Join hesitant fuzzy filters of residuated lattices

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Abstract. The notions of join hesitant fuzzy filters and join hesitant fuzzy G-filters in residuated lattices are introduced, and related properties are investigated. Characterizations of join hesitant fuzzy filters and join hesitant fuzzy G-filters are discussed. Conditions for a join hesitant fuzzy filter to be a join hesitant fuzzy G-filter are provided, and a new join hesitant fuzzy filter is construct by the given join hesitant fuzzy filter.

Keywords: Residuated lattice, join hesitant fuzzy filter, join hesitant fuzzy G-filter.

1. Introduction

Hesiatnt fuzzy sets, as another extension of fuzzy sets, have been proposed in [16]. The motivation for introducing hesitant fuzzy sets is that it is sometimes difficult to determine the membership of an element into a set and in some circumstances this difficulty is caused by a doubt between a few different values.

As a non-classical logic system, residuated lattices are a formal and useful tool for computer science to deal with uncertain and fuzzy information. In [20], Zhu and Xu discussed filter theory of residuated lattices. Moreover, Jun et al. applied the notion of hesitant fuzzy sets to MTL-algebras, BCK/BCI-algebras, EQ-algebras and semigroups (see [4], [5], [6] and [7]). Also, Muhiuddin et al. applied the notion of hesitant fuzzy sets to residuated lattices, lattice implication algebras and BCK/BCI-algebras (see [8], [9], [10], [11], [12], [13] and [14]).

In this paper, we introduce join hesitant fuzzy filters and join hesitant fuzzy G-filters in residuated lattices, and investigate their properties. We consider characterizations of join hesitant fuzzy filters and join hesitant fuzzy G-filters. We provide conditions for a join hesitant fuzzy filter to be a join hesitant fuzzy

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G-filter. Given a join hesitant fuzzy filter, we construct a new join hesitant fuzzy filter.

2. Preliminaries

We display well-known results on residuated lattices and hesitant fuzzy sets. We refer the reader to [1, 2, 3, 15, 16, 17, 18, 19] for further information regarding residuated lattices and hesitant fuzzy sets.

A *residuated lattice* is an algebra $\mathcal{L}:=(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.

(2) $(L, \otimes, 1)$ is a commutative monoid.

(3) $\otimes$ and $\rightarrow$ form an adjoint pair, that is,

$$(\forall x, y, z \in L)\,(x \leq y \rightarrow z \;\Leftrightarrow\; x \otimes y \leq z)\,.$$

In a residuated lattice $\mathcal{L}$, the ordering $\leq$ is defined as follows:

$$(\forall x, y \in L)\,(x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \rightarrow y = 1)$$

and x' will be reserved for $x \rightarrow 0$, and $x'' = (x')'$, etc. for all $x \in L$.

In a residuated lattice L, the following properties are valid.

$$1 \rightarrow x = x,\; x \rightarrow 1 = 1,\; x \rightarrow x = 1. \tag{2.1}$$

$$0 \rightarrow x = 1,\; x \rightarrow (y \rightarrow x) = 1. \tag{2.2}$$

$$x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z = y \rightarrow (x \rightarrow z). \tag{2.3}$$

$$x \leq y \;\Rightarrow\; z \rightarrow x \leq z \rightarrow y,\; y \rightarrow z \leq x \rightarrow z. \tag{2.4}$$

$$z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y). \tag{2.5}$$

$$z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x). \tag{2.6}$$

A nonempty subset F of a residuated lattice $\mathcal{L}$ is called a *filter* of $\mathcal{L}$ if it satisfies the conditions:

$$(\forall x, y \in L)\,(x, y \in F \;\Rightarrow\; x \otimes y \in F)\,. \tag{2.7}$$

$$(\forall x, y \in L)\,(x \in F,\; x \leq y \;\Rightarrow\; y \in F)\,. \tag{2.8}$$

A nonempty subset F of $\mathcal{L}$ is called a *G-filter* of $\mathcal{L}$ if it is a filter of $\mathcal{L}$ that satisfies the following condition:

$$(\forall x, y \in L)\,((x \otimes x) \rightarrow y \in F \;\Rightarrow\; x \rightarrow y \in F)\,. \tag{2.9}$$

Proposition 2.1 ([15])**.** *A nonempty subset F of a residuated lattice $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if it satisfies:*

$$1 \in F. \tag{2.10}$$

$$(\forall x \in F)\,(\forall y \in L)\,(x \rightarrow y \in F \;\Rightarrow\; y \in F)\,. \tag{2.11}$$

Torra [16] defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain.

Definition 2.2 ([16])**.** Let L be a reference set. Then we define *hesitant fuzzy set* on L in terms of a function $\mathcal{H}$ that when applied to L returns a subset of $[0, 1]$.

