# Some classes of mappings on generalized quaternion metric spaces II

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**Abstract.** In connection with the theory of $(m, q)$-isometries mappings on metric spaces ([3]) and the theory of $m$-quaternion-valued $G$-isometric mappings ([1]), we introduce the concept of $(m, \infty)$-generalized isometric mappings on a generalized real-valued metric space. We present some essential properties of these classes of mappings.

**Keywords:** metric space, $G$-metric space, quaternion space, $m$-isometry.

## 1. Introduction and preliminaries results

In [9] and [10], the authors Mustafa and Sims introduced the concept of generalized metric spaces , which are called a real $G$-metric spaces as generalization of metric space $(E, d_{\mathbb{R}})$. A generalization of real metric and $G$-metric spaces to complex-valued metric space $(E, d_{\mathbb{C}})$ and to complex-valued $G$-metric space $(E, G_{\mathbb{C}})$ has been presented by many authors in the last years in papers [2], [4], [5], [8] and [12]. Very recently, in paper [1] the present author introduced and study the concept of quaternion-valued $G$-metric spaces $(E, G_{\mathbf{H}})$.

**Definition 1.1** ([9], [10]). *Let $E$ be an non-empty set and let $G_{\mathbb{R}} : E \times E \times E \longrightarrow \mathbb{R}_+ = [0,\ \infty)$ be a function satisfying the following conditions*

(1) $G_{\mathbb{R}}(u, v, w) = 0$ *if* $u = v = w$*;*

(2) $0 < G_{\mathbb{R}}(u, u, v)$ *for all* $u, v \in E$ *with* $u \neq v$*;*

(3) $G_{\mathbb{R}}(u, u, v) \leq G_{\mathbb{R}}(u, v, w)$ *for all* $u, v, w \in E$ *with* $v \neq w$*;*

(4) $G_{\mathbb{R}}(u, v, w) = G_{\mathbb{R}}(u, w, v) = G_{\mathbb{R}}(v, w, u) = \ldots$ *(symmetry in all three variables);*

(5) $G_{\mathbb{R}}(u, v, w) \leq G_{\mathbb{R}}(u, a, a) + G_{\mathbb{R}}(a, v, w)$*, for all* $u, v, w, a \in E$ *(rectangle inequality).*

*Then the function $G_{\mathbb{R}}$ is called a real-valued generalized metric or, more specifically, a $G$-metric on $E$ and the pair $(E, G_{\mathbb{R}})$ is called a real $G$-metric space.*

Based on this notion, many fixed point results under different conditions have been obtained for a variety of mappings in this new setting.

A $G$-metric space $(E, G_{\mathbb{R}})$ is called symmetric $G$-metric space if $G_{\mathbb{R}}(u,v,v) = G_{\mathbb{R}}(u,u,v)$, for all $u, v \in E$. For $k = 1, \ldots, d$ , let $(E_k, G^k_{\mathbb{R}})$ be $G$-metric spaces and let $E = E_1 \times E_2 \times \ldots \times E_d$, then natural definitions for $G$-metrics on the product space $E$ would be

$$\begin{aligned} G^m_{\mathbb{R}}(u,v,w) &= G^m_{\mathbb{R}}((u_1,\ldots,u_d),(v_1,\ldots,v_d),(w_1,\ldots,w_d)) \\ &= \max_{k\in\{1,\ldots,d\}}\{G^k_{\mathbb{R}}(u_k,v_k,w_k)\} \end{aligned}$$

or

$$\begin{aligned} G^s_{\mathbb{R}}(u,v,w) &= G^s_{\mathbb{R}}((u_1,\ldots,u_d),(v_1,\ldots,v_d),(w_1,\ldots,w_d) \\ &= \sum_{k=1}^{d} G^k_{\mathbb{R}}(u_k,v_k,w_k)) \end{aligned}$$

for all $u = (u_1, \ldots, u_d), v = (v_1, \ldots, v_d), w = (w_1, \ldots, w_d) \in E$.

It was observed in [10] that in general $(E,\ G^m_{\mathbb{R}})$ and $(E,\ G^s_{\mathbb{R}})$ are not $G$-metrics spaces. However, it was proved that $(E,\ G^m_{\mathbb{R}})$ (resp.$(E,\ G^s_{\mathbb{R}})$) is a symmetric $G$-metric space if and only if each $(E_k,\ G^k_{\mathbb{R}})$ is a symmetric $G$-metric space for $k = 1, \ldots, d$.

The set of real quaternions, denoted by $\mathbf{H}$, is defined by

$$\mathbf{H} := \{a_0 + a_1 i + a_2 j + a_3 k,\ \ a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R}\ \},$$

where $i^2 = j^2 = k^2 = ijk = -1,\ ij = k, jk = i,\ ki = j$. Note that $ij = -ji,\ ik = -ki,\ jk = -kj$ and there is an operation on $\mathbf{H}$ called quaternionic conjugation which is defined by

$$\overline{(a_0 + a_1 i + a_2 j + a_3 k)} = a_0 - a_1 i - a_2 j - a_3 k.$$

Every element of $\mathbf{H}$ has an additive inverse $-$ if $q = a_0 + a_1 i + a_2 j + a_3 k \in \mathbf{H}$ then $-q = (-a_0) + (-a_1)i + (-a_2)j + (-a_3)k \in \mathbf{H}$. For all $q \in \mathbf{H}$, $Re(q) := \frac{1}{2}(q + \overline{q})$ is the real part of $q$ and $Im(q) := \frac{1}{2}(q - \overline{q})$ is the imaginary part of $q$.

