# On two-sided group digraphs and graphs

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**Abstract.** In this paper, we consider a generalization of Cayley graphs and digraphs (directed graphs) introduced by Iradmusa and Praeger. For non-empty subsets $L, R$ of group $G$, two-sided group digraph $\overrightarrow{2S}(G; L, R)$ has been defined as a digraph having the vertex set $G$, and an arc from $x$ to $y$ if and only if $y = l^{-1}xr$ for some $l \in L$ and $r \in R$. This article has strived to answer some open problems posed by Iradmusa and Praeger related to these graphs. Further, we determine sufficient conditions by which two-sided group graphs to be non-planar, and then we consider some specific cases on subsets $L, R$. We prove that the number of connected components of $\overrightarrow{2S}(G; L, R)$ is equal to the number of double cosets of the pair $L, R$ when they are two subgroups of $G$.
**Keywords:** Cayley digraph, Cayley graph, group.

## 1. Introduction

Let $G$ be a finite group and $S \subseteq G$ such that $e \notin S$. The Cayley digraph is defined as a digraph with vertex set $G$ and an arc $(x, y)$ (from vertex $x$ to vertex $y$) if and only if $x^{-1}y \in S$ denoted by $\overrightarrow{Cay}(G, S)$. The condition $e \notin S$ yields a digraph with no loops. Moreover, if $S = S^{-1}$ (where $S^{-1} = \{s^{-1} | s \in S\}$), then we have a simple undirected graph [4] called a Cayley graph and denoted by $Cay(G, S)$. In this definition, $S$ can be considered an empty set, by which the related Cayley graph has no edges. It is proved that the Cayley graph is connected if and only if $S$ generates $G$ [4]. There are many applications of Cayley graphs in different fields such as biology, coding theory and computer [3, 7, 9, 11]. So far various generalizations of Cayley graphs have been introduced, for example: generalized Cayley graph [12], quasi-Cayley graphs [6], various kinds of groupoid graphs [13, 14], group action graphs [2], general semigroup graphs [10], and there are many graphs that have been defined on algebraic structures by which many authors have been motivated to reveal some properties of the algebraic structures [1]. In this paper, we study a generalization of Cayley digraphs introduced by Iradmusa and Praeger in 2016 [8]. They named it two-sided group digraph (graph) and denoted by $\overrightarrow{2S}(G;L,R)$ ( $2S(G;L,R)$). Also, they found conditions for the adjacency relation defining the digraphs to be symmetric, transitive or connected, etc. and they posed eight problems in their article [8]. This paper has strived to answer a number of those problems which are as follows. We should emphasize that we have solved only Problem 2 completely.

**Problem 1** ([8])**.** *Decide whether or not $\overrightarrow{2S}(G;L,R)$ can be a regular graph of valency strictly less than $|L|\,|R|$, and, if it is possible, find necessary and sufficient conditions for this to occur.*

**Problem 2** ([8])**.** *Decide whether or not there exist $G, L, R$ satisfying the hypothesis of Theorem 1.7 such that $G = \langle L\rangle\langle R\rangle$, and $\overrightarrow{2S}(G;L,R)$ has connected components of different sizes.*

**Problem 3** ([8])**.** *Find necessary and sufficient conditions on $L$ and $R$ for a two-sided group digraph $\overrightarrow{2S}(G;L,R)$ to be connected, when at least one of $L$ and $R$ is not inverse-closed.*

Let $G$ be a group and $L, R$ be two non-empty subsets of $G$, then the two-sided group digraph $\overrightarrow{2S}(G;L,R)$ is defined with vertex set $G$ and an arc $(x, y)$ from $x$ to $y$' if and only if $y = l^{-1}xr$ for some $l \in L$ and $r \in R$. The connection set of $\overrightarrow{2S}(G;L,R)$ is defined as the set $\hat{S}(L,R) = \{\lambda_{l,r} : l \in L, r \in R\}$, where $\lambda_{l,r}$ is a permutation of the form $\lambda_{l,r} : g \mapsto l^{-1}gr$, for certain $l, r \in G$. Note that if there are no loops and the adjacency relation is symmetric, then $\overrightarrow{2S}(G;L,R)$ will be regarded as a simple graph, and will be named a two-sided group graph. Let $x \in G$ be an arbitrary element; we define an equivalence relation on $L \times R$ as follows: $(l_1, r_1) \sim_x (l_2, r_2)$ if and only if $(x)\lambda_{l_1,r_1} = (x)\lambda_{l_2,r_2}$; then equivalence class containing $(l, r)$ is presented as $C_x(l,r) = \{(l', r') | (x)\lambda_{l',r'} = (x)\lambda_{l,r}, l' \in L, r' \in R\}$ and $C_x$ is the set of all equivalence classes of $\sim_x$. It is obvious when $\Gamma = \overrightarrow{2S}(G;L,R)$ is an undirected graph, then valency$(x)$ is equal to $|C_x|$. In other words, the *valency*$(x)$ is corresponding to a partition of $|L||R|$.

