Some remarks on generalizations of prime submodules

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. Let $S(M)$ be the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule N of M is called $(n-1,n)$-ϕ-prime, if $r_1 \dots r_{n-1}x \in N \setminus \phi(N)$ where $r_1, \dots, r_{n-1} \in R$ and $x \in M$, then there exists $i \in \{1, ..., n-1\}$ such that $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}x \in N$ or $r_1 \dots r_{n-1} \in (N : M)$ $(n \geq 2)$. In this work, $(n-1,n)$-ϕ-prime submodules are studied and some results are obtained. Also, the characterization of $(n-1,n)$-ϕ-prime submodules of a free multiplication module is given.
Keywords: ϕ-prime submodule, ϕ-prime ideal, $(n-1,n)$-ψ-prime ideal, multiplication module, $(n-1,n)$-almost prime, $(n-1,n)$- ϕ-$\mathbb{CP}$ submodule, $(n-1,n)$-ϕ-$\mathbb{FCP}$ module.

1. Introduction

Throughout the paper, all rings are commutative with identity and all modules are unitary. Let M be an R-module and N be a submodule of M. The ideal $\{r \in R | rM \subseteq N\}$ will be denoted by $(N : M)$. Let $S(M)$ be the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule N of M is said to be a $(n-1,n)$-ϕ-prime, if $r_1...r_{n-1}x \in N \setminus \phi(N)$, $r_1, ..., r_{n-1} \in R$ and $x \in M$ $(n \geqslant 2)$, then $r_1...r_{i-1}r_{i+1}...r_{n-1}x \in N$ for some $i \in \{1, ..., n-1\}$ or $r_1...r_{n-1} \in (N : M)$. Without loss of generality, throughout of the paper we will assume $\phi(N) \subseteq N$. If $\phi(N) = \emptyset$ (resp. $\phi(N) = 0$, $\phi(N) = (N : M)N$, $\phi(N) = (N : M)^{m-1}N$ and $\phi(N) = \cap_{i=1}^{\infty}(N : M)^i N$), then the submodule N is called a $(n-1,n)$- prime (resp. $(n-1,n)$-weakly prime, $(n-1,n)$-almost prime, $(n-1,n)$-m-almost prime and $(n-1,n)$-ω -prime). Firstly, Anderson and Bataineh in [4] introduced various generalizations of prime ideals. Let $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R. We call a proper ideal I of R a ψ-prime ideal if $a, b \in R$ with $ab \in I \setminus \psi(I)$, then $a \in I$ or $b \in I$. If $\psi(I) = \emptyset$ (resp. $\psi(I) = 0$, $\psi(I) = I^2$, $\psi(I) = I^m$ and

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$\psi(I) = \cap_{m=1}^{\infty} I^m$), then ideal I is called a prime ideal (resp. weakly prime ideal, almost prime ideal, m-almost prime ideal and ω-prime ideal). Zamani in [11] used this concept for ϕ-prime submodule, in fact a proper submodule N of M is a prime submodule relative to ϕ or ϕ-prime submodule if $rx \in N \setminus \phi(N)$ where $r \in R$ and $x \in M$, then $x \in N$ or $r \in (N : M)$. If $\phi(N) = \emptyset$ (resp. $\phi(N) = 0$, $\phi(N) = (N : M)N$, $\phi(N) = (N : M)^{m-1}N$ and $\phi(N) = \cap_{i=1}^{\infty}(N : M)^m N)$, then a submodule N is a prime submodule (resp. weakly prime submodule, almost prime submodule, m-almost prime submodule and ω-prime submodule). Some properties of ϕ-prime submodules have been studied in [11]. Ebrahimpour and Nekooei defined $(n-1,n)$-ϕ-prime submodule and $(n-1,n)$-ψ-prime ideal (see [7]). A proper ideal I of R is $(n-1,n)$-ψ-prime if $r_1 \dots r_n \in I \setminus \psi(I)$, then $r_1 \dots r_{i-1} r_{i+1} \dots r_n \in I$ for some $i \in \{1, ..., n\}$. A number of results concerning $(n-1,n)$-ϕ- prime submodules have been established in [7]. Also some basic properties of prime submodules have been studied in [1,3, 5, 6, 9].
In this work, we continue the above studies in a special case, by alternation of n and ϕ. Again some other results lead us to conclude some corollaries and propositions and theorems to concerning the properties of $(n-1,n)$-ϕ-prime submodules. Also, for a free multiplication module M, the results are given in Section 4.