For a hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ and a subset τ of $[0, 1]$, the *τ-exclusive set* of $\mathcal{H}$ is denoted by $e(\mathcal{H}; \tau)$, and is defined to be the set

$$e(\mathcal{H}; \tau) := \{x \in L \mid \mathcal{H}(x) \subseteq \tau\}\,.$$

3. Join hesitant fuzzy filters

In what follows, let $\mathcal{L}$ denote a residuated lattice unless otherwise specified, and we take L as a reference set.

Definition 3.1. A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is called a *join hesitant fuzzy filter* of $\mathcal{L}$ if it satisfies:

$$(\forall x, y \in L)\,(x \leq y \;\Rightarrow\; x\mathcal{H} \supseteq y\mathcal{H})\,, \tag{3.1}$$

$$(\forall x, y \in L)\,(x\mathcal{H} \cup y\mathcal{H} \supseteq (x \otimes y)\mathcal{H})\,. \tag{3.2}$$

Proposition 3.2. *Every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ satisfies:*

$$(\forall x \in L)\,(x\mathcal{H} \supseteq 1\mathcal{H})\,. \tag{3.3}$$

$$(\forall x, y \in L)\,(x\mathcal{H} \cup (x \rightarrow y)\mathcal{H} \supseteq y\mathcal{H})\,. \tag{3.4}$$

Proof. Let $x, y \in L$. Since $x \leq 1$, we have $x\mathcal{H} \supseteq 1\mathcal{H}$ by (3.1). Since $x \otimes (x \rightarrow y) \leq y$, it follows from (3.2) and (3.1) that

$$x\mathcal{H} \cup (x \rightarrow y)\mathcal{H} \supseteq (x \otimes (x \rightarrow y))\mathcal{H} \supseteq y\mathcal{H}.$$

This completes the proof. □

Lemma 3.3. *If a hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ satisfies two conditions* (3.3) *and* (3.4)*, then*

$$(\forall x, y, z \in L)\,(x \leq y \rightarrow z \;\Rightarrow\; x\mathcal{H} \cup y\mathcal{H} \supseteq z\mathcal{H})\,, \tag{3.5}$$

$$(\forall x, y, z \in L)\,(x \otimes y \leq z \;\Rightarrow\; x\mathcal{H} \cup y\mathcal{H} \supseteq z\mathcal{H})\,. \tag{3.6}$$

Proof. Assume that $x \leq y \to z$ for all $x, y, z \in L$. Then $x \to (y \to z) = 1$, and so

$$\begin{aligned} x\mathcal{H} \cup y\mathcal{H} &= (x\mathcal{H} \cup 1\mathcal{H}) \cup y\mathcal{H} \\ &= (x\mathcal{H} \cup (x \to (y \to z))\mathcal{H}) \cup y\mathcal{H} \\ &\supseteq y\mathcal{H} \cup (y \to z)\mathcal{H} \supseteq z\mathcal{H}. \end{aligned}$$

Since $x \leq y \to z \;\Leftrightarrow\; x \otimes y \leq z$, we know that (3.5) induces (3.6). □

We consider characterizations of join hesitant fuzzy filters.

Theorem 3.4. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies two conditions* (3.3) *and* (3.4).

Proof. The necessity is from Proposition 3.2.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ that satisfies (3.3) and (3.4). Let x and y be elements of $\mathcal{L}$ such that $x \leq y$. Then $x \to y = 1$ and so

$$x\mathcal{H} = x\mathcal{H} \cup 1\mathcal{H} = x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H}.$$

Since $x \otimes y \leq x \otimes y$ for all $x, y \in L$, it follows from (3.6) that $x\mathcal{H} \cup y\mathcal{H} \supseteq (x \otimes y)\mathcal{H}$ for all $x, y \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 3.5. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.5).

Proof. The necessity is from Lemma 3.3 and Theorem 3.4.

Conversely let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ satisfying (3.5). Since

$$x \leq x \to 1 \;\text{ and } x \to y \leq x \to y,$$

for all $x, y \in L$, it follows from (3.5) that

$$x\mathcal{H} = x\mathcal{H} \cup x\mathcal{H} \supseteq 1\mathcal{H} \text{ and } x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H}$$

for all $x, y \in L$. Hence $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. □

Proposition 3.6. *Every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ satisfies the following condition:*

$$(3.7) \qquad (\forall x, y, z \in L)\,((x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to (x \to z))\mathcal{H}).$$

Proof. Let $x, y, z \in L$. Using (2.3) and (2.5), we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)).$$

It follows from Theorem 3.5 that

$$(x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to (x \to z))\mathcal{H}.$$

This completes the proof. □

Theorem 3.7. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if $\mathcal{H}$ satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\,((x \to (y \to z))\mathcal{H} \cup y\mathcal{H} \supseteq (x \to z)\mathcal{H})\,. \tag{3.8}$$