The sum of $q_1 = a_0 + a_1 i + a_2 j + a_3 k$ and $q_2 = b_0 + b_1 i + b_2 j + b_3 k$ is defined as

$$q_1 + q_2 = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k \in \mathbf{H}.$$

We refer the reader to the paper [11] for more details.

In [6] the authors have considered a partial order on $\mathbf{H}$ as follows:

$$\begin{cases} q_1 \preccurlyeq q_2 \ \text{ if and only if } \ Re(q_1) \le Re(q_2), \\ Im_s(q_1) \le Im_s(q_2),\ q_1, q_2 \in \mathbf{H};\ s = i, j, k; \\ \text{where } Im_i(q_r) = a_{1r},\ Im_j(q_r) = a_{2r}, Im_k(q_r) = a_{3r}, \\ q_r = a_{0r} + a_{1r} i + q_{2r} j + a_{3r} k,\ \ r = 1, 2. \end{cases}$$

Following the partial order defined on $\mathbf{H}$, the present author has been apparently the first one introduce the notion of generalized quaternion metric space.

**Definition 1.2** ([1]). *Let $E$ be an non-empty set and let $G_{\mathbf{H}} : E \times E \times E \to \mathbf{H}$ be a function satisfying the following conditions:*

(1) $G_{\mathbf{H}}(u, v, w) = \mathbf{0_H}$ *if* $u = v = w$*;*

(2) $\mathbf{0_H} \prec G_{\mathbf{H}}(u, u, v)$ *for all* $u, v \in E$ *with* $u \neq v$*;*

(3) $G_{\mathbf{H}}(u, u, v) \preccurlyeq G_{\mathbf{H}}(u, v, w)$ *for all* $u, v, w \in E$ *with* $v \neq w$*;*

(4) $G_{\mathbf{H}}(u, v, w) = G_{\mathbf{H}}(u, w, v) = G_{\mathbf{H}}(v, w, u) = ...$ *(symmetry in all three variables);*

(5) $G_{\mathbf{H}}(u, v, w) \preccurlyeq G_{\mathbf{H}}(u, a, a) + G_{\mathbf{H}}(a, v, w)$, *(for all* $u, v, w, a \in E$, *(rectangle inequality).*

*Then the function $G_{\mathbf{H}}$ is called a quaternion-valued generalized metric or, more specifically, a quaternion-valued $G_{\mathbf{H}}$-metric on $E$ and the pair $(E, G_{\mathbf{H}})$ is called a quaternion-valued $G$-metric space.*

In the paper [3], Bermúdez, Martinón and Müller introduced the notion of $(m, q)$-isometry for maps on a real-valued metric space $(E, d_{\mathbb{R}})$. A map $S : E \to E$ is called an $(m, q)$-isometric mapping for positive integer $m \geq 1$ and for real $q > 0$, if it satisfies

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} d(S^k u, S^k v)^q = 0, \quad \forall \ u, v \in E.$$

**Definition 1.3** ([1]). (i) Let $(E, G_{\mathbb{R}})$ be a real-valued $G$-metric space. A map $S : E \to E$ is called an $(m, q)$-$G$-isometry for some positive integer $m$ and $q > 0$ if, for all $u, v, w \in E$

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbb{R}}(S^r u, S^r v, S^r w)^q = 0.$$

(ii) Let $(E, G_{\mathbf{H}})$ be a quaternion-valued $G$-metric space. A map $S : E \to E$ is called an $m$-quaternion-valued $G$-isometric map for some positive integer $m$ if, for all $u, v, w \in E$

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbf{H}}(S^r u, S^r v, S^r w) = \mathbf{0_H}.$$

In the following, we collect some properties of $(m, q)$-$G$-isometric mappings.

The proof of the following theorem is very similar to ([1], Proposition 3.1, Theorem 3.1, Theorem 3.3) we omit it.

**Theorem 1.1.** *Let $(E, G_{\mathbb{R}})$ be a real-valued $G$-metric space and let $S : E \longrightarrow E$ be an mapping. The following statements hold:*

(1) *if $S$ is an $(m, q)$-$G$-isometry, then $S$ is an $(n, q)$-$G$-isometry for all positive integer $n \geq m$.*

(2) *if $S$ invertible $(m, q)$-$G$-isometry, then $S^{-1}$ is an $(m, q)$-$G$-isometry.*

(3) *if $S$ is an $(m, q)$-$G$-isometry, then $T^n$ is an $(m, q)$-$G$-isometry for all $n = 1, 2, \ldots, .$*

**Theorem 1.2.** *Let* $(X, G_{\mathbb{R}})$ *be a real-valued* $\mathbf{G}$*-metric space and* $S, R : E \to E$ *be a maps such that* $RS = SR$. *If* $S$ *is an* $(m, q)$*-*$G$*-isometry and* $R$ *is an* $(n, q)$*-*$G$*-isometry, then* $SR$ *is an* $(m + n - 1, q)$*-*$G$*-isometry.*

The proof of this theorem is very similar to ([1], Theorem 3.4).