**Definition 1.1** ([8])**.** Let $G$ be a group with identity element $e$ and two subsets $L, R$. Then a pair $(L, R)$ has *2S-graph-property* if both $L$ and $R$ are non-empty, and the following conditions hold:

(i) $L^{-1}xR = LxR^{-1}$ for each $x \in G$;

(ii) $L^x \cap R = \emptyset$ for each $x \in G$;

(iii) $(LL^{-1})^x \cap (RR^{-1}) = \{e\}$ for each $x \in G$.

(i) and (ii) in previous definition guarantee a two-sided digraph with these properties is a simple graph.

**Theorem 1.2** ([8]). *Let $G$ be a group, and $L, R$ be non-empty, inverse-closed subsets of $G$. Then $\Gamma = \overrightarrow{2S}(G; L, R)$ is a two-sided graph, which is regular of valency $|L||R|$, if and only if $(L, R)$ has the 2S-graph-property.*

The authors of [8] posed Problem 1 after Theorem 1.2. Although we didn't determine necessary and sufficient conditions by which a two-sided group graph would be a regular simple graph of valency strictly less than $|L||R|$, for which we present some recognized sufficient conditions and also necessary conditions independently. As some results of above theorem, we have the next corollaries.

**Corollary 1.3.** *Let $G$ be a group and $L, R$ be two non-empty subsets of $G$, that $L^{-1}xR = LxR^{-1}$ and $L^x \cap R = \emptyset$ for each $x \in G$. If $(LL^{-1}) \cap (RR^{-1}) = \{e\}$ and $LL^{-1} \trianglelefteq G$ or $RR^{-1} \trianglelefteq G$, then $\Gamma = 2S(G; L, R)$ is a regular simple graph of valency $|L||R|$.*

**Corollary 1.4.** *Let $G$ be a group and $L, R$ be two non-empty subsets of $G$, that $L^{-1}xR = LxR^{-1}$, $L^x \cap R = \emptyset$ for each $x \in G$. If $\Gamma$ is regular of valency strictly less than $|L||R|$, then the orders of elements of $LL^{-1}$ and $RR^{-1}$ are not relatively prime.*

**Proof.** It is obvious $\Gamma$ is a simple graph. If the orders of elements of $LL^{-1}$ and $RR^{-1}$ are relatively prime, then it is true for $(LL^{-1})^x$ and $RR^{-1}$ for all $x \in G$ and it implies that $\Gamma$ is regular of valency $|L||R|$ by Theorem 1.2, which is a contradiction.

The next theorem answers Problem 1 by using above equivalence relation without presenting certain properties on $L, R$ and $G$.

**Theorem 1.5.** *Let $G$ be a group and $L, R$ be two non-empty subsets of $G$, and $|L| > 1, |R| > 1$. Then $\Gamma = 2S(G; L, R)$ is a regular simple graph of valency strictly less than $|L|\,|R|$, if and only if $L^{-1}xR = LxR^{-1}$, $L^x \cap R = \emptyset$, $\{e\} \subsetneq RR^{-1} \cap (LL^{-1})^x$, and $|C_x| = |C_e|$ for all $x \in G$.*

**Corollary 1.6.** *Let $G$ be a group and $L, R$ are two non-empty subsets of $G$. If at least one of $L$ or $R$ is normal in $G$, $L^{-1}xR = LxR^{-1}$ for each $x \in G$ and $|LL^{-1} \cap RR^{-1}| > 1$. Then $\Gamma = 2S(G; L, R)$ is a regular simple graph of valency strictly less than $|L|\,|R|$.*

**Proof.** By Proposition 4.1 from [8] $\Gamma = 2S(G; L, R)$ is a Cayley digraph and other assumptions guarantee $\Gamma$ is a regular simple graph of valency strictly less than $|L||R|$.

**Theorem 1.7** ([8]). *Let $L, R$ be non-empty, inverse-closed subsets of a group $G$, and let $\Gamma = \overrightarrow{2S}(G; L, R)$. Then $\Gamma$ is connected if and only if*

(1) *$G = \langle L\rangle\langle R\rangle$, and there exist words $w$ in $L$ and $w'$ in $R$, with lengths of opposite parity, such that the evaluation $ww' = e$ in $G$.*

*Further, if $G = \langle L\rangle\langle R\rangle$, but condition (1) does not hold, then $\Gamma$ is disconnected with exactly two connected components.*

Problem 2 is related to above theorem; to solve this problem, we obtain two theorems and one corollary (Theorem 1.8, Corollary 1.9 and Theorem 1.10).