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Then the condition (3.3) is valid. Using (3.4) and (2.3), we have

$$\begin{aligned}(x \to z)\mathcal{H} &\subseteq y\mathcal{H} \cup (y \to (x \to z))\mathcal{H}\\ &= y\mathcal{H} \cup (x \to (y \to z))\mathcal{H}\end{aligned}$$

for all $x, y, z \in L$.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ satisfying (3.3) and (3.8). Taking $x := 1$ in (3.8) and using (2.1), we have

$$\begin{aligned}z\mathcal{H} = (1 \to z)\mathcal{H} &\subseteq (1 \to (y \to z))\mathcal{H} \cup y\mathcal{H}\\ &= (y \to z)\mathcal{H} \cup y\mathcal{H}\end{aligned}$$

for all $y, z \in L$. Thus $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. □

Proposition 3.8. *Every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ satisfies the following condition:*

$$(\forall a, x \in L)\,(a\mathcal{H} \supseteq ((a \to x) \to x)\mathcal{H})\,. \tag{3.9}$$

Proof. If we take $y := (a \to x) \to x$ and $x := a$ in (3.4), then

$$\begin{aligned}((a \to x) \to x)\mathcal{H} &\subseteq a\mathcal{H} \cup (a \to ((a \to x) \to x))\mathcal{H}\\ &= a\mathcal{H} \cup ((a \to x) \to (a \to x))\mathcal{H}\\ &= a\mathcal{H} \cup 1\mathcal{H} = a\mathcal{H}.\end{aligned}$$

This completes the proof. □

Theorem 3.9. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if it satisfies the following conditions:*

$$(\forall x, y \in L)\,(x\mathcal{H} \supseteq (y \to x)\mathcal{H})\,, \tag{3.10}$$

$$(\forall x, a, b \in L)\,(a\mathcal{H} \cup b\mathcal{H} \supseteq ((a \to (b \to x)) \to x)\mathcal{H})\,. \tag{3.11}$$

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Using (3.4), (2.3), (2.1) and (3.3), we have

$$(y \to x)\mathcal{H} \subseteq x\mathcal{H} \cup (x \to (y \to x))\mathcal{H} = x\mathcal{H} \cup 1\mathcal{H} = x\mathcal{H},$$

for all $x, y \in L$.

Using (3.8) and (3.9), we get

$$((a \to (b \to x)) \to x)\mathcal{H} \subseteq ((a \to (b \to x)) \to (b \to x))\mathcal{H} \cup b\mathcal{H} \subseteq a\mathcal{H} \cup b\mathcal{H},$$

for all $a, b, x \in L$.

Conversely, let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ satisfying two conditions (3.10) and (3.11). If we take $y := x$ in (3.10), then $x\mathcal{H} \supseteq (x \to x)\mathcal{H} = 1\mathcal{H}$ for all $x \in L$. Using (3.11) induces

$$y\mathcal{H} = (1 \to y)\mathcal{H} = (((x \to y) \to (x \to y)) \to y)\mathcal{H} \subseteq (x \to y)\mathcal{H} \cup x\mathcal{H},$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. □

Theorem 3.10. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy filter of $\mathcal{L}$ if and only if the nonempty τ-exclusive set of $\mathcal{H}$ is a filter of $\mathcal{L}$ for all $\tau \in \mathcal{P}([0,1])$.*

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ and let $\tau \in \mathcal{P}([0,1])$ be such that $e(\mathcal{H};\tau) \neq \emptyset$. Let $x, y \in L$ be such that $x \in e(\mathcal{H};\tau)$ and $x \to y \in e(\mathcal{H};\tau)$. Then $\tau \supseteq x\mathcal{H}$ and $\tau \supseteq (x \to y)\mathcal{H}$. It follows from (3.3) and (3.4) that $1\mathcal{H} \subseteq x\mathcal{H} \subseteq \tau$ and $y\mathcal{H} \subseteq x\mathcal{H} \cup (x \to y)\mathcal{H} \subseteq \tau$. Hence $1 \in e(\mathcal{H};\tau)$ and $y \in e(\mathcal{H};\tau)$, and therefore $e(\mathcal{H};\tau)$ is a filter of $\mathcal{L}$ by Proposition 2.1.

