A new view of closed-CS-module

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Abstract. This paper give a new fact about the extending module. A module M is called extending if every closed submodule N of M is a direct summand. Study of the concepts complement closed submodule ((Closed-N)^c) is achieved. Also we expose to a new way to obtain generalization of extending module by complement closed submodule.
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1. Introduction

In (1976), Goodearl introduced the definition of complement closed submodule and Dungh, Huynh, Smith and Wisbauer [1], studied the extending modules. Wang [5] studied closed-CS-module. A submodule A of M is called essential submodule if A∩K≠0 for every non-zero submodule K of M, equivalently A is a essential in M if and only if every non-zero element of M has a non-zero multiple in A. Therefore if every submodule is essential in a direct summand of M, then M is called extending module. A module M is called extending if every closed submodule N of M is a direct summand of M. Extending modules has been studied in [1] and [2]. Let Z(M)= $\{I_x \in M : I_x=0$, for some ideal $I \leq_{ess} R\}$. If Z(M)=M, then M is a singular. Thus we can define another set: Let $\frac{M}{N}$ be a quotient module and let $Z(\frac{M}{N})=\{a+I_x \in (\frac{M}{N}) : I_x=0$, for some ideal $I \leq_{ess} R\}$. If $Z(\frac{M}{N})=\frac{M}{N}$, then $\frac{M}{N}$ is singular. Therefore if $Z(\frac{M}{N}) \neq \frac{M}{N}$, this means the quotient module $\frac{M}{N}$ is non singular.

Remark 1.1. (a) We denote (Closed-N)^c to complement closed submodule N of M.

(b) Every semisimple R-module is an extending module. For example Z6 as Z-module.

(c) Not every module M has closed submodule is extending; for example; the module M=Z8⊕Z2 as a Z-module. Let A=(2,1) be the submodule generated by (2,1). Clear that A is closed in M but not a summand. Hence M is not extending.

(d) Let us take (Closed-B)^c belong to A; where A and B are submodules in an R-Module M. Then $\frac{A}{B}$ is essential in M.

(e) Every (Closed-N)^c is closed.

Theorem 1.2. *Any module K is singular if and only if there exists a short exact sequence*

$$0 \longrightarrow N \longrightarrow L \longrightarrow K \longrightarrow 0$$

such that f is an essential monomorphism between N and L.

Definition 1.3. *(see [4]) Let M be a module. Then M is called closed-CS-module (generalization of extending module) if for every submodule N of M; the quotient module $\frac{M}{N}$ is non singular and is direct summand of M. (i.e. M has (Closed-N)^c and direct summand of M).*

This paper, contain two main sections. In the first section we give some properties of (Closed-N)^c and in the second section the closed-CS-module is investigated. We prove if K is maximal (Closed-K)^c of M, then $\frac{M}{K}$ is a projective and K is a direct summand of M. (see Proposition 2.13). On the other hand, we prove that an R-module M is closed-CS-module iff for every (Closed-N)^c of M, there is a decomposition M=$M_1 \oplus M_2$ such that A is a subset of M_1 and A^c =$M_2 \in$M. (see Theorem 3.5).

2. Complement closed submodule

Let N be a submodule of an R-module M (N$\leq$M). Then we can denote (Closed-N)^c of M to the complement closed submodule N and (closed-CS-module) means M has (Closed-N)^c. If every (Closed-N)^c of M is a direct summand, then we obtain a generalization of extending module M (closed-CS-module).

Remark 2.1. If the quotient module $\frac{M}{N}$ is non singular, then N is a (Closed-N)^c.

Definition 2.2. *For N subset of M and L subset of N such that L$\lhd$ N, then M$\equiv$($\frac{N}{L}$). So, if we have N as a module, then N is called generalization of extending module if the quotient module $\frac{N}{L}$ is non singular and is a direct summand in M.*

Note that, if (Closed-N)^c is a subset of M, then N subset of (Closed-K)^c and from the second isomorphism theorem, we have; N subset of (Closed-N)^c +K$\Longleftrightarrow$(N$\cap$K) is a subset of (Closed-K)^c. Also, by the third isomorphism theorem we can say: N is a subset of K and K is a subset of M$\Longrightarrow$K is a subset of (Closed-N)^c of M $\Longleftrightarrow$ $\frac{K}{N}$ is a subset of (Closed-$\frac{K}{N}$)^c.