**Theorem 1.8.** *Let $G$ be a group, and $L, R$ be non-empty, inverse-closed subsets of $G$. Let $G = \langle L\rangle\langle R\rangle$ and at least $L$ or $R$ including a non-identity element of odd order. Then $\Gamma = \overrightarrow{2S}(G; L, R)$ is connected.*

**Corollary 1.9.** *If $G$ is a group of odd order, $L, R$ are non-empty, inverse-closed subsets of $G$ including at least a non-identity element and $G = \langle L\rangle\langle R\rangle$. Then $\Gamma = \overrightarrow{2S}(G; L, R)$ is connected.*

**Theorem 1.10.** *If $G$ is a group of even order, $L, R$ are non-empty, inverse-closed subsets of $G$ and $G = \langle L\rangle\langle R\rangle$ but condition (1) of Theorem 1.7 does not hold. Then $\Gamma = \overrightarrow{2S}(G; L, R)$ has two connected components of the same size.*

**Theorem 1.11.** *Let $L, R$ be non-empty subsets of a group $G$ such that at least one of them is inverse closed and let $\Gamma = \overrightarrow{2S}(G; L, R)$. Then $\Gamma$ is connected if and only if*

(2) *$G = \langle L\rangle\langle R\rangle$, and there exist words $w$ in $L \cup L^{-1}$ and $w'$ in $R \cup R^{-1}$, with lengths of opposite parity, such that $ww' = e$ in $G$.*

Theorem 1.11 is not a complete answer to Problem 3. In fact, the only case which remains to be answered is when both $L, R$ are not inverse-closed.

Moreover, in this paper, we present sufficient conditions by which $\Gamma = \overrightarrow{2S}(G; L, R)$ is a non-planar graph. Also, we consider the case in which both $L, R$ are singleton and, in this case, necessary and sufficient conditions has been found by which $\Gamma = 2S(G; L, R)$ is a matching. Then we consider a particular case when $L, R$ are both subgroups of $G$ and we prove that the number of connected components is equal to the number of double cosets of the pair $(L, R)$ when $\Gamma = 2S(G; L, R)$ is not connected. Further, if $L$ and $R$ are $p$-Sylow and $q$-Sylow subgroups of $G$, respectively, for prime numbers $p \neq q$ and if $L^{\#} = L - \{e\}, R^{\#} = R - \{e\}$, and the pair $(L^{\#}, R^{\#})$ has 2S-graph-property; therefore $\Gamma$, in this case, is a regular simple graph of valency $(|L| - 1)(|R| - 1)$.

For a vertex $x$ of a two-sided group digraph $\overrightarrow{2S}(G; L, R)$, the arcs beginning with $x$, are the pair $(x, y)$ with $y = (x)\lambda$, for some $\lambda \in \hat{S}(L, R)$, such elements $y$ are called out-neighbors of $x$, and the number of distinct out-neighbors of $x$ is called the out-valency of $x$. Similarly, the arcs ending in $x$ are the pairs $(y, x)$ with $(y)\lambda = x$, for some $\lambda \in \hat{S}(L, R)$, such elements $y$ are called in-neighbors of $x$, and the number of distinct in-neighbors of $x$ is called the in-valency of $x$. If there is a constant $c$ such that each vertex $x$ has out-valency $c$ and in-valency $c$, then $\overrightarrow{2S}(G; L, R)$ is regular of valency $c$.

**Remark 1.12.** Let $L$ be a non-empty subset of a group $G$. Then a word $w$ in $L$ is a string $w = l_1 l_2 ... l_k$ with each $l_i \in L$; the integer $k$ is called the length of $w$, denoted by $|w|$, and we often identify $w$ with its evaluation in $G$ (the element of $G$ is obtained by multiplying together the $l_i$ in the given order).

The following remark, from Iradmusa and Praeger [8], is used in our proof. They have shown that the connected components are the sets $\mathcal{C}_\delta = \{g | g \in G, \delta(g) = \delta\}$, for $\delta \in \{0, 1\}$.

**Remark 1.13.** Let $G$ be a group and $L, R$ be non-empty, inverse-closed subsets of $G$, and let $\Gamma = \overrightarrow{2S}(G; L, R)$. If $G = \langle L \rangle \langle R \rangle$ but condition (1) does not hold, then for each $g \in G$, we can present $g = ww'$, where $w$ and $w'$ are words in $L$ and $R$, respectively, the parity of the sum $|w|+|w'|$ is independent of the words $w, w'$, and depends only on $g$. Let $\delta(g) \in \{0, 1\}$, where $\delta(g) \equiv |w| + |w'| \pmod 2$.

## 2. Main results

Let $G$ be a group with two non-empty subsets $L, R$. If $l$ is an arbitrary element of $L$, so $l^{-1}lr = r$ and this relation means $(l, r)$ is an arc in $\Gamma = \overrightarrow{2S}(G; L, R)$, for each $l \in L, r \in R$. Similarly, $(r^{-1}, l^{-1})$ is an arc as well. Thus, $\{l, r\}, \{r^{-1}, l^{-1}\}$ are edges in $\Gamma$, in the case that $\Gamma$ is undirected, so $valency(l) \geq |R|$ and $valency(r^{-1}) \geq |L|$. Hence, if $\Gamma$ is an undirected regular graph, we will have $valency(x) \geq \frac{|L|+|R|}{2}$ for each $x \in G$.

**Proposition 2.1.** Let $G$ be a group with two non-empty subsets $L, R$ and $\Gamma = 2S(G; L, R)$ is a regular, undirected graph, then $\frac{|L|+|R|}{2} \leq valency(x) \leq |L||R|$.