2. Some general results

The following propositions give some properties when we use the defintion $(n-1,n)$-ϕ-prime submodule.

Proposition 2.1. *Let R be a ring and M be an R-module. If N is a proper ϕ-prime submodule of M ((1, 2)-ϕ-prime), then N is $(n-1,n)$-ϕ-prime submodule ($n \geqslant 2$).*

Proof. Let $r_1, \dots, r_{n-1} \in R$ and $m \in M$ with $r_1 \dots r_{n-1} m \in N \setminus \phi(N)$. Assume that $r_1 \dots r_{n-1} \notin (N : M)$. Since N is a ϕ-prime submodule of M, hence $m \in N$, so $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} m \in N$ for some $i \in \{1, ..., n-1\}$.

Proposition 2.2. *Let M be an R-module and $\phi_1, \phi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions where $S(M)$ is the set of all submodules of M with $\phi_1 \leqslant \phi_2$ (i.e., for every submodule N of M, $\phi_1(N) \subseteq \phi_2(N)$). If N is $(n-1,n)$-ϕ_1-prime submodule, then N is $(n-1,n)$-ϕ_2-prime.*

Proof. It is clear.

Proposition 2.3. *Suppose that N is a $(n-1,n)$-ϕ-prime submodule of M, then N is a $(n, n+1)$-ϕ-prime submodule of M.*

Proof. Let $r_1 \dots r_n m \in N \setminus \phi(N)$ where $r_1, ..., r_n \in R$ and $m \in M$. Then $r_1 \dots r_{n-1}(r_n m) \in N \setminus \phi(N)$, so $r_1 \dots r_{n-1} \in (N : M)$ or $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1}(r_n m)$ $\in N$ for some $i \in \{1, ..., n-1\}$.

Thus $r_1 \dots r_{n-1}r_n \in (N : M)$ or $r_1 \dots r_{i-1}r_{i+1} \dots r_n m \in N$ for some $i \in \{1, ..., n\}$.

Example 2.4. We know that if N is a prime submodule of M, then N is a $(n-1, n)$-prime submodule of M $(n \geqslant 2)$ by Proposition 2.1. But the converse of Proposition 2.1, is not true in general. For example, let $M = \mathbb{Z} \oplus \mathbb{Z}$ be a $\mathbb{Z}$-module and $N = \langle(3, 0)\rangle$. Since $(N : M) = 0$ and $3(2, 0) \in N$ but $3 \notin (N : M)$ and $(2, 0) \notin N$, therefore N is not a prime submodule. We show that N is a $(2, 3)$- prime submodule. Suppose that $r_1, r_2 \in \mathbb{Z}$, $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ with $r_1r_2(m, n) \in N$. We have $(r_1r_2m, r_1r_2n) \in \langle(3, 0)\rangle$. If $r_1 = 0$ or $r_2 = 0$, then $0 = r_1r_2 \in (N : \mathbb{Z} \oplus \mathbb{Z})$, so N is a $(2, 3)$-prime submodule. Now, let $r_1 \neq 0$ and $r_2 \neq 0$, hence $0 \neq r_1r_2 \notin (N : \mathbb{Z} \oplus \mathbb{Z})$. Since $(r_1r_2m, r_1r_2n) \in \langle(3, 0)\rangle$, therefore $n = 0$ and $3 \mid r_1r_2m$. If $3 \mid m$, then $r_1(m, 0) \in N$ and $r_2(m, 0) \in N$. If $3 \nmid m$, then $3 \mid r_1r_2$. Hence $3 \mid r_1$ or $3 \mid r_2$. Thus $r_1(m, 0) \in N$ or $r_2(m, 0) \in N$, as required.