For mare details about the concept of $m$-quaternion valued $G$-isometries, the reader can refer to [1].

## 2. $(m, \infty)$-$G$-isometric mappings in generalized real metric space

In this section, we present the definition of $(m, \infty)$-$G$-isometric mapping on a real-valued metric spaces and give the main results of this topic. Similar results for $(m, \infty)$-isometric operators on Banach space were proved by P. Hoffmann et al. in [7].

Let $S : E \to E$ be an $(m, q)$-$G$-isometric mapping. It obvious that for all $u, v, w \in E$

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q = 0$$

$$\Leftrightarrow \sum_{\substack{k\in\{0,\dots,m\}\\ k\ \text{even}}}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q = \sum_{\substack{k\in\{0,\dots,m\}\\ k\ \text{odd}}}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q$$

$$\Leftrightarrow g\Bigg(\sum_{\substack{k\in\{0,\dots,m\}\\ k\ \text{even}}}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q\Bigg)^{\frac{1}{q}}$$

$$= g\Bigg(\sum_{\substack{k\in\{0,\dots,m\}\\ k\ \text{odd}}}\binom{m}{k}G_{\mathbb{R}}(S^k u, S^k v, S^k w)^q\Bigg)^{\frac{1}{q}}.$$

By taking the limit as $q \to \infty$, we make the following definition of an $(m, \infty)$-$G$-isometric mapping.

**Definition 2.1.** *Let* $m$ *be a positive integer* $m \geq 1$. *An mapping* $S$ *acting on a generalized real-valued metric space* $(E, G_{\mathbb{R}})$ *is called an* $(m, \infty)$*-*$G$*-isometry (or* $(m, \infty)$*-generalized isometry) if for all* $u, v, w \in E$

$$\max_{\substack{j\in\{0,\dots,m\}\\ (j\ \text{even})}}\{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\} = \max_{\substack{j\in\{0,\dots,m\}\\ (j\ \text{odd})}}\{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\}.$$

**Remark 2.1.** (i) Every $(1, \infty)$-$G$-isometric mapping $S$ is an $G$-isometric mapping i.e.,$S$ satisfies

$$G_{\mathbb{R}}(Su, Sv, Sw) = G_{\mathbb{R}}(u, v, w) \text{ for all } u, v, w \in E.$$

(ii) An mapping $S : E \to E$ is an $(2, \infty)$-$G$-isometric mapping if and only if

$$G_{\mathbb{R}}(Su, Sv, Sw) = \max\{G_{\mathbb{R}}(S^2 u, S^2 v, S^2 w), G_{\mathbb{R}}(u, v, w)\}, \ \forall\, u, v, w \in E.$$

(iii) An mapping $S : E \to E$ is an $(3, \infty)$-$G$-isometric mapping if and only if

$$\max\{G_{\mathbb{R}}(Su, Sv, Sw), G_{\mathbb{R}}(S^3u, S^3v, S^3w)\}$$
$$= \max\{G_{\mathbb{R}}(S^2u, S^2v, S^2w), G_{\mathbb{R}}(u, v, w)\}, \ \forall\, u, v, w \in E.$$

**Example 2.1.** Let $E = \mathbb{R}$ and let $G_{\mathbb{R}}$ be the $G$-metric on $E \times E \times E$ defined as follows

$$G_{\mathbb{R}}(u, v, w) = |u - v| + |v - w| + |u - w|.$$

Define a map $S : E \to E$ by $Su = u + 2$. Clearly we have $S^k u = u + 2k$ for all $k \in \mathbb{N}$. From which we get $G_{\mathbb{R}}(S^k u, S^k v, S^k w) = |u - v| + |v - w| + |u - w|$. Consequently,

$$\max_{\substack{j\in\{0\dots,m\}\\ (j\ \text{ even})}} \{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\} = \max_{\substack{j\in\{0,\dots,m\}\\ (j\ \text{ odd})}} \{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\}.$$

So, we have that $S$ is an $(m, \infty)$-$G$-isometric mapping.

