En-semi prime subacts over monoids with zero

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Abstract. Let S be a monoids with zero and A_s be a right S-act . In this paper,we introduce the notion of En- semi prime subact of an S-act A_s which is define as : A proper subact B of an S-act A_s is said to be En- semi prime subact, if whenever $f^2(a)S \subseteq B$ for some endomorphism f of an S-act A_s, and $a \in A_s$, then $f(a)S \subseteq B$. An S-act A_s itself is called En-semi prime if the zero subact (θ) of A_s is En-semi prime subact of A_s. Also, we study and gives some related concepts of this notion like: semi prime subact, En-prime subact, En- pure subact and En-radical of subact.

Keywords: En-semi prime subact, En-Prime subact, En-pure subact and En-radadical of subact.

1. Introduction

Firstly we begin with some preliminary definitions and notions: "Recall that a nonempty set A is called a right S-act where S is monoid that is semigroup with identity element 1, if there exists a mapping $\phi : A \times S \to A$ define as $(a, s) \to as$ and satisfying $a.1 = a$ and $a(st) = (as)t$, for all $a \in A$and $s, t \in S$. We call A a right S-act or right act over S and write A_s" [1]. "Similarly, we define a left S-acts A and write ${}_sA$. If S is a commutative monoid, then every left S-act is right S-act. A non empty subset B of a right S-act A_s is called subact of A_s and written by $B \leq A_s$, if $bs \in B$ for all $b \in B$ and $s \in S$. An element $\theta \in A_s$ is called a zero of A_s or fixed element if $\theta_s = \theta$ for all $s \in S$, i.e. $\{\theta\}$ is a one-element subact [1]. In this paper θ is a unique fixed element of all S-act A_s. A nonempty subset I is called an ideal or two sided ideal of S (left and right) if $SI \subseteq I$ and $IS \subseteq I$. A mapping $f : A_s \to B_s$, where A_s and B_s are two right S-acts is called S- homomorphism if $f(as) = f(a)s$, for all $a \in A_s$ and $s \in S$. The set of all S-homomorphism from A in to B denoted by $Hom(A_s, B_s)$ or $Hom_s(A, B)$. An S-homomorphism $f : A_s \to A_s$ is called an endomorphism of A_s. The composition $g \circ f$ of homomorphism $f : A_s \to B_s$ and $g : B_s \to C_s$ of a right S-acts is a homomorphism of a right act,i.e. $g \circ f \in Hom(A_s, B_s)$" [1].

Now, A. A. Estaji and S. Tajnia in [3] introduce the concept of semi prime subact which is define as: "A proper subact B of an Sact A_s is called semi-prime, if whenever $as^k \in B$ for some $s \in S$, $a \in A_s$ and $k \in N$ implies that

$as \in B$" [2]. Shireen in [3] introduce the notion of En-prime subact, where: "A proper subact B of an Sact A_s is called En- prime subact of A_s if for any endomorphism f of A_s and $a \in A_s$ with $f(a)S \subseteq B$ implies that either $a \in B$ or $f(As) \subseteq B$. The right S-act A_s is called En-prime if the zero subact (θ) of A_s is En-prime subact" [3].

In this paper, we introduce the concept of En- semi prime subact as a generalization of En-prime subact, where a proper subact B of an Sact A_s is said to be En- semi prime subact if whenever $f^2(a)S \subseteq B$ for some endomorphism f of an Sact A_s and $a \in A_s$, then $f(a)S \subseteq B$. An Sact A_s itself is called En-semi prime if the zero subact (θ) of A_s is En-semi prime subact of A_s. In section one, we study some basic properties of this notion and the relationships between this notion and some other related concepts like: semi prime, En-prime and maximal subacts.

2. En-semi prime subacts and some related concept

In this section, we introduce the concept of En- semi prime subacts and give some characterizations for this notion.

Definition 2.1. *A proper subact B of an S-act A_s is said to be En- semi prime subact, if whenever $f^2(a)S \subseteq B$ for some endomorphism f of an S-act A_s and $a \in A_s$, then $f(a) \subseteq B$. An S-act A_s itself is called En-semi prime if the zero subact (θ) of A_s is En-semi prime subact of A_s.*

Proposition 2.2. *Let B be a proper subact of an S-act A_s, then B is En- semi prime subact of A_s if and only if $f^k(a)S \subseteq B$ for some endomorphism f of an S-act A_s,$a \in A_s$ and for $k \geq 2$ then $f(a)S \subseteq B$.*

Proof. The proof by induction on the positive integer k. If $k = 2$, then the proposition is true by the definition of En- semi prime subact. Assume that our proposition is true for $k - 1$ which means if $f^{k-1}(a)S \subseteq B$, then $f(a)S \subseteq B$. Now, suppose that $f^k(a)S \subseteq B$ and thus $f^2(f^{k-2}(a))S \subseteq B$ which implies that $f(f^{k-2}(a))S \subseteq B$. Therefore $f(a)S \subseteq B$ by our induction. □

Recall that a proper subact B of an Sact A_s is called semi-prime, if whenever $as^k \in B$ for some $s \in S$, $a \in A_s$ and $k \in N$ implies that as $\in B$ [2].