Proposition 2.5 *Let M be an R-module and suppose that for every $x \in M$, $Ann(x) = 0$. If N is a $(n-1, n)$-weakly prime submodule of M, then N is a $(n-1, n)$-prime submodule .*

Proof. Let $r_1 \dots r_{n-1}x \in N$ where $r_1, ..., r_{n-1} \in R$, $x \in M$ $(n \geqslant 2)$, and suppose that $r_1 \dots r_{n-1} \notin (N : M)$. Since N is a $(n-1, n)$-weakly prime submodule, hence $0 \neq r_1...r_{n-1}x \in N$, implies that $r_1 \dots r_{i-1}r_{i+1} \dots r_{n-1}x \in N$ for some $i \in \{1, ...n-1\}$. Hence N is a $(n-1, n)$-prime submodule. But if $x \neq 0$ and $r_1 \dots r_{n-1}x = 0$, then $r_1 \dots r_{n-1} = 0$, so $0 = r_1 \dots r_{n-1} \in (N : M)$, this is a contradiction.

Proposition 2.6 *Let $\varphi : R \to S$ be a ring homomorphism and M be an S-module. Suppose that N is a $(n-1, n)$-ϕ-prime submodule of S-module M, then N is a $(n-1, n)$-ϕ-prime submodule of R-module M.*

Proof. Let $r_1 \dots r_{n-1}x \in N \setminus \phi(N)$ where $r_1, \dots, r_{n-1} \in R$, $x \in M$ $(n \geqslant 2)$. We know that $r_1 \dots r_{n-1}x = \varphi(r_1 \dots r_{n-1})x = \varphi(r_1) \dots \varphi(r_{n-1})x \in N \setminus \phi(N)$ where $\varphi(r_i) \in S$, for all $i \in \{1, ..., n-1\}$ and $x \in M$ $(n \geqslant 2)$. It is clear that N is a $(n-1, n)$-ϕ-prime submodule of R-module M.

3. Main results

We state the following theorems and propositions which in the proofs of them, we use the definition $(n-1, n)$-ϕ-prime submodule.

The motivation of [7, Theorem 2.7], we introduce function $\phi_{R/I}$. Let M be an R-module and I be an ideal of R. Since $I \subseteq Ann_R(M/IM)$, so M/IM is an R/I-module. We know that $(r + I)(m + IM) = rm + IM$ where $r + I \in R/I$, $m + IM \in M/IM$. Now, for a submodule N of M with $IM \subseteq N$, let $\phi_{R/I} : S(M/IM) \to S(M/IM) \cup \{\emptyset\}$ be defined by $\phi_{R/I}(N/IM) = (\phi(N) + IM)/IM$ for $IM \subseteq N$ and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function.

Theorem 3.1. *Let M be an R-module , I be an ideal of R and N be a submodule of M with $IM \subseteq N$. If N is a $(n-1,n)$-ϕ-prime submodule M, then N/IM is a $(n-1,n)$-$\phi_{R/I}$-prime submodule of M/IM $(n \geq 2)$.*

Proof. See [7, Theorem 2.7].

Theorem 3.2. *Let M be an R-module, L and N be two submodules of M with $L \subseteq N \subset M$. Let $\phi_L : S(M/L) \to S(M/L) \cup \{\emptyset\}$ be defined by $\phi_L(N/L) = (\phi(N) + L)/L$ with $L \subseteq \phi(N)$. If N/L is a $(n-1,n)$-ϕ_L-prime submodule of M/L, then N is a $(n-1,n)$-ϕ-prime submodule of M $(n \geq 2)$.*

Proof. Let $r_1, \ldots, r_{n-1} \in R$, $x \in M$ with $r_1 \ldots r_{n-1}x \in N \setminus \phi(N)$ and $r_1 \ldots r_{n-1} \notin (N : M)$. So $r_1 \ldots r_{n-1}x \in N$ and $r_1 \ldots r_{n-1}x \notin \phi(N)$. Hence $r_1 \ldots r_{n-1}x + L \in N/L$ and $r_1 \ldots r_{n-1}x + L \notin (\phi(N) + L)/L$. Since N/L is a $(n-1,n)$-ϕ_L-prime and $r_1 \ldots r_{n-1} + L \notin (N/L :_R M/L)$, thus there exists $i \in \{1, \ldots, n-1\}$ such that $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}x + L \in N/L$. Therefore $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}x \in N$ and so N is a $(n-1,n)$-ϕ-prime.

Corollary 3.4. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and N be a proper submodule of M. Then N is a $(n-1,n)$-ϕ-prime submodule of M if and only if $\frac{N}{\phi(N)}$ is a $(n-1,n)$-weakly prime submodule of $\frac{M}{\phi(N)}$.*

Proof. It is straightforward.