**Proposition 2.1.** *An mapping $S$ acting on a real valued $G$-metric space $E$ is an $(m, \infty)$-$G$-isometric if and only if $\forall u, v, w \in E, \forall l \in \mathbb{N}_0$*

$$\max_{\substack{j\in\{l,\dots,l+m\}\\ (j\ \ even)}} \{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\} = \max_{\substack{j\in\{l,\dots,l+m\}\\ (j\ \ odd)}} \{G_{\mathbb{R}}(S^j u, S^j v, S^j w)\}.$$

**Proof.** For $l \in \mathbb{N}_0$, substituting $S^l u$, $S^l v$ and $S^l w$ for $u, v$ and $w$ in Definition 2.1 ,we obtained the desired characterizations. ☐

**Lemma 2.1** ([7])**.** *For all $k \in \mathbb{N}_0$ let $\pi(k) = k \bmod 2$ denote the parity of $k$. Let further $m \in \mathbb{N}$ and $a = (a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$. The following are equivalent:*
(1) *a satisfies*

$$\max_{\substack{k\in\{l,\dots,m+l\}\\ k\ even}} a_k = \max_{\substack{k\in\{l,\dots,m+l\}\\ k\ odd}} a_k, \ \ \forall\, l \in \mathbb{N}_0.$$

(2) *a attains a maximum and*

$$\max_{k\in\mathbb{N}_0}(a_k) = \max_{\substack{k\in\{l,\dots,m+l\}\\ \pi(k)=\pi(m-1+l)}} (a_k), \ \ \forall\, l \in \mathbb{N}_0.$$

**Corollary 2.1.** *Let $S : E \to E$ be an mapping on a $G$-metric space $E$ and $m \in \mathbb{N}$. Then $S$ is an $(m, \infty)$-$G$-isometric mapping if and only if, for all $u, v, w \in E$*

$$\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{\substack{k\in\{j,\dots,m+j\}\\ \pi(k)=\pi(m-1+j)}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \ \ \forall\, j \in \mathbb{N}.$$

**Proof.** The proof in an immediate consequence of Lemma 2.1. ☐

**Corollary 2.2.** *Let $S : E \to E$ be an mapping on a $G$-metric space $E$ such is an $(m,\infty)$-$G$-isometric. Then for all $n \in \mathbb{N}_0$*

$$G_{\mathbb{R}}(S^n u, S^n v, S^n w) \leq \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k u, S^k w)\}, \forall\, u,v,w,\in E.$$

**Proof.** From Corollary 2.1, we have

$$\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k u, S^k w)\} = \max_{\substack{k\in\{j,\ldots,m-1+j\}\\ \pi(k)=\pi(m-1+j)}}\{G_{\mathbb{R}}(S^k u, S^k v, , S^k w)\},$$

$$\forall\, u,v,w \in E, \quad \forall\, j \in \mathbb{N}_0.$$

This gives that $\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} < \infty$. Further, we see that for all $n \in \mathbb{N}_0$

$$\begin{aligned} G_{\mathbb{R}}(S^n u, S^n v, S^n w)\} &\leq \max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} \\ &\leq \max_{0\leq k\leq m-1}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \forall\, u,v,w \in E. \end{aligned}$$

□

**Theorem 2.1.** *Let $S$ be an $(m,\infty)$-$G$-isometry on a real-valued $G$-metric space $(E, G_{\mathbb{R}})$ such that $(E, G_{\mathbb{R}})$ is symmetric. Then there exists a real-valued $G$-metric $G_{\mathbb{R}}^{\infty}$ on $E$ such that $S$ is an $G$-isometry on $(E, G_{\mathbb{R}}^{\infty})$. Moreover $G_{\mathbb{R}}^{\infty}$ is given by*

$$G_{\mathbb{R}}^{\infty}(u,v,w) = \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \quad \forall\;\, u,v,w \in E.$$

**Proof.** By the assumption that $S$ is an $(m,\infty)$-$G$-isometry, we have by Corollary 2.2 that

$$\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \;\; \forall\, u,v,w \in E.$$

Define the map $G_{\mathbb{R}}^{\infty} : E \times E \times E \to \mathbb{R}_+$ by

$$G_{\mathbb{R}}^{\infty}(u,v,w) := \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \;\; \forall\, u,,v,w \in E.$$

A simple calculation shows that the map $G_{\mathbb{R}}^{\infty}$ satisfies the conditions (1)-(5) of Definition 1.1. Hence $G_{\mathbb{R}}^{\infty}$ is a real-valued $G$-metric on $E$. Furthermore we have

$$\begin{aligned} G_{\mathbb{R}}^{\infty}(u,v,w) &= \max_{k\in\{0,\ldots,m-1\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} \\ &= \max_{k\in\mathbb{N}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} \\ &= \max_{k\in\{j,\ldots,m-1+j\}}\{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}, \;\forall\, u,v,w \in E, \;\forall\, j \in \mathbb{N}. \end{aligned}$$

Consequently, $G_{\mathbb{R}}^{\infty}(u,v,w) = G_{\mathbb{R}}^{\infty}(Su, Sv, Sw)$. So, $S$ is an isometry on $(E, G_{\infty})$ and the proof is complete. □

**Proposition 2.2.** *Let $S : E \to E$ be an mapping and $m \in \mathbb{N}, m \geq 2$. If $S$ satisfies the following conditions.*

(i) $G_{\mathbb{R}}(S^m u, S^m v, S^m w) = G_{\mathbb{R}}(S^{m-1}u, S^{m-1}v, S^{m-1}w)$, for all $u, v, w \in E$.