Conversely, suppose that $e(\mathcal{H};\tau)$ is a filter of $\mathcal{L}$ for all $\tau \in \mathcal{P}([0,1])$ with $e(\mathcal{H};\tau) \neq \emptyset$. For any $x \in L$, let $x\mathcal{H} = \delta$. Then $x \in e(\mathcal{H};\delta)$ and $e(\mathcal{H};\delta)$ is a filter of $\mathcal{L}$. Hence $1 \in e(\mathcal{H};\delta)$ and so $x\mathcal{H} = \delta \supseteq 1\mathcal{H}$. For any $x, y \in L$, let $x\mathcal{H} = \delta_x$ and $(x \to y)\mathcal{H} = \delta_{x\to y}$. If we take $\delta = \delta_x \cup \delta_{x\to y}$, then $x \in e(\mathcal{H};\delta)$ and $x \to y \in e(\mathcal{H};\delta)$ which imply that $y \in e(\mathcal{H};\delta)$. Thus

$$x\mathcal{H} \cup (x \to y)\mathcal{H} = \delta_x \cup \delta_{x\to y} = \delta \supseteq y\mathcal{H}.$$

Therefore $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. □

Theorem 3.11. *For a hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$, let $\mathcal{H}^*$ be a hesitant fuzzy set of $\mathcal{L}$ which is given as follows:*

$$\mathcal{H}^* : L \to \mathcal{P}([0,1]),\ x \mapsto \begin{cases} x\mathcal{H}, & \text{if } x \in e(\mathcal{H};\tau), \\ [0,1], & \text{otherwise,} \end{cases}$$

where $\tau \in \mathcal{P}([0,1])$ with $\tau \neq [0,1]$. If $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$, then so is $\mathcal{H}^*$.

Proof. Suppose that $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Then $e(\mathcal{H};\tau)$ is a filter of $\mathcal{L}$ for all $\tau \in \mathcal{P}([0,1])$ with $e(\mathcal{H};\tau) \neq \emptyset$ by Theorem 3.10. Thus $1 \in e(\mathcal{H};\tau)$, and so $1\mathcal{H}^* = 1\mathcal{H} \subseteq x\mathcal{H} \subseteq x\mathcal{H}^*$ for all $x \in L$. Let $x, y \in L$. If $x \in e(\mathcal{H};\tau)$ and $x \to y \in e(\mathcal{H};\tau)$, then $y \in e(\mathcal{H};\tau)$. Hence

$$x\mathcal{H}^* \cup (x \to y)\mathcal{H}^* = x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H} = y\mathcal{H}^*.$$

If $x \notin e(\mathcal{H};\tau)$ or $x \to y \notin e(\mathcal{H};\tau)$, then $x\mathcal{H}^* = [0,1]$ or $(x \to y)\mathcal{H}^* = [0,1]$. Thus

$$x\mathcal{H}^* \cup (x \to y)\mathcal{H}^* = [0,1] \supseteq y\mathcal{H}^*.$$

Therefore $\mathcal{H}^*$ is a join hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 3.12. *If $\mathcal{H}$ is a join hesitant fuzzy filter of L, then the set*

$$\mathcal{L}_a := \{x \in L \mid a\mathcal{H} \supseteq x\mathcal{H}\}$$

is a filter of $\mathcal{L}$ for every $a \in L$.

Proof. Since $1\mathcal{H} \subseteq a\mathcal{H}$ for all $a \in L$, we have $1 \in \mathcal{L}_a$. Let $x, y \in L$ be such that $x \in \mathcal{L}_a$ and $x \to y \in \mathcal{L}_a$. Then $x\mathcal{H} \subseteq a\mathcal{H}$ and $(x \to y)\mathcal{H} \subseteq a\mathcal{H}$. Since $\mathcal{H}$ is a join hesitant fuzzy filter of L, it follows from (3.4) that

$$a\mathcal{H} \supseteq x\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq y\mathcal{H}$$

so that $y \in \mathcal{L}_a$. Hence $\mathcal{L}_a$ is a filter of $\mathcal{L}$ by Proposition 2.1. □

Theorem 3.13. *Let $a \in L$ and let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$. Then*

(1) *If $\mathcal{L}_a$ is a filter of L, then $\mathcal{H}$ satisfies the following condition:*

$$(3.12) \quad (\forall a, x, y \in L)\,(a\mathcal{H} \supseteq x\mathcal{H} \cup (x \to y)\mathcal{H} \;\Rightarrow\; a\mathcal{H} \supseteq y\mathcal{H}).$$

(2) *If $\mathcal{H}$ satisfies* (3.3) *and* (3.12)*, then $\mathcal{L}_a$ is a filter of L.*

Proof. (1) Assume that $\mathcal{L}_a$ is a filter of L. Let x and y be elements of $\mathcal{L}$ such that

$$a\mathcal{H} \supseteq x\mathcal{H} \cup (x \to y)\mathcal{H}.$$

Then $x \to y \in \mathcal{L}_a$ and $x \in \mathcal{L}_a$. Using (2.11), we have $y \in \mathcal{L}_a$ and so $a\mathcal{H} \supseteq y\mathcal{H}$.

