Application of differential transformation method for solving prey predator model with holling type I

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Abstract. Nonlinear differential equations are used for describing many phenomena in the real world as prey predator interactions. Prey predator models are classified as one of the most important applications in applied mathematics. In this paper,

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modified structure of prey predator model is used, theoretical properties of the model are presented, the boundedness of the model is shown and the dynamical behavior of the model is proved as globally stable. Semi analytical solution by using differential transformation method (DTM) is obtained for non- dimensional prey predator model with Holling type I in the case of persistence dynamics of the model. The results seem to satisfy biological domain of the problem. We conclude that the results of differential transformation method is in good agreement with numerical results from interpolation method (IM) by using MATHEMATICA program.

Keywords: prey predator model, differential transform method, boundedness, stability.

1. Introduction

Nonlinear differential equations are used to describe many real world phenomena as prey predator interactions. Prey predator models are classified as one of the most important applications in applied mathematics, biology, and ecology sciences. Lotka-Volterra model is considered as the original model to formulate prey-predator interactions [1]. However, numerous extensions of the original model have been applied to describe particular scenarios which involve surroundings and nature of species by many researchers [2]-[11] problems of biological and ecological interest are described in the form of differential equations with appropriate initial or boundary conditions.

Usually, difficulty arises in the solution of nonlinear system, so many numerical methods and semi-analytical methods are developed for finding the solution of these problems by many researchers.

Differential transform method (DTM) is applied for solving specific kind of system of nonlinear differential equations. DTM is considered among the few semi analytical methods to overcome the difficulties that are caused by the nonlinear terms. The method gives an analytical solution in the form of a series for differential equations. It formulize the Taylor series in a totally different approach so it is a semi-numerical and semi-analytic method. Zhou [12] proposed firstly the idea of differential transform (see Ref. [1, 8, 9, 13, 14, 15, 16]) and it was used for solving linear and non-linear initial value problems in electric circuit analysis. Simultaneously, Pukhov [17] also studied differential transformation method. This method depends on transforming IVP or BVP into a recurrence relation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. The method is helpful to obtain both semi analytical and approximate solutions of linear and non-linear IVPs. This method avoids discretization, linearization or perturbation, big computational work and round-off errors.

Several authors have employed DTM for the solution of problems involved in non-linear phenomenon. Moon et al. [18] applied the DTM for solving some non-linear differential equations. Warade and Chopade [19] solved initial value problems involving fourth order ordinary differential equations by DTM. Chang and Chang [20] introduced developed algorithm for calculating one-dimensional differential transform of nonlinear functions. Iftikhar et al. [21] used the DTM for solving linear and nonlinear thirteenth order boundary value problems. Hatami and Ganji [22] used Lagrangian and high accuracy multi- step differential transformation method on the motion of a spherical particle. Hussain et al. [23] obtained analytical solution of magnetohydrodynamic flow of Newtonian fluids over a stretching sheet by using DTM. Hatami and Jing [24] used DTM for solving the coupled nonlinear differential equations in fluid mechanics problems. Sepasgozar et al. [25] applied DTM to obtain the solution of momentum and heat transfer equations of non-Newtonian fluid flow in an axisymmetric channel with porous wall. Through the literature, we observe that DTM is used for solving different real applications as semi analytical method.

Our objective for this paper is to introduce semi analytical solution of real world problem. Differential transform method (DTM) is applied for solving different style of prey predator model with Holling type I, which has been used in Alebraheem and Abu Hasan, [3]. The logistic law describes the growth rates of the model of the prey and predator. The persistence is considered as one of the main important dynamic behaviors. The new idea here we use differential transform method to solve the model with persistence dynamic behaviors. The persistence condition of this model is determined to give biological meaning and validation of the parametric values. We introduce comparison between differential transform method and approximation method in case of persistence dynamics of the model.