(ii) $G_{\mathbb{R}}(S^m u, S^m v, S^m w) \geq G_{\mathbb{R}}(S^k u, S^k v, S^k w)$ for $k = 0, \ldots, m-2$, for all $u, v, w, \in E$, then $S$ is an $(m, \infty)$-$G$-isometry.

**Proof.** By the assumptions (i) and (ii), we have for all $u, v, w \in E$,

$$G_{\mathbb{R}}(S^m, S^m v, S^m w) = G_{\mathbb{R}}(S^{m-1}u, S^{m-1}v, S^{m-1}w)$$

and

$$G_{\mathbb{R}}(S^m u, S^m v, S^m w) \geq G_{\mathbb{R}}(S^k u, S^k v, S^k w), k = 0. \ldots, m-2.$$

From which we conclude that

$$\max_{\substack{k \in \{0,\ldots,m\} \\ k \text{ even}}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{\substack{k \in \{0,\ldots,m\} \\ k \text{ odd}}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}.$$

This implies that $S$ is an $(m, \infty)$-$G$-isometry by Definition 2.1. □

The following lemma gives a characterization of $(2, \infty)$-$G$-isometric mapping on a real-valued $G$-metric space.

**Lemma 2.2.** *Let $S : E \longrightarrow E$ be an mapping on a $G$-metric space $E$. Then $S$ is an $(2, \infty)$-$G$-isometric mapping if and only if $S$ satisfies the following conditions*

$$\begin{cases} G_{\mathbb{R}}(S^2u, S^2v, S^2w) = G_{\mathbb{R}}(Su, Sv, Sw), & \forall\ u, v, w \in E, \\ G_{\mathbb{R}}(S^2u, S^2v, S^2w) \geq G_{\mathbb{R}}(u, v, w), & \forall\ u, v, w \in E. \end{cases} \tag{2.1}$$

**Proof.** Firstly, assume that $S$ is an $(2, \infty)$-$G$-isometry, then we have for all $u, v, w \in E$,

$$G_{\mathbb{R}}(Su, Sv, Sw) = \max\{G_{\mathbb{R}}(S^2u, S^2v, S^2w), G_{\mathbb{R}}(u, v, w)\}$$

and it follows that for all $u, v, w \in E$

$$G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w)$$

and

$$G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(S^2u, S^2v, S^2w).$$

Replacing $u \longrightarrow Su$, $v \longrightarrow Sv$ and $w \longrightarrow Sw$ we obtain that

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) = \max\{G_{\mathbb{R}}(Su, Sv, Sw), G_{\mathbb{R}}(S^3u, S^3v, S^3w)\}, \ \forall\, u, v, w \in E.$$

Thus, for all $u, v, w \in E$

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) \geq G_{\mathbb{R}}Su, Sv, Sw).$$

So,

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) = G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w), \ \ \forall\, u, v, w \in E.$$

This gives (2.1).

Conversely assume that $S$ satisfies (2.1) in this case we have that $S$ is an $(2,\infty)$-$G$-isometry by Proposition 2.2. □

**Theorem 2.2.** *Let $S : E \longrightarrow E$ be an mapping on a $G$-metric space $E$. If $S$ is an $(2,\infty)$-$G$-isometric mapping, then $S^n$ is an $(2,\infty)$-$G$-isometric mapping.*

**Proof.** Let $S$ be an $(2,\infty)$-isometric mapping. We need to prove that $S^n$ is an $(2,\infty)$-$G$-isometric mapping for all positive integer $n$. By Lemma 2.2 it suffices to show that

$$G_{\mathbb{R}}(S^{2n}u, S^{2n}v, S^{2n}w) = G_{\mathbb{R}}(S^nu, S^nv, S^nw) \geq G_{\mathbb{R}}(u, v, w), \ \ \forall \ u, v, w \in E.$$

Firstly, we prove by mathematical induction on $n$ that

$$G_{\mathbb{R}}(S^{2n}u, S^{2n}v, S^{2n}w) = G_{\mathbb{R}}(S^nu, S^nv, S^nw), \ \ \forall \ u, v, w \in E.$$

For $n = 1$ it is true since $S$ is an $(2,\infty)$-$G$-isometry. Assume that this equality is true for $n$ and prove it for $n + 1$. In fact, we have

$$\begin{aligned} G_{\mathbb{R}}(S^{2n+2}u, S^{2n+1}v, S^{2n+2}w) &= G_{\mathbb{R}}(S^{2n}S^2u, S^{2n}S^2v, , S^{2n}S^2v) \\ &= G_{\mathbb{R}}(S^nS^2u, S^nS^2v, S^nS^2w) \\ &= G_{\mathbb{R}}(S^{n+1}u, S^{n+1}v, S^{n+1}w), \ \ \forall \ u, v, w \in E. \end{aligned}$$

Thus by induction, we proved that $G_{\mathbb{R}}(S^{2n}u, S^{2n}v, S^{2n}w) = G_{\mathbb{R}}(S^nu, S^nv, , S^nw)$, $\forall u, v, w \in E$ holds for all $n = 1, 2 \dots$.
It remains to show that for all $u, v, w \in E : G_{\mathbb{R}}(S^nu, S^nv, S^nw) \geq G_{\mathbb{R}}(u, v, w, )$, for all $n = 1, 2, \dots$.