(2) Suppose that $\mathcal{H}$ satisfies (3.3) and (3.12). Then $1 \in \mathcal{L}_a$ by (3.3). Let x and y be elements of $\mathcal{L}$ such that $x \in \mathcal{L}_a$ and $x \to y \in \mathcal{L}_a$. Then $a\mathcal{H} \supseteq x\mathcal{H}$ and $a\mathcal{H} \supseteq (x \to y)\mathcal{H}$, which imply that $a\mathcal{H} \supseteq x\mathcal{H} \cup (x \to y)\mathcal{H}$. Thus $a\mathcal{H} \supseteq y\mathcal{H}$ by (3.12), and so $y \in \mathcal{L}_a$. Therefore $\mathcal{L}_a$ is a filter of $\mathcal{L}$ by Proposition 2.1. □

4. Join hesitant fuzzy G-filters

Definition 4.1. A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is called a *join hesitant fuzzy G-filter* of $\mathcal{L}$ if it is a join hesitant fuzzy filter of $\mathcal{L}$ such that

$$(4.1) \quad (\forall x, y \in L)\,(((x \otimes x) \to y)\mathcal{H} \supseteq (x \to y)\mathcal{H})\,.$$

Note that the condition (4.1) is equivalent to the following condition:

$$(4.2) \quad (\forall x, y \in L)\,((x \to (x \to y))\mathcal{H} \supseteq (x \to y)\mathcal{H})\,.$$

Example 4.2. Let $L := [0, 1]$ (unit interval). For any $a, b \in L$, define

$$a \vee b = \max\{a, b\},\;\; a \wedge b = \min\{a, b\},$$
$$a \to b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise,} \end{cases} \quad \text{and} \quad a \otimes b = \min\{a, b\}.$$

Then $\mathcal{L} := (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice. Let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ defined by

$$\mathcal{H} : L \to \mathcal{P}([0,1]),\ x \mapsto \begin{cases} (0, \frac{1}{2}], & \text{if } x \in [\frac{1}{2}, 1], \\ [0,1], & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$.

Theorem 4.3. *Let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$. Then $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ if and only if it is a join hesitant fuzzy filter of $\mathcal{L}$ that satisfies the following condition:*

$$(\forall x, y, z \in L)\, ((x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to z)\mathcal{H})\,. \tag{4.3}$$

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. Then $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Note that $x \leq 1 = (x \to y) \to (x \to y)$, and thus $x \to y \leq x \to (x \to y)$ for all $x, y \in L$. It follows from (3.1) that $(x \to y)\mathcal{H} \supseteq (x \to (x \to y))\mathcal{H}$. Combining this and (4.2), we have

$$(x \to y)\mathcal{H} = (x \to (x \to y))\mathcal{H}, \tag{4.4}$$

for all $x, y \in L$. Using (3.7) and (4.4), we have

$$(x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to z)\mathcal{H},$$

for all $x, y, z \in L$.

Conversely, let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$ that satisfies the condition (4.3). If we put $y = x$ and $z = y$ in (4.3) and use (2.1) and (3.3), then

$$\begin{aligned}(x \to y)\mathcal{H} &\subseteq (x \to (x \to y))\mathcal{H} \cup (x \to x)\mathcal{H} \\ &= (x \to (x \to y))\mathcal{H} \cup 1\mathcal{H} \\ &= (x \to (x \to y))\mathcal{H},\end{aligned}$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. □

Theorem 4.4. *Let $\mathcal{H}$ be a hesitant fuzzy set of $\mathcal{L}$ that satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\, (x\mathcal{H} \cup ((y \to z) \to (x \to y))\mathcal{H} \supseteq y\mathcal{H})\,. \tag{4.5}$$

Then $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$.

Proof. If we take $z := 1$ in (4.5) and use (2.1), then

$$\begin{aligned}x\mathcal{H} \cup (x \to y)\mathcal{H} &= x\mathcal{H} \cup (1 \to (x \to y))\mathcal{H} \\ &= x\mathcal{H} \cup ((y \to 1) \to (x \to y))\mathcal{H} \\ &\supseteq y\mathcal{H}.\end{aligned}$$

Hence $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$ by Theorem 3.4. Let $x, y, z \in L$. Since

$$x \to (y \to z) \leq (x \to y) \to (x \to (x \to z))$$

by (2.3), (2.4) and (2.5), we have

$$(x \to (y \to z))\mathcal{H} \supseteq ((x \to y) \to (x \to (x \to z)))\mathcal{H}$$

by (3.1). It follows that

$$\begin{aligned}(x \to y)\mathcal{H} \cup (x \to (y \to z))\mathcal{H} &\supseteq (x \to y)\mathcal{H} \cup ((x \to y) \to (x \to (x \to z)))\mathcal{H} \\ &\supseteq (x \to (x \to z))\mathcal{H} \\ &\supseteq (((x \to z) \to z) \to (x \to z))\mathcal{H} \\ &= (((x \to z) \to z) \to (1 \to (x \to z)))\mathcal{H} \\ &\supseteq (x \to z)\mathcal{H}.\end{aligned}$$

Therefore $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ by Theorem 4.3. □

The following example shows that any join hesitant fuzzy G-filter may not satisfy the condition (4.5).