2. Mathematical model

The system of equations is written in non-dimensional form as

$$\frac{da}{dt} = a(1-a) - \alpha ab, \tag{2.1}$$

$$\frac{db}{dt} = -ub + e\alpha ab - e\alpha b^2 \tag{2.2}$$

The biological meanings of the parameters are explained as follows: The intrinsic growth rate of prey is 1. α measures efficiency of the search and the capture of predators is b, u is the death rates of predators. $f(a) = \alpha a$ denotes the functional response which is defined as consumption rate of prey by a predator, $g(a) = e\alpha a$, $g(a)$ represents numerical response of the predators y that characterize changing in the population of predators through prey consumption. e represents efficiency of converting consumed prey into predator births. Since the biological meaning is taken into consideration, all the parameters and initial conditions of the model are supposed positive values. The initial conditions of system (1) are:

$$a(0) = a_0, b(0) = b_0 \quad \text{where} \quad 0 < a_0, b_0 < 1. \tag{2.3}$$

However, the initial conditions are $0 < a_0, b_0 \leq 1$ because the carrying capacity of non dimensional system (2.3) is 1, so the maximum values of initial conditions are 1, while they are greater than zero because of biological meaning.

3. The boundedness of the model

3.1 Theorem

The solution of the system (2.1-2.2) for $t \geq 0$ in R^3_+ is bounded.

Proof. We show that the first equation of the system (2.1-2.2) is bounded through

$$\frac{da}{dt} \leq a(1+a). \tag{3.1}$$

The solution of the equation 2.2 is $a(t) = \frac{1}{(1+qe^(-t))}$, $q = \frac{(1-a_0)}{a_0}$ is the constant of integration. then $a(t) \leq 1 \;\; \forall \;\; t \geq 0$. Then, we prove that $a(t)+b(t) \leq L.\forall t \geq 0$. Let $D(t) = a(t) + b(t)$. The time derivative of the function D

$$\begin{aligned} \frac{dD}{dt} &= \frac{da}{dt} + \frac{db}{dt} \\ &= ((1-a) - \alpha b)a + (-u + e\alpha a - e\alpha b)b. \end{aligned} \tag{3.2}$$

However, the solutions initiating remain in nonnegative quadrant in R^3_+ and all the parameters are positive; it can be assumed the following

$$\frac{dD}{dt} \leq ((1-a))a + (-u + e\alpha a - e\alpha b)b. \tag{3.3}$$

It can be concluded that

$$\max_{(R_+)} a(1-a) = \frac{1}{4}. \tag{3.4}$$

By substituting in (3.3), it become as follows

$$\frac{dD}{dt} \leq \frac{1}{4} + (-u + e\alpha a - e\alpha b)b, \tag{3.5}$$

$$\frac{dD}{dt} \leq \frac{1}{4} + (-u + e\alpha a - e\alpha b)b + D(t) - D(t). \tag{3.6}$$

The equation (3.6) can be written as follows

$$\frac{dD}{dt} + D(t) \leq \frac{1}{4} + a + (-u + e\alpha a - e\alpha b + 1)b. \tag{3.7}$$

Since $a(t) \leq 1$, then

$$\frac{dD}{dt} + D(t) \leq \frac{5}{4} + (-u + e\alpha - e\alpha b + 1)b. \tag{3.8}$$

But

$$\max_{(R_+)} (-u + e\alpha - e\alpha b + 1)b = \frac{(1 + e\alpha - u)^2}{4e}. \tag{3.9}$$

So Eq. (3.8) becomes:

$$\frac{dD}{dt} + D(t) \le L, \tag{3.10}$$

where

$$L = \frac{1}{4}(5 + \frac{(1+e\alpha-u)^2}{4e\alpha}). \tag{3.11}$$

Consequently,

$$D(t) \le L + \sigma e^{-t}, \tag{3.12}$$

where σ is a constant of integration. $t \to \infty$; Then $D(t) \le L$.

4. The dynamic behavior

One of the main properties of dynamic systems is stability. The stability is studied to determine properties of solutions or equations in differential equations, consequently the dynamic behavior will be explained. The system has three non-negative equilibrium points that are: The first point is $E_0 = (0, 0)$ exists without conditions on parameters. The second point is $E_1 = (1, 0)$ exists without conditions on parameters. The third point is $E_2 = (\hat{a}, \hat{b}) = (\frac{(u+e)}{(e\alpha+e)}, \frac{(e\alpha-u)}{(e\alpha^2+e\alpha)})$, which is called persistence point. The equilibrium point E_2 is positive under the following condition: $e\alpha > u$ We are interested to study the dynamic behaviors in case the permanent coexistence of prey predator system, so we study the coexistence point $E_2 = (\hat{a}, \hat{b}) = (\frac{(u+e)}{(e\alpha+e)}, \frac{(e\alpha-u)}{(e\alpha^2+e\alpha)})$, it represents the permanent coexistence (i.e. persistence) of prey predator system.