Proposition 2.3. *Let B be a proper subact of an S-act A_s. If B is En- semi prime subact of A_s, then B is semi prime subact.*

Proof. Suppose that $as^k \in B$, where $a \in A_s$, $s \in S$ and $k \in N$. Define $f : A_s \to A_s$ by $f(c) = cs$ for all $c \in A_s$ and $s \in S$. Now, $f(a) = as$ and so $f^k(a) = as^k \in B$. Hence $f^k(a)S \subseteq B$, but B is En-semi prime subact of A_s then $f(a)S \subseteq B$. Therefore $as \in B$. □

Remark 2.4. The converse of previous proposition is not true in general and we can show that by the following example: Let $Z \oplus Z$ be an $(Z, .)$act with multiplication by integers as operation and let $6Z \oplus Z$ be a subact of $Z \oplus Z_{(z,.)}$ which is semi prime subact of $Z \oplus Z_{(z,.)}$. We define a function as: $f : Z \oplus Z \to Z \oplus Z$ by $f(x, y) = (y, x)$, for all $x, y \in Z$. Now, $f^2(0, 3) = (0, 3) \in 6Z \oplus Z$ and hence $f^2(0, 3)Z \subseteq 6Z \oplus Z$. But $f(0, 3) = (3, 0) \notin 6Z \oplus Z$ which implies that $f(0, 3)Z \nsubseteq 6Z \oplus Z$. Thus $6Z \oplus Z$ is not En- semi prime subact of $Z \oplus Z_{(z,.)}$.

Recall that a proper subact B of an Sact A_s is called En- prime subact of A_s if for any endomorphism f of A_s and $a \in A_s$ with $f(a)S \subseteq B$ implies that either $a \in B$ or $f(A_s) \subseteq B$. The right S-act A_s is called En-prime if the zero subact (θ) of A_s is En-prime subact [3].

Proposition 2.5. *Every En-prime subact of an Sact A_s is En-semi prime subact of A_s.*

Proof. Let B be a proper subact of an S-act A_s and suppose that $f^2(a)S \subseteq B$ for some endomorphism f of an S-act A_s and $a \in A_s$. We have to prove that $f(a)S \subseteq B$. Now, $f^2(a)S = f(f(a))S \subseteq B$, but B is En- prime subact of A_s, then either $f(a) \in B$ or $f(A_s) \subseteq B$. That follows in any case $f(a) \in B$. Therefore $f(a)S \subseteq B$. □

Proposition 2.6. *Let A_s be an S-act and B a proper subact of A_s. If $B = \cap P_i$, where P_i is En- prime subact of A_s, then B is En- semi prime subact of A_s.*

Proof. Let $f^2(a)S \subseteq B$ for some endomorphism f of an Sact A_s and $a \in A_s$, then $f^2(a)S \subseteq P_i$ for each i. But P_i is En- prime subact of A_s, then by prop.(2.5) P_i is En- semi prime subact of A_s. Thus $f(a)S \subseteq P_i$ for each i which implies that $f(a)S \subseteq \cap P_i = B$. Therefore B is En- semi prime subact of A_s. □

Corollary 2.7. *The intersection of En-semi prime subacts of an S-act A_s is En-semi prime subact of A_s*

Proposition 2.8. *The union of any two En- semi prime subacts of A_s is En-semi prime subact of A_s.*

Proof. Let B_1 and B_2 be any two En- semi prime subacts of an Sact A_s. Suppose that for some endomorphism f of A_s and $a \in A_s$ we have $f^2(a)S \subseteq B_1 \cup B_2$. Now, since we have $f^2(a)S \subseteq B_1 \cup B_2$, then either $f^2(a)S \subseteq B_1$ or $f^2(a)S \subseteq B_2$. But B_1 and B_2 are En- semi prime subacts of A_s then either $f(a)S \subseteq B_1$ or $f(a)S \subseteq B_2$. Thus $f(a)S \subseteq B_1 \cup B_2$. □

Recall that a subact B of an S-act A_s is called fully invariant subact if $f(B) \subseteq B$ for every endomorphism f of A_s and A_s is called duo act if every subact of A_s is fully invariant .

Proposition 2.9. *Every maximal subact of duo act is En- semi prime subact.*

Proof. Let B be a maximal subact of duo act A_s . Then by [[3], corl.(2.7)] we have B is Enprime subact of A_s and by prop.(2.5) we get B is Ensemi prime subact of A_s. □

Definition 2.10. *A subact B of an S-act A_s is called En- pure subact if $f(A_s) \cap B = f(B)$ for any endomorphism f of A_s.*

Example 2.11. The one element subact zero θ and the act itself are En-pure subacts.