**Remark 2.2.** Let $L, R$ be non-empty subsets of group $G$, and $\Gamma = \overrightarrow{2S}(G; L, R)$ be a complete digraph (without regarding directions among all arcs). Then for each $e \neq g \in G$: $(e, g)$ or $(g, e)$ is an arc, so $l^{-1}er = g$ or $l^{-1}gr = e$, for some $l \in L$ and $r \in R$. Thus $l^{-1}r = g$ or $g = lr^{-1}$; therefore, $g \in L^{-1}R \cup LR^{-1}$, and it follows that $G = \langle L^{-1}R \cup LR^{-1} \rangle$. In particular, if $L$ and $R$ be inverse-closed, in this case, we have $G = \langle LR \rangle$; however the reverse is not true in general. The following example illustrates this point.

**Example 2.3.** Let $G = S_3$, $L = \{(12)\}$, $R = \{(123), (132)\}$ and $\Gamma = \overrightarrow{2S}(G; L, R)$. Figure 1 displays this graph. Clearly, $G = S_3 = \langle LR \rangle$, and $\Gamma$ is not complete.

**Theorem 2.4.** *Let $G$ be a group, and $|G| = p^\alpha q^\beta m$, where $p$ and $q$ are distinct prime numbers and $\gcd(m, p) = 1, \gcd(m, q) = 1$. Let $L$ and $R$ be $p$-Sylow subgroup and $q$-Sylow subgroup of $G$, respectively. Suppose that $L^\# = L - \{e\}, R^\# = R - \{e\}$ and $\Gamma = \overrightarrow{2S}(G; L^\#, R^\#)$, then pair $(L^\#, R^\#)$ has the 2S-graph-property; therefore, $\Gamma$ is a simple graph and it is regular of valency $(p^\alpha - 1)(q^\beta - 1)$.*

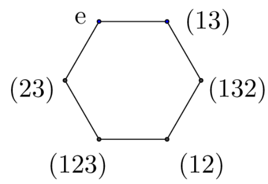


Figure 1: $2S(S_3, L, R)$.

**Proof.** It is clear that $L^{\#-1}gR^{\#} = L^{\#}gR^{\#-1}$. In addition, $L^{\#g} \cap R^{\#} = \emptyset$, for each $g \in G$, since $L$ contains elements of a $p$-power order and $R$ contains elements of a $q$-power order and $p \neq q$. Finally, we have $(L^{\#}L^{\#-1})^g \cap (R^{\#}R^{\#-1}) \subseteq L^g \cap R = \{e\}$. So pair $(L^{\#}, R^{\#})$ satisfies the third condition as well, therefore $\Gamma$ is a graph. Thus by Theorem 1.2 $\Gamma$ is regular of valency $(p^\alpha - 1)(q^\beta - 1)$.

A graph is planar if it can be drawn in such a way that no edges cross each other; by Kuratowski's theorem [6] we know that a graph is planar if and only if it contains no subgraph that is a subdivision of either $K_5$ or $K_{3,3}$; based on this theorem we have the next result.

**Theorem 2.5.** *Let $L, R$ be non-empty subsets of a group $G$ and let $\Gamma = 2S(G; L, R)$ be a two-sided group (undirected) graph. If we have $L \cap L^{-1}LR = \emptyset$, $R \cap L^{-1}RR = \emptyset$, $|L| \geq 3$ and $|R| \geq 3$, then $\Gamma$ is non-planar.*

**Proof.** If $\{l, l'\}$ is an edge, for two arbitrary $l, l' \in L$, then $l' = l_1^{-1}lr_1$ for some $l_1 \in L$, $r_1 \in R$, thus $L \cap L^{-1}LR \neq \emptyset$ is a contradiction. So, for each $l, l' \in L$, $\{l, l'\}$ is not an edge. Similarly, assumption $R \cap L^{-1}RR \neq \emptyset$ implies that $\{r, r'\}$ is not an edge, for each $r, r' \in R$. And also for each $r \in R$ and $l \in L$ we have $r = l^{-1}lr$, so $\{r, l\}$ is an edge; therefore, $\Gamma$ contains a complete bipartite graph $K_{|L|,|R|}$ as a subgraph. Since $|L| \geq 3$ and $|R| \geq 3$, so $\Gamma$ contains $K_{3,3}$, hence $\Gamma$ is non-planar. It is reminded that an independent set is a set of vertices in a graph, no two of which are adjacent.

**Lemma 2.6.** *Let $L, R$ be two subsets of a group $G$, and $\Gamma = \overrightarrow{2S}(G; L, R)$. If $S$ is an independent subset of $G$, then $LS \cap SR = \emptyset$.*

**Proof.** Let $x \in LS \cap SR$ then $x = ls_1 = s_2r$ for some $s_1, s_2 \in S$ and $l \in L$, $r \in R$. Therefore we have $s_1 = l^{-1}s_2r$ so $s_1$ connected to $s_2$, and this is a contradiction.