Lemma 2.3. *Let M be an R-module and let Bα in Λ, be an independent family of submodules of M and Aα is a subset of Bα, for all α in Λ. Then $\oplus$Aα is a subset of (Closed-N)^c of $B_\alpha\alpha$ if and only if Aα is a subset of (Closed-N)^c of Bα, for all α in Λ.*

Proof. Suppose that $\oplus$Aα is a subset of $\oplus$Bα. We have, $\frac{\oplus B\alpha}{\oplus A\alpha} \cong \frac{B\alpha}{A\alpha}$. Then A$\alpha$ subset of (Closed-N)^c of Bα, for all α in Λ. Conversely, Aα is a subset of (Closed-N)^c of Bα, for all α in Λ . Then $\frac{B\alpha}{A\alpha}$ is non-singular, for all αinΛ and hence $\oplus$ $\frac{B\alpha}{A\alpha}$ is non-singular. But $\oplus\frac{B\alpha}{A\alpha}\cong\frac{\oplus B\alpha}{\oplus A\alpha}$. So A$\oplus\alpha$ is a subset of (Closed-N)^c of $\oplus$ Bα. □

Theorem 2.4. *Let M be an R-module and let N and K are submodules of M. Then (N∩K) is a subset of (Closed-N)^c in M.*

Proof. Let N be a subset of (closed-CS-module) and let K be a subset of (closed-CS-M). We must prove that (N∩K) is a subset of (Closed-N)^c in M. Let us take an element m∈M such that m+(N∩K) belong to Z($\frac{M}{N}$∩K). Thus Annihilator of (m+N∩K) is a subset of (eR). Since Annihilator of (m+N∩K) is a subset of Annihilator of (m+N), then Annihilator of (m+N) is a subset of (eR). We have Z($\frac{M}{N}$)=0, therefore m+N=N. Similar, we get m+K=K. Thus m belong to N∩K and then Z($\frac{M}{N\cap K}$)=0. □

Lemma 2.5. *Let L and K be a submodules of an R-module M. If L is a subset of (Closed-K)^c and K is a subset of (closed-CS-module), then L is a subset of (closed-CS-module).*

Proof. Let L be a subset of (Closed-K)^c and let K be a subset of (closed-CS-module). Let us take short exact sequence:

$$0 \longrightarrow (\frac{K}{L}) \longrightarrow (\frac{M}{L}) \longrightarrow (\frac{M}{L})/(\frac{K}{L}) \longrightarrow 0.$$

Such that i is the inclusion map from ($\frac{K}{L}$) into ($\frac{M}{L}$) and π is the natural epimorphism from ($\frac{M}{L}$) into ($\frac{M}{L}$)/($\frac{K}{L}$). Since L is a subset of K and K is a subset of (closed-CS-module), then ($\frac{K}{L}$) is a subset of (Closed-N)^c of ($\frac{M}{L}$), (see Theorem 2.4). Since ($\frac{K}{L}$) and ($\frac{M}{L}$)/($\frac{K}{L}$) are non-singular, then $\frac{M}{L}$ is non-singular. □

Let M be an R-module such that L subset of K and K subset of M. If K subset of (Closed-N)^c of M, then L need not be (Closed-N)^c. See the following example:

Example 2.6. Consider Z as Z-module, it is clear that Z subset of (Closed-N)^c of Z. But Z(2Z ⊆Z)=Z(Z_2)= Z_2 is singular. On the other hand, if L subset of (Closed-N)^c of M, then K need not be (Closed-K)^c.