Proposition 2.12. *Let A_s be an S-act A_s such that every subact of A_s is En-pure subact of A_s, then each proper subact of A_s is En- semi prime subact of A_s.*

Proof. Let B be a proper subact of an Sact A_s and let $f^2(a)S \subseteq B$, where f an endomorphism of A_s and $a \in A_s$. Now, $f(a) \in f(A_s) \cap f(a)S = f(f(a)S) = f^2(a)S \subseteq B$. Hence $f(a) \in B$ and therefore $f(a)S \subseteq B$. □

Definition 2.13. *The intersection of all En-prime subacts of an S-act A_s containing a subact B of A_s is said to be En- radical of B and denoted by En-rad (B) . If B is not contained in any En- prime subact of A_s, then we put En-rad$(B) = A_s$.*

Recall that a proper subact B of an S-act A_s is said to be prime subact of A_s, if for every $s \in S$ and $a \in A_s$, $as \in B$ implies that $a \in B$ or $s \in (B : A_s)$ [2]. Recall that for a subact B of an Sact A_s, $rad_A(B)$ is the intersection of all prime subacts of A_s containing B and $rad(B) = B$, if B is not contained in any prime subact of A_s [2].

Proposition 2.14. *If B is a subact of an S-act A_s then:*

1. *$B \subseteq En - rad(B)$.*
2. *$rad(B) \subseteq En - rad(B)$.*

Proof.

1. It is clear.
2. Let C be En-prime subact of an S-act A_s containing B. Then By [**??**, prop.(2.3)] C is a prime subact of A_s, hence $rad(B) \subseteq C$. Also, $rad(B) \subseteq \cap C$ for all En- prime subact C containing B. Therefore $rad(B) \subseteq En - rad(B)$. □

Proposition 2.15. *Let B be a subact of an S-act A_s . If En-rad$(B) \neq A_s$, then En-rad(B) is En- semi prime subact of A_s.*

Proof. Let $f^2(a)S \subseteq En - rad(B)$, where f an endomorphism of A_s and $a \in A_s$. Now, we have $f^2(a)S \subseteq \cap P_i$, where P_i is En- prime subact of A_s containing B. Then $f^2(a)S \subseteq P_i$ for all i which implies that $f(a)S \subseteq P_i$ for all i. Therefore $f(a)S \subseteq \cap P_i. = En - rad(B)$. □

Definition 2.16. *Let B be a subact of an S-act A_s, then we define: En(B)=\{ $f(a)$, where f any endomorphism of A_s and $a \in A_s$ such that $f^k(a) \in B$ for some $k \in \mathbb{N}$ \}.*

Recall that let B be a subact of an S-act A_s, then $E(B) = \{as : s \in S$ and $as^k \in B$ for some $k \in \mathbb{N}\}$ [2].

Proposition 2.17. *Let B be a subact of an S-act A_s, then:*

1. *$E(B) \subseteq En(B)$ and thus $B \subseteq En(B)$.*
2. *$En(B) \subseteq P$ for all En-prime subact P containing B and thus $En(B) \subseteq En - rad(B)$.*

Proof. 1. Let $x \in E(B)$, then $x = as$ where $a \in A_s$ and $s \in S$ and there exists a positive integer k such that $as^k \in B$. Define $f : A_s \to A_s$ by $f(b) = bs$ for all $b \in A_s$ and $s \in S$. Now, $x = f(a)$ and $f^k(a) = as^k \in B$ and hence $x \in En(B)$.

2. Suppose that $x \in En-(B)$, then there exists an endomorphism of A_s and $a \in A_s$ such that $x = f(a)$ and for some positive integer k we have $f^k(a) \in B$ and thus $f^k(a)S = f^{k-1}(f(a)) \subseteq B \subseteq P$ But P is En- prime subact of an S-act A_s, then either $f(a) \in P$ or $f^{k-1}(A_s) \subseteq P$, which follows that $x = f(a) \in P$. Consequently, $En(B) \subseteq En - rad(B)$. □

Proposition 2.18. *Let B be a subact of an S-act A_s, then B is En-semi prime subact of A_s if and only if $En(B) = B$.*

Proof. Let $x \in En(B)$, then there exists an endomorphism f of A_s and $a \in A_s$ such that $x = f(a)$ and for some positive integer k we have $f^k(a) \in B$ and thus $f^k(a)S \subseteq B$. But B is En-semi prime subact of A_s, then $f(a)S \subseteq B$ wich implies that $x = f(a) \in B$. Therefore $En(B) \subseteq B$ and the result follows from prop.(2.17)(1). Conversely, suppose that $En(B) = B$ and let $f^2(a)S \subseteq B$ for some endomorphism f of A_s and $a \in A_s$, then $f^2(a) \in B$ and thus $f(a) \in En - (B) = B$. Hence $f(a)S \subseteq B$. □

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