**Corollary 2.7.** *Let $L, R$ be two subsets of group $G$, and $\Gamma = \overrightarrow{2S}(G; L, R)$. If $S$ is an independent subset of $G$, then $|LS| + |SR| \leq |G|$.*

**Proof.** According to Lemma 2.6 we have : $LS \cap SR = \emptyset$; therefore we conclude: $|LS| + |SR| \leq |G|$.

In this part, we introduce some notations about two-sided group digraph. Let $G$ be a group, and $L, R$ be two non-empty subsets of $G$, and $\Gamma = \overrightarrow{2S}(G; L, R)$ be a two-sided group digraph of $G$ with respect to $L, R$. Let $A = Aut(\Gamma)$, $Aut(G, L, R) = \{\alpha \in Aut(G) | L^\alpha = L, R^\alpha = R\}$ , $A_1$ and $1^A$ be the stabilizer and the orbit of identity. It is reminded that $\mathcal{R}(G)$ and $\mathcal{L}(G)$ are considered as right and left representation respectively. Obviously, we have following results.

**Proposition 2.8.** (1) $\mathcal{R}(N_G(R))$, $\mathcal{L}(N_G(L))$ *are subgroups of* $A$.
(2) $N_G(L)N_G(R) \subseteq 1^A$.

**Proof.** It is clear.

**Corollary 2.9.** *Let $G$ be a group and $L, R$ be non-empty subsets of $G$, and let $\Gamma = \overrightarrow{2S}(G; L, R)$ be a two-sided group digraph. Then $Aut(G, L, R) \leq Aut(G) \cap A_1 \leq Aut(G, L^{-1}R)$.*

**Proof.** The first part of above inequality is clear. Now, assume that $\varphi \in Aut(G) \cap A_1$, therefore $\varphi$ is a group homomorphism which keeps the adjacency relation and $\varphi(1) = 1$. Since 1 is connected to all elements of $L^{-1}R$; therefore, $\varphi(1) = 1$ connected to $\varphi(L^{-1}R)$. Thus $\varphi(L^{-1}R) \subseteq L^{-1}R$. It follows that $\varphi \in Aut(G, L^{-1}R)$.

## 2.1 Proof of Theorems

**Proof of Theorem 1.5.** First, if all given conditions are satisfied, it is clear, $\Gamma$ is an undirected regular graph of valency strictly less than $|L|\,|R|$ by Lemma 3.1 [8] and argument which was presented before definition 1.1. Conversely, let $\Gamma$ be a regular (undirected) graph of valency strictly less than $|L|\,|R|$. By Lemma 3.1 [8], it is clear that $L^{-1}xR = LxR^{-1}$, $L^x \cap R = \emptyset$ and $|C_x| = |C_e|$ for each $x \in G$. On the other hand, for each $x \in G$, we have $\left|L^{-1}xR\right| < |L|\,|R|$, so there exist $(l_1, r_1), (l_2, r_2) \in L \times R$ such that $(l_1, r_1) \neq (l_2, r_2)$ $(|L| > 1, |R| > 1)$ and $l_1^{-1}xr_1 = l_2^{-1}xr_2$ $(l_1, l_2 \in L$ *and* $r_1, r_2 \in R)$, then $x^{-1}l_2l_1^{-1}x = r_2r_1^{-1}$. If $r_2r_1^{-1} = e$, then $r_1 = r_2$, and so $l_2 = l_1$ is a contradiction; therefore, $r_2r_1^{-1} \neq e$. Similarly if $l_1 = l_2$ then $r_1 = r_2$, i.e. $\{e\} \subsetneqq RR^{-1} \cap (LL^{-1})^x$.

**Example 2.10.** Let $G = S_3$, $L = \{(12), (23)\}$ and $R = \{(123), (132)\}$. It's easy to see that, this example has the mentioned above properties. This graph has been drawn by Figure 2, as you can see : $valency(x) = 3 < |L|\,|R| = 4$.

The next theorem gives a sufficient condition by which a simple two-sided group graph is regular of valency less than $|L||R|$.

**Theorem 2.11.** *Let $G$ be a group, and $L$, $R$ be two non-empty subsets of $G$. If $G$ factorizes as $G = N_G(L)N_G(R)$ and $|LL^{-1} \cap RR^{-1}| > 1$, $\Gamma = 2S(G; L, R)$ be a simple graph, then $\Gamma$ is regular of valency less than $|L|\,|R|$.*

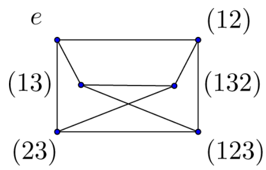


Figure 2: $2S(S_3, L, R)$.

**Proof.** By Theorem 1.13 of [8] $\Gamma$ is vertex-transitive, thus $\Gamma$ is regular. It is clear the valency of $\Gamma$ is less than $|L|\,|R|$.

**Theorem 2.12.** *Let $G$ be a group, and $L$, $R$ are two non-empty subsets of $G$, $|G| = pm$ where $p$ is a prime number, $gcd(p, m) = 1$, $|Syl_p(G)| > 1$, $LL^{-1} = RR^{-1}$ and $LL^{-1} \subseteq P$, for some $P \in Syl_p(G)$, $|L| > 1, |R| > 1$ and $L^{-1}xR = LxR^{-1}$, $L^x \cap R = \emptyset$ for each $x \in G$, then $\Gamma = 2S(G; L, R)$ is a non-regular simple graph.*

**Proof.** It is clear $\Gamma$ is a simple graph and $(LL^{-1})^x \subseteq P^x$, for all $x \in G$. Since $|Syl_p(G)| > 1$, so there is $x \in G$ such that $P^x \neq P$; therefore $|(LL^{-1})^x \cap RR^{-1}| \leq |P^x \cap P| = 1$. On the other hand, we have $|LL^{-1} \cap RR^{-1}| > 1$, thus $\Gamma$ is not regular.