Theorem 3.5. *Let $f : M \to M'$ be an R-module epimorphism, $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \to S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*

(1) If N is a $(n-1,n)$-ϕ-prime submodule of M with $ker f \subseteq N$ and $f(\phi(N)) \subseteq \phi'(f(N))$, then $f(N)$ is a $(n-1,n)$-ϕ'-prime submodule of M'.

(2) If L is a $(n-1,n)$-ϕ'-prime submodule of M' and $f^{-1}(\phi'(L)) \subseteq \phi(f^{-1}(L))$, then $f^{-1}(L)$ is a $(n-1,n)$-ϕ-prime submodule of M.

Proof. (1) Let $r_1, \ldots, r_{n-1} \in R$ and $m' \in M'$ with $r_1 \ldots r_{n-1}m' \in f(N) \setminus \phi'(f(N))$. There exists $m \in M$ such that $f(m) = m'$. We have $r_1 \ldots r_{n-1}f(m) \in f(N)$ and $r_1 \ldots r_{n-1}f(m) \notin \phi'(f(N))$. It follows that $r_1 \ldots r_{n-1}m \in N$ and $r_1 \ldots r_{n-1}m \notin \phi(N)$, because $r_1 \ldots r_{n-1}f(m) \notin f(\phi(N))$. Thus $r_1 \ldots r_{n-1}m \in N \setminus \phi(N)$, so $r_1 \ldots r_{n-1} \in (N : M)$ or $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in N$ for some $i \in \{1, ..., n-1\}$. Therefore $r_1 \ldots r_{n-1} \in (f(N) : M')$ or $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}f(m) \in f(N)$ for some $i \in \{1, ..., n-1\}$.

(2) Let $r_1 \ldots r_{n-1}m \in f^{-1}(L) \setminus \phi(f^{-1}(L))$ where $r_1, \ldots, r_{n-1} \in R$ and $m \in M$. We have $r_1 \ldots r_{n-1}m \in f^{-1}(L)$ and $r_1 \ldots r_{n-1}m \notin \phi(f^{-1}(L))$. It follows that $r_1 ... r_{n-1}f(m) \in L \setminus \phi'(L)$. So $r_1 \ldots r_{n-1} \in (L : M')$ or $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}f(m) \in L$ for some $i \in \{1, ..., n-1\}$, because L is a $(n-1,n)$-ϕ'-prime submodule of M'. Thus $r_1 \ldots r_{n-1} \in (f^{-1}(L) : M)$ or $r_1 \ldots r_{i-1}r_{i+1} \ldots r_{n-1}m \in f^{-1}(L)$ for some $i \in \{1, ..., n-1\}$, as required.

Theorem 3.6. *Let M be a free R-module with a basis $\{m_\alpha\}_{\alpha\in\Lambda}$, $\phi : S(M) \to S(M)\cup\{\emptyset\}$ and $\psi : \mathcal{I}(R) \to \mathcal{I}(R)\cup\{\emptyset\}$ be two functions where $\mathcal{I}(R)$ is the set of all ideals of R. If P is a $(n-1,n)$-ψ-prime ideal of R with $\psi(P)M \subseteq \phi(PM)$, then PM is a $(n-1,n)$-ϕ-prime submodule of M and $(PM : M) = P$.*