Example 4.5. The join hesitant fuzzy G-filter $\mathcal{H}$ of $\mathcal{L}$ in Example 4.2 does not satisfy the condition (4.5) since

$$\tfrac{2}{3}\mathcal{H} \cup ((\tfrac{1}{3} \to \tfrac{1}{4}) \to (\tfrac{2}{3} \to \tfrac{1}{3}))\mathcal{H} = \tfrac{2}{3}\mathcal{H} \cup 1\mathcal{H} = \tau \not\supseteq U = \tfrac{1}{3}\mathcal{H}.$$

Proposition 4.6. *For a join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$, the condition* (4.5) *is equivalent to the following condition.*

$$(\forall x, y \in L)\left(((x \to y) \to x)\mathcal{H} \supseteq x\mathcal{H}\right). \tag{4.6}$$

Proof. Assume that the condition (4.5) is valid. It follows from (3.3) and (2.1) that

$$\begin{aligned}((x \to y) \to x)\mathcal{H} &= 1\mathcal{H} \cup ((x \to y) \to x)\mathcal{H} \\ &= 1\mathcal{H} \cup ((x \to y) \to (1 \to x))\mathcal{H} \\ &\supseteq x\mathcal{H}\end{aligned}$$

for all $x, y \in L$.

Conversely, suppose that the condition (4.6) is valid. It follows from (2.3) and (3.4) that

$$\begin{aligned}x\mathcal{H} \cup ((y \to z) \to (x \to y))\mathcal{H} &= x\mathcal{H} \cup (x \to ((y \to z) \to y))\mathcal{H} \\ &\supseteq ((y \to z) \to y)\mathcal{H} \supseteq y\mathcal{H}\end{aligned}$$

for all $x, y \in L$. □

Combining Theorem 4.4 and Proposition 4.6, we have the following result.

Theorem 4.7. *Every join hesitant fuzzy filter satisfying the condition* (4.6) *is a join hesitant fuzzy G-filter.*

Proposition 4.8. *Every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ with the condition* (4.5) *satisfies the following condition.*

$$(\forall x, y \in L)\,(((x \to y) \to y)\mathcal{H} \supseteq ((y \to x) \to x)\mathcal{H})\,. \tag{4.7}$$

Proof. Let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$ that satisfies the condition (4.5) and let $x, y \in L$. Since $x \to ((y \to x) \to x) = (y \to x) \to (x \to x) = (y \to x) \to 1 = 1$, that is, $x \leq (y \to x) \to x$, we have $((y \to x) \to x) \to y \leq x \to y$ by (2.4). It follows from (2.6), (2.3) and (2.4) that

$$\begin{aligned}(x \to y) \to y &\leq (y \to x) \to ((x \to y) \to x)\\ &= (x \to y) \to ((y \to x) \to x)\\ &\leq (((y \to x) \to x) \to y) \to ((y \to x) \to x).\end{aligned}$$

Using (3.1), (3.3), (2.1), (2.3) and (4.5), we have

$$\begin{aligned}((x \to y) \to y)\mathcal{H} &\supseteq ((((y \to x) \to x) \to y) \to ((y \to x) \to x))\mathcal{H}\\ &= 1\mathcal{H} \cup (1 \to ((((y \to x) \to x) \to y) \to ((y \to x) \to x)))\mathcal{H}\\ &= 1\mathcal{H} \cup ((((y \to x) \to x) \to y) \to (1 \to ((y \to x) \to x)))\mathcal{H}\\ &\supseteq ((y \to x) \to x)\mathcal{H}.\end{aligned}$$

Hence the condition (4.7) is valid. □

Corollary 4.9. *Every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ with the condition* (4.6) *satisfies the condition* (4.7).

Proposition 4.10. *Every join hesitant fuzzy G-filter $\mathcal{H}$ of $\mathcal{L}$ with the condition* (4.7) *satisfies the condition* (4.5).

Proof. Let $\mathcal{H}$ be a join hesitant fuzzy G-filter of $\mathcal{L}$ that satisfies the condition (4.7). For any $x, y, z \in L$, we have

$$\begin{aligned}z\mathcal{H} \cup ((x \to y) \to (z \to x))\mathcal{H} &= z\mathcal{H} \cup (z \to ((x \to y) \to x))\mathcal{H}\\ &\supseteq ((x \to y) \to x)\mathcal{H}\\ &\supseteq ((x \to y) \to ((x \to y) \to y))\mathcal{H}\\ &\supseteq ((x \to y) \to y)\mathcal{H}\\ &\supseteq ((y \to x) \to x)\mathcal{H}\end{aligned}$$

by (2.3), (3.4), (2.6), (3.1), (4.2) and (4.7). Since

$$(x \to y) \to x \leq y \to x \leq z \to (y \to x),$$

it follows from (3.1) that $((x \to y) \to x)\mathcal{H} \supseteq (z \to (y \to x))\mathcal{H}$ and so from (3.4) that

$$\begin{aligned} z\mathcal{H} \cup ((x \to y) \to (z \to x))\mathcal{H} &\supseteq z\mathcal{H} \cup ((x \to y) \to x)\mathcal{H} \\ &\supseteq z\mathcal{H} \cup (z \to (y \to x))\mathcal{H} \\ &\supseteq (y \to x)\mathcal{H}. \end{aligned}$$