**Proof of Theorem 1.8.** Let $e \neq l \in L$ be an arbitrary element of odd order, and suppose m is its order. If $r \in R$ is an arbitrary element, so we have $l^m rr^{-1} = e$, and it means condition (1) of Theorem 1.7 holds, hence $\Gamma$ is connected. Similarly, if $R$ includes an element of odd order, then condition (1) holds.

**Proof of Theorem 1.10.** By Theorem 1.7 $\Gamma$ is disconnected with exactly two connected components, and according to Remark 1.13 these two connected components are $\mathcal{C}_0$, and $\mathcal{C}_1$. We show that $|\mathcal{C}_0| = |\mathcal{C}_1|$. It is adequate to define function $\phi : \mathcal{C}_0 \to \mathcal{C}_1$ such that $\phi(g) = lg$, where $l \in L$ is an arbitrary element but fixed in $L$. It is clear that $\phi$ is well-defined, for $g \in \mathcal{C}_0$ and $g = x_g y_g$, we have $lg = lx_g y_g$, so $|lg| = |lx_g| + |y_g| = |x_g| + |y_g| \pm 1 \equiv 1(\text{mod}2)$, because $|x_g| + |y_g| \equiv 0(\text{mod}2)$, and it means $lg \in \mathcal{C}_1$. It is obvious $\phi$ is a one-to-one map, hence $|\mathcal{C}_0| = |\mathcal{C}_1|$.

Corollary 1.9 and Theorem 1.10 answer Problem 2, i.e. there are no $G, L, R$ satisfying the hypothesis of Theorem 1.7 such that $G = \langle L\rangle\langle R\rangle$, and $\overrightarrow{2S}(G; L, R)$ has connected components of different sizes.

It should be reminded, Theorem 1.11 is a generalization of Theorem 1.7.

**Proof of Theorem 1.11.** If $\Gamma$ is connected, then by Lemma 3.4 [8] condition (2) holds. Conversely, suppose that condition (2) holds. First, since $G = \langle L\rangle\langle R\rangle$, then we can write $g = x_g y_g$ for every $g \in G$ in which $x_g$ and $y_g$ are words in

$L \cup L^{-1}$ and $R \cup R^{-1}$, respectively. By condition (2) there are words $x_e$ in $L \cup L^{-1}$ and $y_e$ in $R \cup R^{-1}$ with lengths of opposite parity, such that $x_e y_e = e$. It implies, as it has been done in [8], for each $g \in G$ we can find $x_g$ and $y_g$ with the same length. Now, we suppose $L$ is inverse-closed, then $x_g = l_k l_{k-1} ... l_1$ and $y_g = r_1 ... r_k$ such that $l_i \in L$ and $r_i \in R \cup R^{-1}$. If we put $g_i = l_i^{-1} g_{i-1} r_i$ and $g_0 = e$, then there is a path from $e$ to $g$ in $\Gamma$, because either $(g_{i-1}, g_i)$ or $(g_i, g_{i-1})$ is an arc in $\Gamma$, and it depends on $r_i \in R$ or $r_i \in R^{-1}$. By a similar argument we can obtain a path from $e$ to $g$ when $R$ is inverse-closed, thus $\Gamma$ is connected.

### 2.2 Considering some specific case

Now, we consider the case in which $L$, $R$ are singleton. Let $L, R$ be non-empty subsets of group $G$, and $\Gamma = \overrightarrow{2S}(G; L, R)$. It can be proved if $|L| = 1$ (or $|R| = 1$) then $\Gamma$ is a regular digraph of valency $|R|$ ($|L|$). Furthermore, if $L = \{l\}, R = \{r\}, l \neq r$, then $(L, R)$ has 2S-graph-property if and only if $l^2 = r^2, l^2 \in Z(G)$ and $r \neq x^{-1} l x$ for each $x \in G$; in particular $l \neq r$.

**Example 2.13.** Let $G = D_8 = \langle a, b |\ a^4 = b^2 = e,\ bab = a^{-1} \rangle$, be the dihedral group of order eight and $L = \{a^2\}$, $R = \{b\}$. Then $(a^2)^2 = b^2 = e$ and also $l^2 = (a^2)^2 = a^4 = e \in Z(D_8)$, and b, $a^2$ are not conjugate. Finally, $\Gamma$ can be presented as follows.

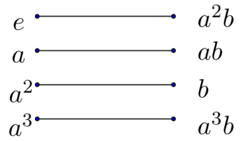


Figure 3: $\Gamma = 2S(D_8; L, R)$.