Proof. Since P is a proper ideal of R, so PM is a proper submodule of M. Let $r_1,\dots,r_{n-1} \in R$, $x \in M$ with $r_1\dots r_{n-1}x \in PM \setminus \phi(PM)$. Since M is a free R-module with a basis $\{m_\alpha\}_{\alpha\in\Lambda}$, therefore $PM = \{\sum_{f.s} s_i m_i | s_i \in P, m_i \in \{m_\alpha\}_{\alpha\in\Lambda}\}$. We have $r_1\dots r_{n-1}x \in PM$ and $r_1\dots r_{n-1}x \notin \phi(PM)$ with $x = \sum_{f.s} r'{}_\alpha m_\alpha$ ($r'_\alpha \in R, m_\alpha \in \{m_\alpha\}_{\alpha\in\Lambda}$). Thus $\sum_{f.s}(r_1\dots r_{n-1}r'_\alpha)m_\alpha = \sum_{f.s} s_\alpha m_\alpha$, so $r_1\dots r_{n-1}r'_\alpha = s_\alpha \in P$ for every $\alpha \in \Lambda$. But $r_1\dots r_{n-1}r'_\alpha \notin \psi(P)$, otherwise $r_1\dots r_{n-1}r'_\alpha \in \psi(P)$. Thus for every $m_\alpha \in \{m_\alpha\}_{\alpha\in\Lambda}$, we have $r_1\dots r_{n-1}r'_\alpha m_\alpha \in \psi(P)m_\alpha$. So for every $\alpha \in \Lambda$, $r_1\dots r_{n-1}\sum_{f.s} r'_\alpha m_\alpha \in \psi(P)M$. Since $\psi(P)M \subseteq \phi(PM)$, therefore $r_1\dots r_{n-1}\sum_{f.s} r'_\alpha m_\alpha \in \phi(PM)$, so $r_1\dots r_{n-1}x \in \phi(PM)$. This is a contratiction. We showed that for every $\alpha \in \Lambda$, $r_1\dots r_{n-1}r'_\alpha \in P\setminus\psi(P)$. Since P is a $(n-1,n)$-ψ-prime ideal, so there exists $i \neq \alpha \in \{1,\dots n-1\}$, $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}r'_\alpha \in P$. For every $m_\alpha \in \{m_\alpha\}_{\alpha\in\Lambda}$, we have $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}r'_\alpha m_\alpha \in Pm_\alpha$. Therefore $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x \in PM$. But if $i = \alpha$, we have $r_1\dots r_{n-1} \in P$, so $r_1\dots r_{n-1} \in (PM : M)$. Thus PM is $(n-1,n)$-ϕ-prime submodule of M. It is clear that $(PM : M) = P$.

Now, the following corollary is given as a result of the above theorem. We recall that an R-module M is a multiplication module if for every submodule N of M, $N = IM$ for some ideal I of R (see [2], [8], [10]).

Corollary 3.7. *Let M be a free multiplication R-module and N be a proper submodule of M. Let $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}$ is the set of all ideals of R and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function with $\psi(N : M)M \subseteq \phi((N : M)M)$. If $(N : M)$ is $(n-1,n)$-ψ-prime ideal of R, then N is a $(n-1,n)$-ϕ-prime submodule of M.*

Proof. Since $N = (N : M)M$, by Theorem 3.6 the proof is clear.

Definition 3.8. A proper submodule P of an R-module M is called *compactly packed* (or abbreviated by $\mathbb{CP}$ submodule) if for each family $\{P_\alpha\}_{\alpha\in\Lambda}$ of prime submodules of M with $P \subseteq \cup_{\alpha\in\Lambda}P_\alpha$, then $P \subseteq P_\beta$ for some $\beta \in \Lambda$. Whenever $P \subseteq \cup_{\alpha\in\Lambda}P_\alpha$ implies that there exist $\alpha_1\dots\alpha_n \in \Lambda$ such that $P \subseteq_{i=1}^n P_{\alpha_i}$, P is said *finitely compactly packed* (or abbreviated by $\mathbb{FCP}$) submodule. A module M is said to be $\mathbb{CP}$ ($\mathbb{FCP}$), if every proper submodule of M is a $\mathbb{CP}$ ($\mathbb{FCP}$) submodule. We will call a proper submodule N of M as ϕ-$\mathbb{CP}$ if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of ϕ-prime submodules of M with $N \subseteq \cup_{\alpha\in\Lambda}N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \Lambda$. Whenever $N \subseteq \cup_{\alpha\in\Lambda}N_\alpha$ implies that there exist $\alpha_1\dots\alpha_n \in \Lambda$ such that $N \subseteq_{i=1}^n N_{\alpha_i}$, N is said ϕ-$\mathbb{FCP}$ submodule. A module M is said to be ϕ-$\mathbb{CP}$ (ϕ-$\mathbb{FCP}$) if every proper submodule is a ϕ-$\mathbb{CP}$ (ϕ-$\mathbb{FCP}$).