Example 2.7. Let 0 subset of 2Z and 2Z subset of Z. Clearly 0 subset of (closed-CS-Z). But Z($\frac{Z}{2Z}$)=Z(Z_2)=Z_2 is singular. Also, an epimorphic image of an (Closed-N)^c need not be (closed-CS-module). We have the natural epimorphism π:Z $\longrightarrow$ $\frac{Z}{4Z}$. That is means 0 subset of (Closed-N)^c of Z. On the other hand, since $\frac{Z}{4Z} \cong Z_4$ is a singular imply the image of zero always equal zero and moreover it is not (closed-CS-$\frac{Z}{4Z}$).

Proposition 2.8. *Let λ:M $\longrightarrow$ N be an epimorphism and L subset of (closed-CS-module). If ker(f) subset of L, then f(L) subset of (Closed-N)^c.*

Proof. Assume that L subset of (closed-CS-module). To show that f(L) subset of (Closed-N)^c. Let n belong to N such that Annihilator(n+f(L)) subset of eR. Since f is an epimorphism, then n=f(m), for some m ∈ M. Since ker(f) subset of L, then Annihilator(n+f(L)) subset of Annihilator(m+L) and hence Annihilator(n+f(L)) subset of eR. But L subset of (Closed-N)^c of M, so m ∈ L. Thus n=f(m)∈f(L). □

Theorem 2.9. *Let λ:M $\longrightarrow$ N be an R-homomorphism and K (Closed-N)^c, then for every singular submodule L of M, f(L) subset of K .*

Proof. Let μ:N $\longrightarrow \frac{N}{K}$ be the natural epimorphism. Let $\mu\circ\lambda$: M $\longrightarrow \frac{N}{K}$. Now $\mu\circ\lambda|_L$:L $\longrightarrow \frac{N}{K}$. But N is a singular and $\frac{N}{K}$ is non-singular. Thus $\mu\circ\lambda|_L$=0. So $\mu(\lambda$(L)) =0 and hence λ(L) subset of ker(μ)=K. □

As a result from Theorem 2.9, we introduce the following good corollary.

Corollary 2.10. *If N is a module and K subset of (Closed-N)^c. Then $\frac{Hom(M,N)}{M}$ subset of K, such that Z(M)=M.*

Example 2.11. Suppose that M is an R-module. Let L subset of (closed-CS-module). Then Z(M)=Z(L).

Proof. We must prove that Z(M) is a subset of Z(L). Let i:Z(M)$\longrightarrow$M be the inclusion map and μ:M$\longrightarrow\frac{M}{L}$ be the natural epimorphism from M into $\frac{M}{L}$. We take the map $\mu\circ$i:Z(M)$\longrightarrow\frac{M}{L}$. Since Z(M) is a singular and $\frac{M}{L}$ is non-singular, then $\mu\circ$i=0. So $\mu\circ$i:(Z(M))=μ(Z(M))=0. Thus Z(M) is a subset of ker(μ)=L. We know that Z(L)=Z(M)$\cap$A. So Z(L)=Z(M). □

Theorem 2.12. *Let M be an R-module and let L $\subseteq$ K $\subseteq$ M and N $\subseteq$ (closed-CS-module), then $\frac{M}{K}$ is a singular if and only K subset of (closed-CS-module).*

Proof. Let L subset of (Closed-N)^c of M and $\frac{M}{K}$ is singular. By the third isomorphism theorem $\frac{M}{K}\cong(\frac{M}{L})/(\frac{K}{L})$. Since $\frac{M}{L}$ is non-singular, then $(\frac{K}{L})$ subset of (closed-CS-$\frac{M}{N}$). Let μ:M$\longrightarrow\frac{M}{N}$ be the natural epimorphism. We have K=$\mu^{-1}(\frac{K}{L})$ is a subset of $\mu^{-1}(\frac{M}{L})$=M. The converse is clear by [3]. □

Proposition 2.13. *Let M be an R-module and K is maximal (Closed-K)^c of M. Then $\frac{M}{K}$ is projective and K is a direct summand of M.*