**Proposition 2.14.** *Let $G$ be a group and $L$, $R$ be non-empty subsets of $G$.*

(1) $\Gamma = \overrightarrow{2S}(G; L, R)$ *is matching if and only if $L$ and $R$ are single-member having 2S-graph-property.*

(2) *If $L, R$ are single-member subsets and pair $(L, R)$ has the 2S-graph-property then the order of $G$ is even.*

**Proof.** (1) Due to 2S-graph-property conditions and by Theorem 1.2, $\Gamma = \overrightarrow{2S}(G; L, R)$ is a regular graph of valency $|L|\,|R| = 1$, and it implies that graph $\Gamma$ is a matching. Conversely, by Theorem 1.2 if $\Gamma = \overrightarrow{2S}(G; L, R)$ is a matching, thus $\Gamma$ is a regular graph of valency 1 i.e. $|L|\,|R| = 1$, and so pair $(L, R)$ has 2S-graph-property.

(2) By part (1), in this case, graph $\Gamma = \overrightarrow{2S}(G; L, R)$ is a matching and it concludes $|G|$ is even.

**Proposition 2.15.** *Let $G$ be a group, $L = \{l\}, R = \{r\}$ be single-member subsets of $G$ and $n \geq 3$ is an integer number. Then digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ has a cycle of length $n$ (thus girth$\Gamma \leq n$) if and only if $l^n g = gr^n$ for some $g \in G$, and $n$ is the least integer with this property.*

**Proof.** Suppose that digraph $\Gamma$ has a cycle of length $n$, and this cycle is $x_1 x_2 ... x_n x_1$ in which $x_i \neq x_j$, when $i \neq j$. Because of $\Gamma = \overrightarrow{2S}(G; L, R)$ is a regular digraph of valency one, so we have: $x_2 = l^{-1} x_1 r$, $x_3 = l^{-1} x_2 r$, ..., $x_1 = l^{-1} x_n r$ and then $x_1 = \underbrace{l^{-1} l^{-1} ... l^{-1}}_{n-time} x_1 \underbrace{r ... rr}_{n-time} = l^{-n} x_1 r^n$ , thus $l^n x_1 = x_1 r^n$.

Conversely, if $l^n g = gr^n$ for some $g \in G$, then $g = x_n \in G$ is satisfied with $x_n = l^{-n} x_n r^n$, and if we set $l^{-1} x_n r = x_1$, $l^{-1} x_1 r = x_2$, ..., $l^{-1} x_{n-2} r = x_{n-1}$ then $x_1 x_2 ... x_n x_1$ is a cycle of length $n$.

Let $L, R$ be subgroups of group $G$ and $\Gamma = \overrightarrow{2S}(G; L, R)$. Because of $L = L^{-1}$ and $R = R^{-1}$, adjacency relation in $\Gamma$ is symmetric. However, the pair $(L, R)$ doesn't have 2S-graph-property, because $\{e\} \subseteq L^x \cap R$ for each $x \in G$, so each vertex of $\Gamma$ has a loop. In this case, because adjacency relation is symmetric, let us call $\Gamma$ a graph for simply in spite of having loop on each vertex, and also we use words such as complete graph, regular graph, connected graph and domination number though we know, it is not a simple graph.

Let $L, R$ be subgroups of group $G$ and $H = \{\lambda_{l,r} | l \in L, r \in R\}$. It is clear $H$ is a group (with the composition operation), and for each $x \in G$ and $\lambda_{l,r} \in H$ we have: $(x)\lambda_{l,r} = l^{-1} xr$, i.e. $H$ acts on $G$, and stabilizer of $x$ is $stab_H(x) = \{\lambda_{l,r} \in H | l^{-1} xr = x\} \leq H$, and the orbit of $x$; $orbit(x) = \{l^{-1} xr | \lambda_{l,r} \in H\} = LxR$ is a double coset of $L$ and $R$, for each $x \in G$ and then $valency(x)$, in $\Gamma = \overrightarrow{2S}(G; L, R)$, is equal to $\frac{|H|}{|stab_H(x)|} = |LxR|$, and also an orbit is a connected component which is a complete subgraph with a loop on each vertex. In particular, the $orbit(e) = \{l^{-1} r | l \in L, r \in R\} = LR$, thus $|orbit(e)| = |LR| = \frac{|L||R|}{|L \cap R|}$, $stab_H(e) = \{\lambda_{r,r} \in H | r \in R \cap L\}$, so $|stab_H(e)| = |R \cap L|$, and hence $|H| = |L||R|$. In this case, if $\Gamma$ is regular, then $|orbit(x)| = |orbit(e)| = |LR|$ for each $x \in G$. In other words, if $\Gamma$ is a regular graph, then all double coset of $L$ and $R$ are the same size. In one specific case, if we consider $L = \{e\}$, then the connected component is the left coset $R$ and the number of connected components is $\frac{|G|}{|R|}$ and $\Gamma$ is a regular graph of valency $|R|$. By considering the action of $H$ on $G$, we have: the kernel of this action contains all $\lambda_{l,l}$ such that $l \in L \cap R \cap Z(G)$, and it is faithful if and only if $L \cap R \cap Z(G) = \{e\}$, because $\lambda_{l,r}$ belongs to the kernel, if and only if $(x)\lambda_{l,r} = x$ for each $x \in G$, therefore $l^{-1} xr = x$ for each $x \in G$. Specially, if $x \in Z(G)$, then $l^{-1} xr = x$ and it concludes that $l = r$, it means $l \in L \cap R$. Now, $l^{-1} xl = x$ for each $x \in G$ implies that $l \in L \cap R \cap Z(G)$. If $G = LR$, then $H$ acts on $G$ transitively, and $\Gamma$ is a complete graph. Therefore, we have the following theorem.