Therefore

$$z\mathcal{H} \cup ((x \to y) \to (z \to x))\mathcal{H} \supseteq (y \to x)\mathcal{H} \cup ((y \to x) \to x)\mathcal{H} \supseteq x\mathcal{H}.$$

Hence the condition (4.5) is valid. □

Theorem 4.11. *Let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$. Then $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ if and only if the following condition holds:*

$$(\forall x \in L)\,((x \to (x \otimes x))\mathcal{H} = 1\mathcal{H})\,. \tag{4.8}$$

Proof. Suppose that $\mathcal{H}$ is a join hesitant fuzzy G-filter of L. Since $x \to (x \to (x \otimes x)) = 1$ for all $x \in L$, we have $(x \to (x \to (x \otimes x)))\mathcal{H} = 1\mathcal{H}$. It follows from (4.3) and (2.1) that

$$(x \to (x \otimes x))\mathcal{H} \subseteq (x \to (x \to (x \otimes x)))\mathcal{H} \cup (x \to x)\mathcal{H} = 1\mathcal{H}$$

and so from (3.3) that $(x \to (x \otimes x))\mathcal{H} = 1\mathcal{H}$ for all $x \in L$.

Conversely, let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$ which satisfies the condition (4.8) and let $x, y \in L$. Since

$$x \to (x \to y) = (x \otimes x) \to y \leq (x \to (x \otimes x)) \to (x \to y)$$

by (2.3) and (2.5), it follows from (3.1) that

$$(x \to (x \to y))\mathcal{H} \supseteq ((x \to (x \otimes x)) \to (x \to y))\mathcal{H}.$$

Hence, we have

$$\begin{aligned} (x \to y)\mathcal{H} &\subseteq ((x \to (x \otimes x)) \to (x \to y))\mathcal{H} \cup (x \to (x \otimes x))\mathcal{H} \\ &\subseteq (x \to (x \to y))\mathcal{H} \cup (x \to (x \otimes x))\mathcal{H} \\ &= (x \to (x \to y))\mathcal{H} \cup 1\mathcal{H} \\ &= (x \to (x \to y))\mathcal{H} \end{aligned}$$

by using (3.4), (4.8) and (3.3). Hence $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. □

Theorem 4.12. *A hesitant fuzzy set $\mathcal{H}$ of $\mathcal{L}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ if and only if it is a join hesitant fuzzy filter of $\mathcal{L}$ with an additional condition:*

$$(\forall x, y \in L)\,((x \to y)\mathcal{H} = (x \to (x \to y))\mathcal{H})\,. \tag{4.9}$$

Proof. Suppose that $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. Then $\mathcal{H}$ is a join hesitant fuzzy filter of $\mathcal{L}$. Let $x, y \in L$. Since $x \to y \leq x \to (x \to y)$, we have

$$(x \to y)\mathcal{H} \supseteq (x \to (x \to y))\mathcal{H}$$

by (3.1). Hence $(x \to y)\mathcal{H} = (x \to (x \to y))\mathcal{H}$ by using (4.2).

Conversely, let $\mathcal{H}$ be a join hesitant fuzzy filter of $\mathcal{L}$ with the condition (4.9). It follows from Proposition 3.6 that

$$(x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to (x \to z))\mathcal{H} = (x \to z)\mathcal{H}$$

for all $x, y, z \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ by Theorem 4.3. $\square$

Proposition 4.13. *Every join hesitant fuzzy G-filter $\mathcal{H}$ of $\mathcal{L}$ satisfies the following conditions:*

$$(\forall x, y, z \in L)\,((x \to (y \to z))\mathcal{H} \supseteq ((x \to y) \to (x \to z))\mathcal{H})\,. \tag{4.10}$$

$$(\forall x, y, z \in L)\,((x \to (y \to z))\mathcal{H} = ((x \to y) \to (x \to z))\mathcal{H})\,. \tag{4.11}$$