Proof. Since K is maximal submodule of M, then $\frac{M}{K}$ is simple and hence semisimple. But $\frac{M}{K}$ is non-singular, therefore $\frac{M}{K}$ is projective. Now consider the following short exact sequence 0$\longrightarrow$K$\longrightarrow$M$\longrightarrow\frac{M}{K}\longrightarrow$0; where i is the inclusion map and π is the natural epimorphism from M into $\frac{M}{K}$. Since $\frac{M}{K}$ is projective, then the sequence is splits, (see [6]). Thus K is a direct summand of M. Let M be an R-module and N subset of M. Recall that the resdual of M in N (denoted by [N:M]) is defined as follows: [N:M]=r$\in$R, rM$\subseteq$N, (see [7]). □

3. Closed-CS-module

In this section, we introduce main theorems which explain the new ways to obtain a generalization of extending module.

Proposition 3.1. *Let M be a (Closed-N)^c and N $\leq$ M, then the quotient module is a (Closed-N)^c of M*

Proof. Let $\frac{K}{N}$ subset of (Closed-N)^c of $\frac{M}{N}$. Then by Theorem 2.4 and Lemma 2.5, K is a subset of (Closed-N)^c in M. But M is a closed-CS-module. (i.e. has (Closed-N)^c) of M, therefore M=N⊕K, K is a subset of M. Since N is a subset of K, then one can easily show that $\frac{M}{N}$=($\frac{K}{N}$)⊕($\frac{K+N}{N}$). Thus $\frac{M}{N}$ is a closed-CS-module. □

Recall that a module M is called closed-CS-module if for any submodule N of M, there is a direct summand K of M such that N is a subset of K and $\frac{K}{N}$ is singular.

Let N subset of (Closed-N)^c. Since M is (Closed-N)^c, then there exists a direct summand K of M such that N is a subset of K and Z($\frac{K}{N}$)=($\frac{K}{N}$); ($\frac{K}{N}$ is a singular). But $\frac{K}{N}$ is a subset of $\frac{M}{N}$, so is non-singular. Thus K=N. So any (Closed-M)^c is closed-CS-module.

Theorem 3.2. *An R-module M is a closed-CS-module if and only if for every N submodule of M, (Closed-N)^c, there is a decomposition M=M_1⊕M_2 such that N is a subset of M_1 and M_2 is a complement of N in M.*

Proof. ⟹ Clear.

⟸ Let N be a subset of (Closed-N)^c, then by our assumption, there exists decomposition M=M_1⊕M_2 such that N is a subset of M_1 and M_2 is a complement of N in M. So N⊕M_2 is a subset of (Closed-N)^c of M. Thus N is a subset of (Closed-N)^c of M_1 and hence Z($\frac{M_1}{N}$)=$\frac{M_1}{N}$; $\frac{M_1}{N}$ is singular). But N is a subset of M_1 and N is a subset of (Closed-N)^c of M, therefore N is a subset of (Closed-N)^c of M_1, (see Theorem 2.4). Thus N=M_1. □

Corollary 3.3. *Every (Closed-L)^c of closed-CS-module M is closed-CS-module.*

Proof. Let M be a closed-CS-module and let N be a subset of M . We must prove that N is a closed-CS-module. Let K subset of (Closed-N)^c, then by Theorem 2.4, L is a subset of (Closed-N)^c of M. But M is a closed-CS-module, therefore L is a direct summand of M and hence K is a direct summand of A. □

Lemma 3.4. *An R-module M is closed-CS-module if and only if every (Closed-N)^c of M is essential in a direct summand.*

Proof. ⟹ Clear.