**Theorem 2.16.** *Let $G$ be a group and $L, R$ be subgroups of $G$, and $H = \{\lambda_{l,r} | l \in L, r \in R\}$. Then the group $H$ acts on $G$, $\Gamma = \overrightarrow{2S}(G; L, R)$ is a graph with one* *loop on each vertex,* $valency(x) = \frac{|H|}{|stab_H(x)|}$ *for each* $x \in G$ *and* $|H| = |L||R|$. *In particular* $\Gamma$ *is regular if and only if* $valency(x) = |LR|$ *for each* $x \in G$. *Graph* $\Gamma$ *is connected if and only if* $G = LR$ *and otherwise, the number of connected components is equal to the number of double coset of the pair* $(L, R)$.

A dominating set for a graph is a subset $D$ of its vertices such that every vertex which is not in $D$ is adjacent to at least one member of $D$. The domination number $\gamma(G)$ is the number of vertices in the smallest dominating set for the graph [5].

**Corollary 2.17.** *Let* $G$ *be a group, and* $L, R$ *be subgroups of* $G$, *and* $\Gamma = \overrightarrow{2S}(G; L, R)$. *Then domination number of* $\Gamma$ *is the number of double coset of the pair* $(L, R)$ .

**Proof.** Since $L, R \leq G$, then connected components graph $\Gamma$ are complete, therefore domination number of each connected component of graph $\Gamma$ is one, so domination number of $\Gamma$ is the number of connected components. It is trivial that the domination set contains one representative of each double coset of $L$ and $R$.

**Theorem 2.18.** *Let* $L, R$ *be two subgroups of a group* $G$. *Then:*

(1) $\Gamma = \overrightarrow{2S}(G; L, R)$ *is a regular graph with one loop on each vertex, of valency strictly less than* $|L|\,|R|$, *if and only if* $|L \cap R| > 1$.

(2) *The valency of* $e$ *in graph* $\Gamma = \overrightarrow{2S}(G; L, R)$ *is one, if and only if* $L = R = \{e\}$.

**Proof.** By Theorem 2.16, part (1) is clear.

If valency $(e) = 1$, then $\frac{|L||R|}{|L \cap R|} = 1$, therefore $L = R = \{e\}$, and the converse is clear. In this case $H = \{\lambda_{e,e}\} = \{id\}$ and $orbit(x) = \{x\}$ i.e. the graph $\Gamma$ has only loops on each vertex.

## References


[1] A. Abdollahia, S. Akbari, H.R. Maimani, *Non-commuting graph of a group,* Journal of Algebra, 298 (2006), 468-492.

[2] Fred Annexstein, Marc Baumslag and Arnold L. Rosenberg, *Group action graphs and parallel architectures,* SIAM J. Comput., 19 (1990), 544-569.

[3] Vineet Bafna and Pavel A. Pevzner, *Genome sorting by reversals,* SIAM J. Comput., 25 (1996), 272-289.

[4] Norman L. Biggs, *Algabraic graph theory,* second ed., Cambrige Univ. Press, Cambridge, 1993.

[5] Gary Chartrand and Linda Lesniak, *Graphs & Digraphs,* Chapman & Hall/CRS Press, Florida, 1996.

[6] Ginette Gauyacq, *On quasi-Cayley graphs,* Discrete Appl. Math., 77 (1997), 43-58.

[7] Gena Hahn and Gert Sabidussi (eds.), *Graph symmetry,* NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, vol. 497, Dordrecht, Kluwer Academic Publishers Group, 1997, Algebraic Methods and Applications.

[8] Moharram N. Iradmusa, Cheryl E. Praeger, *Two-sided group digraphs and graphs,* Journal of Graph Theory, 82 (2016), 279-295.

[9] John Kececioglu and David Sankoff, *Exact and approximation algorithms for sorting by reversals, with application to genome rearrangement,* Algorithmica, 13 (1995), 180-210.

[10] Andrei V. Kelarev and Cheryl E. Praeger, *On transitive Cayley graphs of groups and semigroups,* European J. Combin., 24 (2003), 59-72.

[11] S. Lakshmivarahan, Jung Sing Jwo and S. K. Dall, *Symmetry in interconnection networks based on Cayley graphs of permutation groups: a survey,* Parallel Comput., 19 (1993), 361-407.

[12] Dragan Marusic, Raffaele Scapellato and Norma Zagaglia Salvi, *Generalized Cayley graphs,* Discrete Math., 102 (1992), 2797-285.

[13] Eric Mwambene, *Representing graphs on groupoids: symmetry and form,* Ph.D. Thesis, University of Vienna, 2001.

[14] Eric Mwambene, *Cayley graphs on left quasi-groups and groupoids representing k-generalised Petersen graphs*, *Group action graphs and parallel architectures,* Discrete Math., 309 (2009), 2544-2547.

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