Also, we call a proper submodule N of M as $(n-1,n)$-ϕ-$\mathbb{CP}$ if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of $(n-1,n)$-ϕ-prime submodules of M with $N \subseteq \cup_{\alpha\in\Lambda}N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \Lambda$. Whenever $N \subseteq \cup_{\alpha\in\Lambda} N_\alpha$ implies that there exist $\alpha_1 \dots \alpha_n \in \Lambda$ such that $N \subseteq_{i=1}^n N_{\alpha_i}$, then N is called $(n-1,n)$-ϕ-$\mathbb{FCP}$ submodule. A module M is said to be $(n-1,n)$-ϕ-$\mathbb{CP}$ ($(n-1,n)$-ϕ-$\mathbb{FCP}$) if every proper submodule is a $(n-1,n)$-ϕ-$\mathbb{CP}$ ($(n-1,n)$-ϕ-$\mathbb{FCP}$).

Proposition 3.9. *Let M be an R-module, ϕ_1 and $\phi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions where $S(M)$ is the set of all submodules of M with $\phi_1 \leq \phi_2$ (i.e., for every submodule N, $\phi_1(N) \subseteq \phi_2(N)$). If M is a $(n-1,n)$-ϕ_2-$\mathbb{CP}$ ($(n-1,n)$-ϕ_2-$\mathbb{FCP}$) module, then M is a $(n-1,n)$-ϕ_1-$\mathbb{CP}$ ($(n-1,n)$-ϕ_1-$\mathbb{FCP}$) module.*

Proof. Let N be a proper submodule of M with $N \subseteq \cup_{\alpha\in\Lambda} N_\alpha$ where N_α is a $(n-1,n)$-ϕ_1-prime submodule of M. By proposition 2.2., N_α is a $(n-1,n)$-ϕ_2-prime submodule of M, so $N \subseteq N_\beta$ for some $\beta \in \Lambda$, because M is a $(n-1,n)$-ϕ_2-$\mathbb{CP}$ module. Since N_β is a $(n-1,n)$-ϕ_1-prime submodule of M, so N is a $(n-1,n)$-ϕ_1-$\mathbb{CP}$ submodule of M. Thus M is a $(n-1,n)$-ϕ_1-$\mathbb{CP}$ module. Similarly, we can prove that M is a $(n-1,n)$-ϕ_1-$\mathbb{FCP}$ module.

Proposition 3.10. *Every $(n,n+1)$-ϕ-$\mathbb{CP}$ ($\mathbb{FCP}$) module is a $(n-1,n)$-ϕ-$\mathbb{CP}$ ($\mathbb{FCP}$) module.*

Proof. Apply Proposition 2.3.

Theorem 3.11. *Let $f : M \to M'$ be an R-module epimorphism, $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \to S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*

(i) If M is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ ($\mathbb{CP}$) module such that for every $(n-1,n)$-ϕ'-prime submodule L of M' with $f^{-1}(\phi'(L)) \subseteq \phi(f^{-1}(L))$, then M' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ ($\mathbb{CP}$) module.

(ii) If M' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ ($\mathbb{CP}$) module such that every $(n-1,n)$-ϕ-prime submodule N of M with $ker f \subseteq N$ and $f(\phi(N)) \subseteq \phi'(f(N))$, then M is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ ($\mathbb{CP}$) module.

Proof. (i) Let N' be a proper submodule of M' such that $N' \subseteq \cup_{\alpha\in\Lambda} L'_\alpha$, where L'_α is a $(n-1,n)$-ϕ'-prime submodule of M' for each $\alpha \in \Lambda$. We have $f^{-1}(N') \subseteq \cup_{\alpha\in\Lambda} f^{-1}(L'_\alpha)$. Since L'_α is a $(n-1,n)$-ϕ'-prime submodule of M' for each $\alpha \in \Lambda$ and $f^{-1}(\phi'(L'_\alpha) \subseteq \phi(f^{-1}(L'_\alpha))$, by Theorem 3.5. (2), $f^{-1}(L'_\alpha)$ is a $(n-1,n)$-ϕ-prime submodule of M for each $\alpha \in \Lambda$. But M is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ module, thus there exist $\alpha_1, \dots, \alpha_n \in \Lambda$ such that $f^{-1}(N') \subseteq \cup_{i=1}^n f^{-1}(L'_{\alpha_i})$, hence $f^{-1}(N') \subseteq f^{-1}(\cup_{i=1}^n L'_{\alpha_i})$. Since f is an epimorphism R-module, so $N' \subseteq \cup_{i=1}^n L'_{\alpha_i}$. Therefore we showed that N' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ submodule of M'. Thus M' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ module. Similarly, we can prove that N' is a $(n-1,n)$-ϕ'-$\mathbb{CP}$ submodule of M'. So M' is a $(n-1,n)$-ϕ'-$\mathbb{CP}$ module.