⟸ let N subset of (Closed-N)^c, we need to show that N is a direct summand of M. Since N subset of (Closed-N)^c of M, then by our assumption N is a subset of (Closed-N)^c of M, where D is a direct summand of M. Thus Z($\frac{D}{N}$)=$\frac{D}{N}$; ($\frac{D}{N}$ is singular). But $\frac{D}{N}$ subset of $\frac{M}{N}$, therefore $\frac{D}{N}$ is non-singular. Thus N=D and hence M is closed-CS-module. □

Theorem 3.5. *An R-module M is closed-CS-module if and only if for every (Closed-N)^c of M; there exists a decomposition M=M_1⊕M_2 such that N is a subset of M_1 and N⊕M_2 is a subset of (Closed-N)^c of M.*

Proof. $\Longrightarrow$ Clear .

$\Longleftarrow$ Let N be a subset of (Closed-N)^c of M, we need to show that N is a direct summand of M. Since N is a subset of (Closed-N)^c of M, then by assumption there exists a decomposition M=$M_1 \oplus M_2$ such that N⊆M_1 and (N⊆M_2) is a subset of (Closed-N)^c of M. So $\frac{M}{(N \oplus M_2)}$ is a singular. But N⊕M_1 and A are subset of (Closed-N)^c of M, therefore by Theorem 2.4, N is a subset of (Closed-N)^c of M_1. Since M_2 is a subset of (Closed-N)^c of M_2, then by Lemma 2.3, (N⊕M_2) is a subset of (Closed-N)^c of $M_1 \oplus M_2$=M. So $\frac{M}{(N \oplus M_2)}$ is non-singular. Thus M=N⊕M_2. □

Proposition 3.6. *An R-module M is a closed-CS-module if and only if for every direct summand A of the injective hull E(M) of M such that (A∩M)^c is a subset of (closed-CS-module), then (A∩M) is a direct summand of M.*

Proof. $\Longrightarrow$ Clear .

$\Longleftarrow$ Let N be a subset of (Closed-N)^c of M and let K be a relative complement of N, then (N⊕K) is a subset of (Closed-N)^c of M. Since M is a subset of (Closed-N)^c of E(M), then (N⊕K) is a subset of (Closed-N)^c of E(M). Thus E(N)⊕E(K)=E(N⊕K)=E(M). Since E(N) is a summand of E(M), then by our assumption E(N)∩M is a summand of M. Now N is a subset of (Closed-N)^c of E(N) and M is a subset of (Closed-N)^c of M, thus N=(N∩M) is a subset of (Closed-N)^c of E(M)∩M. Hence by Lemma 3.5, M is closed-CS-module. □

Theorem 3.7. *Let R be a ring, then R is a closed-CS-module if and only if every cyclic non-singular R-module is projective.*

Proof. Let R be a closed-CS-ring and M=Ra, a∈M be a nonsingular R-module. Let the following be a short exact sequence.

$$0 \longrightarrow \text{ Annihilator}\,(a) \longrightarrow R \longrightarrow Ra \longrightarrow 0,$$

where i is the inclusion homomorphisim and f is a map defined by f(r)= ra, r∈ R. So f is an epimorphisim and ker(f) equal Annihilator of (a). Hence from the first isomorphisim theorem, Annihilator of (a)R≅Ra. But Ra is non-singular, therefore Annihilator of (a) subset of (Closed-N)$_c$ of R. Since R is closed-CS-ring, then Annihilator of (a) is a direct summand of R, so the sequence is split. Thus R is equivalent to Annihilator of (a)⊕Ra. Since R is projective, then Ra is projective. Conversely, let A be a (Closed-N)^c of I, I an ideal in R, then $\frac{R}{A}$ is non-singular. Since R is cyclic, then $\frac{R}{A}$ is cyclic. By our assumption $\frac{R}{A}$ is a projective. Now consider the following short exact sequence:

$$0 \longrightarrow A \longrightarrow R \longrightarrow AR \longrightarrow 0,$$

where i is the inclusion homomorphisim and π is the natural epimorphisim from R into Ra. Since $\frac{R}{A}$ is projective, then the sequence is split. Thus A is a summand of R. Also a direct sum of closed-CS-module need not to be closed-CS-modules (see [4]). □

Proposition 3.8. *Let M and N be closed-CS-modules such that Annihilator of M+Annihilator of N equal R. Then M⊕N is closed-CS-module.*

Proof. Let A be a $(\text{Closed-N})_c$ submodule of M⊕N.