Indeed, since $G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w, ), \forall u, v, w \in E$, we have by using the same inequality that for all $u, v, w \in E$

$$\begin{aligned} G_{\mathbb{R}}(S^nu, S^nv, S^nw) &= G_{\mathbb{R}}(SS^{n-1}u, SS^{n-1}v, SS^{n-1}w) \\ &\geq G_{\mathbb{R}}(S^{n-1}u, S^{n-1}v, S^{n-1}w) \\ &= G_{\mathbb{R}}(SS^{n-2}u, SS^{n-2}v, SS^{n-2}w) \\ &\geq G_{\mathbb{R}}(S^{n-2}u, S^{n-2}v, S^{n-2}w) \\ &\geq \cdots \\ &\geq G_{\mathbb{R}}(Su, Sv, Sw) \\ &\geq G_{\mathbb{R}}(u, v, w). \end{aligned}$$

By induction on $n$ it follows that

$$G_{\mathbb{R}}(S^{2n}u, S^{2n}v, , S^{2n}w) = G_{\mathbb{R}}(S^nu, S^nv, S^nw) \geq G_{\mathbb{R}}(u, v, w), \ \ \forall \ u, v, w \in E.$$

Thus $S^n$ is an $(2,\infty)$-$G$-isometry. □

**Theorem 2.3.** *Let $T$ and $S$ are two mappings acting on a $G$-metric space $T, S : E \longrightarrow E$ such that $TS = ST$. If $T$ is an $(m, \infty)$-$G$-isometry and $S$ is an $(2, \infty)$-$G$-isometry, then $TS$ is an $(m, \infty)$-$G$-isometry.*

**Proof.** Firstly, assume that $T$ and $S$ are both $(2, \infty)$-$G$-isometry.

Since $S$ is an $(2, \infty)$-$G$-isometry, we have by Lemma 2.2

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) = G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w), \ \forall\, u, v, w \in E.$$

It follows that for all $u, v, w \in E$ we have

$$\begin{aligned}
&G_{\mathbb{R}}((TS)^2u, (TS)^2v, (TS)^2w)\\
&= G_{\mathbb{R}}(T^2S^2u, T^2S^2v, , T^2S^2w) = G_{\mathbb{R}}(TS^2u, TS^2v, TS^2w)\\
&= G_{\mathbb{R}}(S^2Tu, S^2Tv, S^2Tw) = G_{\mathbb{R}}(TSu, TSv, TSw)\\
&\geq G_{\mathbb{R}}(Su, Sv, Sw) \ \ (\text{since } T \text{ is an } \ (2, \infty) - \text{isometry})\\
&\geq G_{\mathbb{R}}(u, v, w) \ \ (\text{since } S \text{ is an } (2, \infty) - \text{isometry}).
\end{aligned}$$

This implies that,

$$G_{\mathbb{R}}((TS)^2u, (TS)^2v, (TS)^2w) = G_{\mathbb{R}}(TSu, TSv, TSw) \geq G_{\mathbb{R}}(u, v, w), \forall u, v, w \in E$$

thus, we have $TS$ is an $(2, \infty)$-$G$-isometry by the statement in Lemma 2.2.

If we assume that $T$ is an $(m, \infty)$-$G$-isometry for $m > 2$ and that $S$ is an $(2, \infty)$-$G$-isometry, we have by this fact

$$G_{\mathbb{R}}(S^2u, S^2v, S^2w) = G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w) \ \ \forall\, u, v, w \in E$$

and also for all $k = 1, 2, \ldots$

$$G_{\mathbb{R}}(S^ku, S^kv, S^kw) = G_{\mathbb{R}}(Su, Sv, Sw) \geq G_{\mathbb{R}}(u, v, w) \ \ \forall \ \ u, v, w \in E.$$

Thus we have for all $u, v, w \in E$

$$\begin{aligned}
G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw) &= G_{\mathbb{R}}(T^kS^ku, T^kS^kv, T^kS^kw)\\
&= G_{\mathbb{R}}(ST^ku, ST^kv, ST^kw)\\
&\geq G_{\mathbb{R}}(T^ku, T^kv, T^kw).
\end{aligned}$$

Using the above inequality, for all $u, v, w \in E$, we have

$$\begin{aligned}
&\max_{\substack{k\in\{1,\ldots,m\}\\ k \text{ even}}} \{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\}\\
&= \max_{\substack{k\in\{1,\ldots,m\}\\ k \text{ even}}} \{G_{\mathbb{R}}(T^kSu, T^kSv, T^kSw)\} \geq \max_{\substack{k\in\{1,\ldots,m\}\\ k \text{ even}}} G_{\mathbb{R}}(T^ku, T^kv, T^kw).
\end{aligned}$$