4.1 Theorem

The persistence equilibrium point E_2 is globally asymptotically stable inside the positive quadrant of a-b plane.

Proof. Let $G(a,b) = \frac{1}{ab}$. G is a Dulac function, it is continuously differentiable in the positive quadrant of x-y plane $A = (a,b) \mid a > 0, b > 0$, Hsu [26].

$$\begin{aligned} N_1(a,b) &= a(1-a) - \alpha ab, \\ N_2(a,b) &= -ub + e\alpha ab - \alpha e b^2. \end{aligned} \tag{4.1}$$

Thus, $\Delta(GN_1, GN_2) = \frac{(\partial(GN_1))}{\partial a} + \frac{(\partial(GN_2))}{\partial b} = \frac{-1}{b} - \frac{e\alpha}{a}$. It is observed that $\Delta(GN_1, GN_2)$ is not identically zero and does not change sign in the positive quadrant of a-b plane. So by Bendixson-Dulac criterion, there is no periodic solution inside the positive quadrant of $a - b$ plane. E_2 is globally asymptotically stable inside the positive quadrant of $a - b$ plane. In general, we conclude that the dynamic behavior of this system is stable. Kolmogorov analysis [27] is applied to find the persistence condition of the system (2.1-2.2), so the persistence condition is

$$0 < \frac{u}{e\alpha} < 1. \tag{4.2}$$

4.2 Corollary

The persistence dynamic behavior of the system (2.1-2.2) is globally asymptotically stable.

5. Approximate analytical solution

The basic definitions and rules [13] of the DTM are summarized below:

5.1 Basic definitions and concepts

If $u(t)$ is analytic in the domain T, then it will be differentiated continuously with respect to time t, as shown in equation

$$\frac{(d^n u(t))}{(dt^n)} = \phi(t,n), \quad \forall \ t \epsilon T. \tag{5.1}$$

If $\phi(t,n) = \phi(t_i,n)$, where n belongs to the set of non-negative integers, denoted as the n-domain. The Eq. (5.1) can be rewritten as:

$$U(n) = \phi(t_i,n) = [\frac{(d^n u(t))}{(dt^n)}] \mid_{(t=t_i)}, \tag{5.2}$$

where the spectrum of $u(t)$ at $t = ti.$ is denoted by $U(n)$. $u(t)$ can be represented by Taylor's series as follows:

$$u(t) = \sum_{n=0}^{\infty} [\frac{(t-t_i)^n}{n!}] U(n). \tag{5.3}$$

The Eq. 5.2 is inverse of $U(n)$. The combination of eq. 5.2 and eq. 5.3 yields:

$$u(t) = \sum_{n=0}^{\infty} [\frac{(t-t_i)^n}{n!}] U(n) = D^{-1} U(n). \tag{5.4}$$

The symbol "D" denotes the differential transformation process. By using the differential transformation, a differential equation $u(t)$ becomes:

$$u(t) = \sum_{n=0}^{\infty} [\frac{(t-t_i)^n}{n!}] U(n) + R_{(k+1)}(t), \tag{5.5}$$

where $R_{(k+1)}(t)$ is the remainder. If $F(n)$ and $G(n)$ are transformed functions corresponding to the given functions $f(t)$ and $g(t)$, the operation properties of DTM are as follows:

Base function	Transformed Functions
$q(t) = u(t) \pm v(t)$	$Q(n) = U(n) \pm V(n)$
$q(t) = \alpha u(t)$	$Q(n) = \alpha U(n)$
$q(t) = \frac{du(t)}{dt}$	$Q(n) = (n+1)U(n+1)$
$q(t) = \frac{d^2u(t)}{d(t)}$	$Q(n) = (n+1)(n+2)U(n+2)$
$q(t) = \frac{d^m u(t)}{d(t)_m}$	$Q(n) = (n+1)(n+2)...(n+m)U(n+m)$
$q(t) = u(t)v(t)$	$Q(n) = \sum_l^k = 0v(l)U(n-l)$
$q(t) = t^m$	$Q(n)\delta(n-m), \delta(n-m)$
$q(t) = exp(\gamma t)$	$Q(n) = \frac{\gamma^n}{n!}$
$Q(T) = (1+t)^m$	$Q(n) = \frac{m(m-1)...(m-n+1)}{n!}$
$q(t) = sin(\omega t + \alpha)$	$Q(n) = \frac{\omega^n}{n!} sin((\frac{\pi n}{2} + \omega))$
$q(t) = cos(\omega t + \alpha)$	$Q(n) = \frac{\omega^n}{n!} sin((\frac{\pi n}{2} + \omega))$