Since Annihilator of M+Annihilator of N=R, then by the same way of the prove [9, Proposition 4.2, CH.1], A=C⊕D, where C is a submodule of M and D is a submodule of N. Since A=(C⊕D) is a subset of $(\text{closed-N})^c$ of M⊕N, then C and D are $(\text{Closed-N})^c$ of M and N respectively by Lemma 2.3. But M and N are closed-CS-modules, therefore C is a summand of M and D is a summand of N. So A=C⊕D is a summand of M⊕N. Thus M⊕N is a closed-CS-module. Recall that a submodule N of R-module M is called a fully invariant submodule of M, if for every endomorphism f:M⟶M, f(N) subset of N, (N is fully invariant) (see [8]). □

Corollary 3.9. *Let $M=\oplus M_i$ be an R-module, such that every $(\text{Closed-N})^c$ of M is fully invariant, then M is closed-CS-module if and only if M_i is closed-CS-module; i∈I.*

Proof. ⟹ Clear.

⟸ let S be a $(\text{Closed-N})^c$ of M. For each i∈I, let π_i:M ⟶ M_i be the projection map. Let x∈S, then $x=\sum m_i$, $m_i \in M_i$ and $m_i=0$ for all but finite many element of i∈I, $\pi_i(x)=m_i$. Since we have $(\text{Closed-S})^c$, then by our assumption, S is fully invariant and hence $\pi_i(x)=m_i$ $S\cap M_i$. So $x\in\bigotimes(S\cap M_i)$. Thus S subset of $\oplus(S\cap M_i)$. But $\oplus(S\cap M_i)$ subset of S, therefore S= $\oplus(S\cap M_i)$. Since S is a subset of $(\text{Closed-M})^c$, then by Theorem 2.4, $(S\cap M_i)$ is a subset of $(\text{Closed-N})^c$ of M_i ∀ i∈I. But M_i closed-CS-modules for all i∈I, therefore $(S\cap M_i)$ is a direct summand of M_i. Thus S is a direct summand on M. □

An R-module M is called a distributive module if A∩(B+C)=(A∩B)+(A∩C), for all submodules A, B and C of M, (see [9]).

Corollary 3.10. *Let $M=M_1\oplus M_2$ be distributive R-module. Then M is closed-CS-module if and only if M_1 and M_2 are closed-CS-module.*

Proof. ⟹ Clear.

⟸ Let K be a subset of $(\text{closed-N})^c$ in M. Since $M=M_1\oplus M_2$, then $K= K\cap(M_1\oplus M_2)$. But M is a distributive, therefore $K=(K\cap M_1)\oplus(K\cap M_2)$. By Lemma 2.3, $(K\cap M_1)$ is a subset of $(\text{Closed-N})^c$ of M_1 and $(K\cap M_2)$ is a subset of $(\text{Closed-N})^c$. Since M_1 and M_2 are closed-CS-modules, then $(K\cap M_1)$ is a direct summand of M_1 and $(K\cap M_2)$ is a direct summand of M_2. Clearly that $K=(K\cap M_1)\oplus(K\cap M_2)$ is a direct summand of M. □

Corollary 3.11. *Let M be an R-module and let N be a subset of (closed-CS-M. Then [N:M] is a subset of (closed-CS-R).*

Proof. Let N be a subset of closed-CS-module. Assume that [N:M] is not closed-CS-module in R. So there exists r∈R such that [N:M]≠ r+[N:M]∈ Z($\frac{N}{[N:M]}$). Thus rM not subset of N and hence there exists m_0∈M such that rm_0 not in N. One can easily show that Annihilator of (r+[N:M]) is a subset of Annihilator of (rm_0+N).Since Annihilator of (r+[N:M]) is a subset of eR, then Annihilator of (rm_0+N) is a subset of eR. But $\frac{M}{N}$ is non-singular, therefore rm_o+N=N which is contradiction. □

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