So that

$$\max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ even}}} \{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\} \geq \max_{\substack{k\in\{0,\ldots,m\\ k \text{ even}}} \{G_{\mathbb{R}}(T^ku, T^kv, , T^kw)\}.$$

On the other hand, it is obvious that for all $u, v, w \in E$

$$\max_{\substack{k\in\{1,\ldots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}(T^kSu, T^kSv, T^kSw)\} \le \max_{\substack{k\in\{1,\ldots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}(T^ku, T^kv, T^kw)\}.$$

We get for all $u, v, w \in E$.

$$\max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\} \le \max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}(T^ku, T^kv, T^kw)\}.$$

From the above inequality, we obtain for all $u, v, w \in E$

$$\max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kv)\} = \max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ even}}} G_{\mathbb{R}}(T^ku, T^kv, T^kw)$$

By a similar way we have also for all $u, v, w \in E$

$$\max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\} = \max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}(T^ku, T^kv, T^kw)\}.$$

Using the fact that $T$ is an $(m,\infty)$-isometry, we deduce that for all $u, v, w \in E$.

$$\begin{aligned}&\max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ even}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\}\\ &= \max_{\substack{k\in\{0,\ldots,m\}\\ k \text{ odd}}}\{G_{\mathbb{R}}((TS)^ku, (TS)^kv, (TS)^kw)\}.\end{aligned}$$

The conclusion that $TS$ is an $(m,\infty)$-$G$-isometric mapping follows immediately from Definition 2.1. □

**Proposition 2.3.** *Let $S : E \to E$ be an $(m,\infty)$-$G$-isometry mapping on a real-valued generalized metric space $E$. Then $S$ is an $(m+1,\infty)$-$G$-isometric mapping.*

**Proof.** Assume that $S$ is an $(m,\infty)$-$G$-isometry, then it follows that

$$\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} = \max_{\substack{k\in\{j,\ldots,m-1+j\}\\ \pi(k)=\pi(m-1+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}$$

$\forall, u, v, w \in E, \forall j \in \mathbb{N}$. This implies that for all $u, v, w \in E$ and $\forall j \in \mathbb{N}$ we have

$$\begin{aligned}\max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} &= \max_{\substack{k\in\{j,\ldots,m-1+j\}\\ \pi(k)=\pi(m-1+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}\\ &\le \max_{\substack{k\in\{j,\ldots,m+j\}\\ \pi(k)=\pi(m+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} \le \max_{k\in\mathbb{N}_0}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}.\end{aligned}$$

Consequently

$$\max_{k\in\mathbb{N}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\} = \max_{\substack{k\in\{j,\ldots,m+j\}\\ \pi(k)=\pi(m+j)}}\{G_{\mathbb{R}}(S^ku, S^kv, S^kw)\}.$$

So, $S$ is an $(m+1,\infty)$-$G$-isometry. □

**Proposition 2.4.** *Let $S$ be an mapping acting on a real-valued $G$-metric space $(E, G_{\mathbb{R}})$ such that $S^n$ is an $G$-isometry for odd integer $n$, then $S$ is an $(m,\infty)$-$G$-isometry for $m \geq 2n-1$.*

**Proof.** As an consequence of Proposition 2.3, it suffices to show that $S$ is an $(2n-1,\infty)$-$G$-isometric mapping.

Indeed, by the assumption that $S^n$ is an $G$-isometry, it follows that

$$G_{\mathbb{R}}(S^{k+n}u, S^{n+k}v, S^{n+k}w) = G_{\mathbb{R}}(S^k u, S^k v, S^k w), \quad \forall\, u,v,w \in E,\ \forall\, k \in \mathbb{N}_0.$$

On the other hand , since $n$ is odd integer we have for all $k \in \mathbb{N}$ , $k$ is even if and only if $n+k$ is odd. By assumption, $S^n$ is an $G$-isometry it follows that

$$\{G_{\mathbb{R}}(S^k u, S^k v, S^k w),\ k \in \{0,1,...,2n-1\},\ k \text{ even }\}$$
$$=$$
$$\{G_{\mathbb{R}}(S^k u, S^k v, S^k w),\ k \in \{0,1,...,2n-1\},\ k \text{ odd }\}.$$

and it follows that for all $u,v,w \in E$

$$\max_{\substack{k\in\{0,\dots,2n-1\}\\ k \text{ even}}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{\substack{k\in\{0,\dots,2n-1\}\\ k \text{ odd}}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}.$$

Consequently, $S$ is an $(2n-1,\infty)$-$G$-isometry. □

**Theorem 2.4.** *If $S : E \to E$ is an invertible mapping on a real-valued $G$-metric space $E$ such is an $(m,\infty)$ -$G$-isometry, then $S^{-1}$ is an $(m,\infty)$-$G$-isometry.*

**Proof.** Assume that $S$ is an $(m,\infty)$-$G$-isometry, then we have by Definition 2.1 for all $u,v,w \in E$

$$\max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\} = \max_{\substack{k\in\{0,\dots,m\}\\ k \text{ odd}}} \{G_{\mathbb{R}}(S^k u, S^k v, S^k w)\}.$$