Proof. Let $\mathcal{H}$ be a join hesitant fuzzy G-filter of $\mathcal{L}$. Using (2.3), (4.3), (2.5) and (3.3), we have

$$\begin{aligned}((x \to y) \to (x \to z))\mathcal{H} &= (x \to ((x \to y) \to z))\mathcal{H} \\ &\subseteq (x \to (y \to z))\mathcal{H} \cup (x \to ((y \to z) \to ((x \to y) \to z)))\mathcal{H} \\ &= (x \to (y \to z))\mathcal{H} \cup ((y \to z) \to ((x \to y) \to (x \to z)))\mathcal{H} \\ &= (x \to (y \to z))\mathcal{H} \cup 1\mathcal{H} \\ &= (x \to (y \to z))\mathcal{H}\end{aligned}$$

for all $x, y, z \in L$. Thus (4.10) holds. Since $(x \to y) \to (x \to z) \leq x \to (y \to z)$ for all $x, y, z \in L$, it follows from (3.1) that $((x \to y) \to (x \to z))\mathcal{H} \supseteq (x \to (y \to z))\mathcal{H}$ and so that

$$(x \to (y \to z))\mathcal{H} = ((x \to y) \to (x \to z))\mathcal{H}$$

for all $x, y, z \in L$ by using (4.10). $\square$

Proposition 4.14. *Assume that $\mathcal{L}$ satisfies the divisibility, that is, $x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$. If $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ satisfying* (4.11)*, then the following equality is true.*

$$(\forall x, y, z \in L)\,(((x \otimes y) \to z)\mathcal{H} = ((x \wedge y) \to z)\mathcal{H})\,. \tag{4.12}$$

Proof. Using the divisibility and (2.3), we have

$$(x \wedge y) \to z = (x \otimes (x \to y)) \to z = (x \to y) \to (x \to z)$$

for all $x, y, z \in L$. It follows from (2.3) and (4.11) that

$$\begin{aligned}((x \otimes y) \to z)\mathcal{H} &= (x \to (y \to z))\mathcal{H} \\ &= ((x \to y) \to (x \to z))\mathcal{H} \\ &= ((x \wedge y) \to z)\mathcal{H}\end{aligned}$$

for all $x, y, z \in L$. □

Theorem 4.15. *Let $\mathcal{L}$ satisfy the divisibility, that is, $x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$. Then every join hesitant fuzzy filter $\mathcal{H}$ of $\mathcal{L}$ satisfying the condition (4.12) is a join hesitant fuzzy G-filter of $\mathcal{L}$.*

Proof. Using Proposition 3.6, (2.3) and (4.12), we have

$$\begin{aligned}&(x \to (y \to z))\mathcal{H} \cup (x \to y)\mathcal{H} \supseteq (x \to (x \to z))\mathcal{H} \\ &= ((x \otimes x) \to z)\mathcal{H} = ((x \wedge x) \to z)\mathcal{H} = (x \to z)\mathcal{H}\end{aligned}$$

for all $x, y, z \in L$. Therefore $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$ by Theorem 4.3. □

Theorem 4.16. *Let $\mathcal{H}$ and $\mathcal{G}$ be join hesitant fuzzy filters of $\mathcal{L}$ such that $1\mathcal{H} = 1\mathcal{G}$ and $\mathcal{H} \supseteq \mathcal{G}$, i.e., $x\mathcal{H} \supseteq x\mathcal{G}$ for all $x \in L$. If $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$, then so is $\mathcal{G}$.*

Proof. Assume that $\mathcal{H}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. Using (2.3) and (2.1), we have

$$x \to (x \to ((x \to (x \to y)) \to y)) = (x \to (x \to y)) \to (x \to (x \to y)) = 1$$

for all $x, y \in L$. Thus

$$\begin{aligned}(x \to ((x \to (x \to y)) \to y))\mathcal{G} &\subseteq (x \to ((x \to (x \to y)) \to y))\mathcal{H} \\ &= (x \to (x \to ((x \to (x \to y)) \to y)))\mathcal{H} \\ &= 1\mathcal{H} = 1\mathcal{G}\end{aligned}$$

by hypotheses and (4.4), and so

$$(x \to ((x \to (x \to y)) \to y))\mathcal{G} = 1\mathcal{G}$$

for all $x, y \in L$ by (3.3). Since $\mathcal{G}$ is a join hesitant fuzzy filter of $\mathcal{L}$, it follows from (3.4), (2.3) and (3.3) that

$$\begin{aligned}(x \to y)\mathcal{G} &\subseteq (x \to (x \to y))\mathcal{G} \cup ((x \to (x \to y)) \to (x \to y))\mathcal{G} \\ &= (x \to (x \to y))\mathcal{G} \cup (x \to ((x \to (x \to y)) \to y))\mathcal{G} \\ &= (x \to (x \to y))\mathcal{G} \cup 1\mathcal{G} \\ &= (x \to (x \to y))\mathcal{G}\end{aligned}$$

for all $x, y \in L$. Therefore $\mathcal{G}$ is a join hesitant fuzzy G-filter of $\mathcal{L}$. □

Acknowledgements

The authors are thankful to the anonymous referees for their valuable comments and suggestions.

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Accepted: 18.12.2017