Table 1. Some basic transformations related to Differential Transform Method

By applying differential transformation method (DTM), the system (2.1) and (2.2) is transformed to yield recurrence relation

$$\begin{aligned} (n+1)A(n+1) &= A(n) - \sum_{(n_{1=0})}^{n} A(n_1)A(n-n_1) \\ &- \alpha \sum_{(n_{1=0})}^{n} A(n_1)B(n-n_1) \end{aligned} \tag{5.6}$$

$$\begin{aligned} (n+1)B(n+1) &= -uB(n) + e\alpha \sum_{(n_{1=0})}^{n} A(n_1)B(n-n_1) \\ &- e\alpha \sum_{(n_{1=0})}^{n} B(n_1)B(n-n_1). \end{aligned} \tag{5.7}$$

We choose the following appropriate values$A(0) = 0.5, B(0) = 0.2, e = 0.5, \alpha = 1.32, u = 0.52$, when $n = 0, 1, 2, 3, 4, 5, 6$.

Inverse relation

$$\begin{aligned} a(t) &= 0.5 + 0.118t + 0.005676t^2 - 0.0057915t^3 + 0.0019652t^4 \\ &- 0.00088551t^5 \end{aligned} \tag{5.8}$$

$$\begin{aligned} b(t) &= 0.2 - 0.0644t + 0.0224068t^2 - 0.0057254t^3 + 0.0013109t^4 \\ &- 0.00613653t^5 \end{aligned} \tag{5.9}$$

6. Results

In this section, the system of coupled non -linear differential equations (2.1) and (2.2) have been solved analytically by applying differential transform method.

The values of the parameters of interest have been chosen appropriately for satisfying the persistence of system (2.1)-(2.2). The accuracy of the results has been checked and found in good agreement by their comparison with the numerical results from interpolation method (IM) which is obtained by using MATHEMATICA program (see tables 2 and 3). In addition, the graphical patterns of the results do correspond to the biological configuration of the problem.

Time	DTM	IM	Δ=|DTM-IM|
0.0	0.5	0.5	0.00
0.1	0.511851	0.511850	0.000001
0.2	0.523784	0.523774	0.00001
0.3	0.535768	0.535732	0.000036
0.4	0.547779	0.547687	0.000092
0.5	0.559790	0.559602	0.000188
0.6	0.571778	0.571443	0.000335
0.7	0.583718	0.583175	0.000543
0.8	0.595582	0.594769	0.000813
0.9	0.607342	0.606194	0.001148
1.0	0.618964	0.617425	0.001539

Table 2. Comparative results of the prey equation (2.1) with absolute error obtained through DTM and IM.

Time	DTM	IM	Δ=|DTM-IM|
0.0	0.2	0.2	0.00
0.1	0.193778	0.193778	0.00
0.2	0.187971	0.187972	0.000001
0.3	0.182538	0.182552	0.000014
0.4	0.177429	0.177489	0.00006
0.5	0.172576	0.172758	0.000182
0.6	0.167882	0.168336	0.000454
0.7	0.163219	0.164201	0.000982
0.8	0.158415	0.160334	0.001919
0.9	0.153252	0.156715	0.003463
1.0	0.147456	0.153328	0.005872

Table 3. Comparative results of the predator equation (2.2) with absolute error obtained through DTM and IM

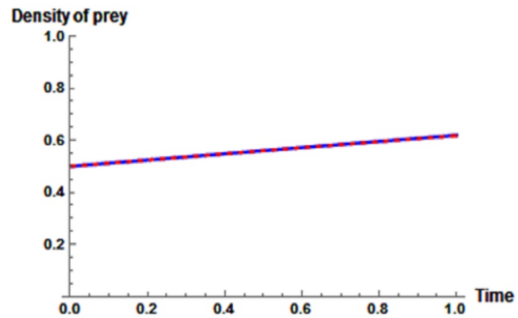

Figure 1: I M results of prey equation (1.a).