Replacing $u \to S^{-m}u$ , $v \to S^{-m}v$ and $w \to S^{-m}w$ we obtain for all $u,v,w \in E$

$$\max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}} \{G_{\mathbb{R}}(S^{k-m}u, S^{k-m}v, S^{k-m}w)\} = \max_{\substack{k\in\{0,\dots,m\}\\ k \text{ odd}}} \{G_{\mathbb{R}}(S^{k-m}u, S^{k-m}v, S^{k-m}w)\}$$

or equivalently

$$\max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}} \{G_{\mathbb{R}}((S^{-1})^{m-k}u, (S^{-1})^{m-k}v, (T^{-1})^{m-k}w)\}$$
$$= \max_{\substack{k\in\{0,\dots,m\}\\ k \text{ odd}}} \{G_{\mathbb{R}}((S^{-1})^{m-k}u, (S^{-1})^{m-k}v, (S^{-1})^{m-k}w)\}$$

this gives

$$\max_{\substack{k\in\{0,\dots,m\}\\ k \text{ even}}} \{G_{\mathbb{R}}((S^{-1})^k u, (S^{-1})^k v, (S^{-1})^k w)\}$$
$$= \max_{\substack{0\in\{0,\dots,m\}\\ k \text{ odd}}} \{G_{\mathbb{R}}((S^{-1})^k u, (S^{-1})^k v, (S^{-1})^k w)\}, \quad \forall\, u,v,w \in E.$$

Thus, $S^{-1}$ is an $(m,\infty)$-$G$-isometry. □

**Theorem 2.5.** *For $k = 1, 2, ..., d$, let $(E_k, G_k)$ be a $G$-metric space which is symmetric, and let $S_k : E_k \to E_k$ be a map, $m_k \geq 1$. Denote by $E = E_1 \times E_2 \times ... \times E_d$ the product space endowed with the product $G$-metric*

$$G_{\mathbb{R}}((u_1, u_2, \ldots, u_d), (v_1, v_2, \ldots, v_d), (w_1, \ldots, w_k)) := \max_{1\leq k\leq d} g\{G^k_{\mathbb{R}}(u_k, v_k, w_k)g\}.$$

*Let $S := S_1 \times S_2 \times ... \times S_d : E \to E$ be a mapping defined by*

$$S(u_1, \ldots, u_d) := (S_1u_1, S_2u_2, \ldots, S_nu_d).$$

*If each $S_k$ is an $(m_k, \infty)$- $G$-isometry for $k = 1, 2, \ldots, d$, then $S$ is an $(m, \infty)$-$G$-isometry, where $m = \max\{m_1, \ldots, m_d\}$.*

**Proof.** Let $m = \max\{m_1, m_2, \ldots, m_d\}$ and consider for all $u, v, w \in E$

$$\begin{aligned}
&\max_{\substack{j\in\{0,\ldots,m\}\\ j\ \text{even}}} \{G_{\mathbb{R}}(S^ju, S^jv, (S^jw)\} \\
&= \max_{\substack{j\in\{0,\ldots,m\}\\ j\ \text{even}}} g(\max_{1\leq k\leq d} \{G^k_{\mathbb{R}}(S^j_ku_k, S^j_kv_k, S^j_kw_k)\}g) \\
&= \max_{1\leq k\leq d} g(\max_{\substack{j\in\{0,\ldots,m\}\\ j\ \text{even}}} \{G^k_{\mathbb{R}}((S^j_ku_k, S^j_kv_k, S^j_kw_k)\}g)
\end{aligned}$$

By the assumption that each $S_k$ is an $(m_k, \infty)$-$G$-isometry for each $k = 1, 2, ..., d$, it follows that $S_k$ is an $(m, \infty)$-$G$-isometry for $k = 1, 2, \ldots, d$ (by Proposition 2.3 ). Then we have

$$\begin{aligned}
&\max_{\substack{j\in\{0,\ldots,.,m\}\\ j\ \text{even}}} \{G_{\mathbb{R}}(S^ju, S^jv, S^jw)\} \\
&= \max_{1\leq k\leq d} g(\max_{\substack{j\in\{0,\ldots,m\}\\ j\ \text{odd}}} \{G^k_{\mathbb{R}}(S^j_ku_k, S^j_kv_k, S^j_kw_k)\}g) \\
&= \max_{\substack{j\in\{0,\ldots,m\}\\ j\ \text{odd}}} g(\max_{1\leq k\leq d} \{G^k_{\mathbb{R}}(S^j_ku_k, S^j_kv_k, S^j_kw_k)\}g).
\end{aligned}$$

Thus, we have

$$\max_{\substack{j\in\{0,\ldots,m\}\\ j\ \text{even}}} \{G_{\mathbb{R}}(S^ju, S^jv, S^jw)\} = \max_{\substack{j\in\{0,\ldots,m\}\\ j\ \text{odd}}} \{G_{\mathbb{R}}(S^ju, S^jv, S^jw)\}.$$

Consequently, $S$ is an $(m, \infty)$-$G$-isometric mapping and the proof is completed. □

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