(ii) Assume that L is a proper submodule of M with $L \subseteq \cup_{\alpha\in\Lambda} N_\alpha$ where N_α is a $(n-1,n)$-ϕ-prime submodule of M for each $\alpha \in \Lambda$. We have $f(L) \subseteq f(\cup_{\alpha\in\Lambda} N_\alpha) = \cup_{\alpha\in\Lambda} f(N_\alpha)$. Since N_α is a $(n-1,n)$-ϕ-prime submodule of M, $f(\phi(N_\alpha)) \subseteq \phi'(f(N_\alpha))$ and $ker f \subseteq N_\alpha$ for each $\alpha \in \Lambda$, by Theorem 3.5. (1),

$f(N_\alpha)$ is a $(n-1,n)$-ϕ'-prime submodule of M'. Since M' is a $(n-1,n)$-ϕ'-$\mathbb{FCP}$ module, so there exist $\alpha_1,\dots,\alpha_n \in \Lambda$ such that $f(L) \subseteq \cup_{i=1}^{n} f(N_{\alpha_i})$. Now, we prove that $L \subseteq \cup_{i=1}^{n} N_{\alpha_i}$.

Let $x \in L$, so $f(x) \in f(\cup_{i=1}^{n} N_{\alpha_i})$, hence $f(x) = f(t)$ for some $t \in \cup_{i=1}^{n} N_{\alpha_i}$. So $x - t \in kerf \subseteq N_{\alpha_j}$ and $t \in N_{\alpha_j}$ for some $\alpha_j \in \{\alpha_1,\dots,\alpha_n\}$. Therefore $x \in N_{\alpha_j}$, so $x \in \cup_{i=1}^{n} N_{\alpha_i}$. It follows that L is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ submodule of M. Thus M is a $(n-1,n)$-ϕ-$\mathbb{FCP}$ module. Similarly, we can prove that L is a $(n-1,n)$-ϕ-$\mathbb{CP}$ submodule of M. So M is a $(n-1,n)$-ϕ-$\mathbb{CP}$ module.

4. The generalization of prime submodules of free multiplication modules

Let M be a free multiplicaton R-module. We study several relations between various generalizations of $(n-1,n)$-ϕ-prime submodules among $(n-1,n)$-almost prime submodules, $(n-1,n)$-prime submodules and $(n-1,n)$-m-almost prime submodules.

Proposition 4.1. *Let M be a free multiplication R-module and N be a proper submodule of M. If N is a $(n-1,n)$-almost prime submodule of M such that $(N:M)^2$ is a prime ideal of R, then N is a $(n-1,n)$-prime submodule of M.*

