcup_{i=1}^{m} Cl_{\tau^*}(U_{n_i}).$$

Thus (X, τ^*) is almost regular countably compact space. Conversely, Assume that is (X, τ^*) is almost regular countably compact and $\{U_n : n \in \mathbb{N}\}$ is a countable regular cover of (X,τ). We know that for every $U_n \in \tau$, we have $U_n \subseteq Int_\tau(Cl_\tau(U_n))$, and by using the same lemma, we get $\{Int_\tau(Cl_\tau(U_n)) : n \in \mathbb{N}\}$ is a countable regular cover of (X,τ^*). Since (X,τ^*) is almost regular countably compact, there is a finite subfamily $\{nt_\tau(Cl_\tau(U_{n_i}))\}_{i=1}^m$such that.

$$X = \bigcup_{i=1}^{m} Cl_{\tau^*}(Int_\tau(Cl_\tau(U_{n_i}))).$$

Since $Int_\tau(Cl_\tau(U_{n_i}))$ is an open set in (X,τ) for each n_i and by using 4 in Lemma 1.1, we get

$$X = \bigcup_{i=1}^{m} Cl_{\tau}(Int_\tau(Cl_\tau(U_{n_i}))).$$

Thus, $X = \bigcup_{i=1}^{m} Cl_\tau(U_{n_i})$. This proves that (X,τ) is almost regular countably compact space. □

Corollary 3.1. *Almost regular countably compact property is a semiregular property.*

The next proposition shows that nearly regular countably compact property is a semiregular property where the proof is omitted because it is very similar to the proof of Proposition 0.3

Proposition 0.4. *A topological space (X,τ) is nearly regular countably compact space if and only if (X,τ^*) is nearly regular countably compact space.*

It is a direct result that is every nearly countably compact space is almost countably compact, and so it is almost regular countably compact space. Each of the following two propositions proves that the converse is true but with one more different strong condition in each of which.

Proposition 0.5. *Let X be an almost regular countably compact space and extremally disconnected then it is nearly countably compact space.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable open cover of X. Since X is extremally disconnected, we have $\{Cl(U_n) : n \in \mathbb{N}\}$ is a regular cover of X. Now from the assumption, X is almost regular countably compact and so it has a finite subfamily $\{Cl(U_{n_i})\}_{i=1}^m$ such that $X = \bigcup_{i=1}^{m} Cl(U_{n_i})$. Again since X is extremally disconnected, we get

$$X = \bigcup_{i=1}^{m} Int(Cl(U_{n_i})).$$

This proves that X is nearly countably compact space and the proof is done. □

Proposition 0.6. *Let X be an almost regular countably compact space and almost regular then it is nearly countably compact space.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable regularly open cover of X. That means for each $x \in X$ there is n_x such that $x \in U_{n_x}$. Since X is almost regular and by using theorem 2.2 in [9], there exists a regularly open set H_{n_x} such that $x \in H_{n_x} \subseteq Cl(H_{n_x}) \subseteq U_{n_x}$. Again, because X is almost regular and $x \in H_{n_x}$, then there is another regularly open set O_{n_x} such that $x \in O_{n_x} \subseteq Cl(O_{n_x}) \subseteq H_{n_x} \subseteq Cl(H_{n_x}) \subseteq U_{n_x}$. It is obvious that the subset $Cl(O_{n_x})$ is regularly closed subset where $X = \bigcup_{x\in X} Int(Cl(O_{n_x})) = \bigcup_{x\in X} O_{n_x}$. This makes the family $\{H_{n_x} : n_x \in \mathbb{N}\}_{x\in X}$ a countable regular cover of X. Since X is almost regular countably compact space, then there is a finite subfamily $\{H_{n_{x_i}} : n_{x_i} \in \mathbb{N}$ and $x_i \in X\}_{i=1}^{m}$ for some $m \in \mathbb{N}$ such that

$$X = \bigcup_{i=1}^{m} Cl(H_{n_{x_i}}).$$

Thus $X = \bigcup_{i=1}^{m} U_{n_{x_i}}$. This proves that X is nearly countably compact space and completes the proof. □

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