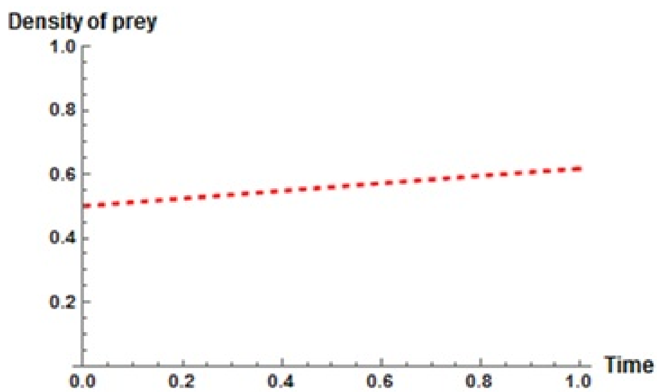

Figure 2: DTM results of prey equation (1.a).

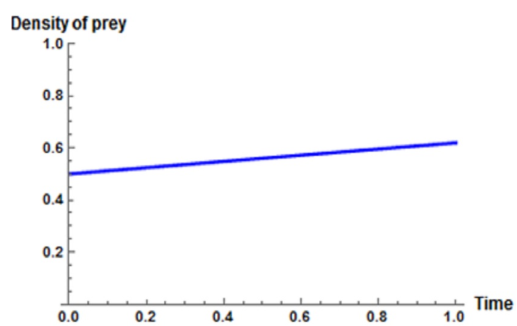

Figure 3: Comparative results of prey equation (1.a) when using IM and DTM.

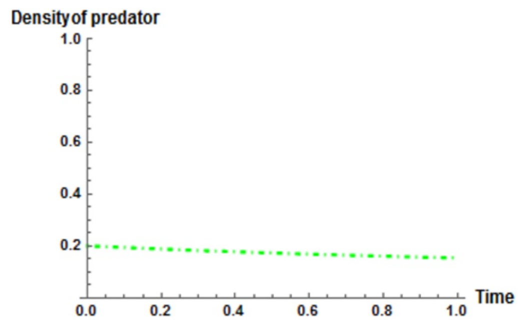

Figure 4: I M results of predator equation (1.b).

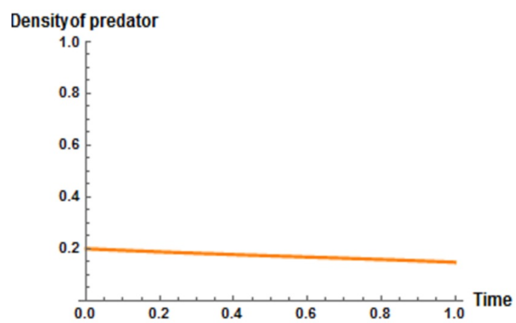

Figure 5: DTM results of predator equation (1.b).

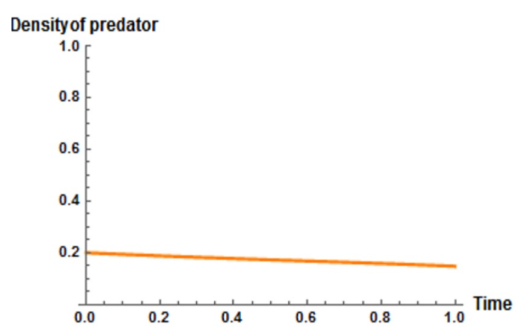

Figure 6: Comparative results of predator equation (1.b) when using IM and DTM.

7. Discussion

Fig 1 represents results of prey equation (2.1) through using IM by executing Mathematica, while the results of prey equation (2.1) using DTM are denoted through Fig 2. We notice through Fig 3 the comparison between both methods. In the same manner, Fig 4 represents results of predator equation (2.2) through using IM by executing Mathematica program, while the results of predator equation (2.2) using DTM are denoted through Fig 5. We notice that Fig 6 describes the comparison between both methods. The results show excellent approximations and the figures explain very close correspondence for both equations with only five terms of differential transformation method.

8. Conclusions

The prey predator model with Holing type I, has been considered for semi analytical solution, using differential transform method. The non-linear coupled equations have been solved smoothly without any rigorous computational work. DTM has been employed for solving the prey predator model in the persistence dynamics. The method worked well with efficacy and efficiency. We obtained close correspondence results of the model between the interpolation method (IM) and differential transform method (DTM) with only five terms. It is concluded that DTM can be used to solve applied problems of non-linear phenomena similar to this work.

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