Proof. Let M be a free multiplication R-module with a basis $\{x_\alpha\}_{\alpha\in\Lambda}$ and N be a proper submodule of M such that N is a $(n-1,n)$-almost prime submodule with $r_1 \dots r_{n-1}x \in N$ where $r_1,\dots,r_{n-1} \in R$ and $x \in M$. If $r_1\dots r_{n-1}x \notin (N:M)N$, we have $r_1\dots r_{n-1}x \in N \setminus (N:M)N$, so $r_1\dots r_{n-1} \in (N:M)$ or $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x \in N$ for some $i \in \{1,...,n-1\}$. Thus N is a $(n-1,n)$-prime submodule of M. If $r_1\dots r_{n-1}x \in (N:M)N$, because $N = (N:M)M$, hence $r_1\dots r_{n-1}x \in (N:M)^2M$. Suppose that $r_1\dots r_{n-1} \notin (N:M)$, we prove that $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x \in N$ for some $i \in \{1,...,n-1\}$. On the other hand, we have $x = \sum_{f.s} {r_\alpha}' x_\alpha$ that $\{x_\alpha\}_{\alpha\in\Lambda}$ is a basis for M. Also, we get $r_1\dots r_{n-1}x = \sum_{f.s} {r_\alpha}'' x_\alpha$ with $r''_\alpha \in (N:M)^2$. Thus $\sum_{f.s}(r_1\dots r_{n-1}r'_\alpha)x_\alpha = \sum_{f.s} r''_\alpha x_\alpha$, so $r_1\dots r_{n-1}r'_\alpha = r''_\alpha$, for all $\alpha \in \Lambda$. Since M is a free module, so $r_1\dots r_{n-1}{r_\alpha}' = r''_\alpha$ for all $\alpha \in \Lambda$. Thus $r_1\dots r_{n-1}{r_\alpha}' \in (N:M)^2$, because $r_1\dots r_{n-1} \notin (N:M)$, hence $r_1\dots r_{n-1} \notin (N:M)^2$. But $(N:M)^2$ is a prime ideal of R, so ${r_\alpha}' \in (N:M)^2$ for every $\alpha \in \Lambda$. Therefore $x = \sum_{f.s} {r_\alpha}' x_\alpha \in (N:M)^2M$. Since $(N:M)^2M \subseteq (N:M)M = N$, so $x \in N$ and hence $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x \in N$ for some $i \in \{1,\dots,n-1\}$.

Corollary 4.2. *Let M be a free multiplication R-module. If N is a $(n-1,n)$-m-almost prime submodule of M such that $(N:M)^m$ is a prime ideal of R, then N is a $(n-1,n)$-prime submodule of M.*

Proof. The proof is similar to the proof of Proposition 4.1.

Corollary 4.3. *Let M be a free multiplication R-module and I be a proper ideal of R such that $(IM:M)^2$ is a prime ideal of R. If IM is a $(n-1,n)$-*

almost prime submodule of M, then IM is a $(n-1,n)$-prime submodule of M.

Proof. Apply Proposition 4.1.

Corollary 4.4. *Let M be a free multiplication R-module and I be a proper ideal of R such that $(IM:M)^m$ is a prime ideal of R. If IM is a $(n-1,n)$-m-almost prime submodule of M, then IM is a $(n-1,n)$-prime submodule of M.*

Proof. Apply Corollary 4.2.

Proposition 4.5. *Let M be a faithful finitely generated R-module with a basis $\{x_\alpha\}_{\alpha\in\Lambda}$ and I be a proper radical ideal of R such that I^2 is a prime ideal of R. If IM is a $(n-1,n)$-almost prime submodule of M, then IM is a $(n-1,n)$-prime submodule.*

Proof. Since M is a faithful finitely generated R-module and I is a radical ideal of R, so $(IM:M)=I$. Let $r_1\dots r_{n-1}x\in IM$ where $r_1,\dots,r_{n-1}\in R$ and $x\in M$. If $r_1\dots r_{n-1}x\notin(IM:M)IM$, then $r_1\dots r_{n-1}x\in IM\setminus(IM:M)IM$. Hence $r_1\dots r_{n-1}\in(IM:M)$ or $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x\in IM$ for some $i\in\{1,\dots,n-1\}$. Now, assume that $r_1\dots r_{n-1}x\in(IM:M)IM$ and $r_1\dots r_{n-1}\notin(IM:M)$. Because $(IM:M)=I$, we have $r_1\dots r_{n-1}x\in I^2M$. On the other hand we get $x=\sum_{f.s}r_\alpha{}'x_\alpha$, since $\{x_\alpha\}_{\alpha\in\Lambda}$ is a basis is for M. Therefore $\sum_{f.s}(r_1\dots r_{n-1}r_\alpha{}')x_\alpha\in I^2M$, hence $\sum_{f.s}(r_1\dots r_{n-1}r_\alpha{}')x_\alpha=\sum_{f.s}r''_\alpha x_\alpha$ where $r''_\alpha\in I^2$ for all $\alpha\in\Lambda$. Thus $r_1\dots r_{n-1}r'_\alpha=r''_\alpha$ for all $\alpha\in\Lambda$, hence $r_1\dots r_{n-1}r'_\alpha\in I^2$. Because $r_1\dots r_{n-1}\notin(IM:M)=I$, so $r_1\dots r_{n-1}\notin I^2$. Since I^2 is a prime ideal of R, hence $r'_\alpha\in I^2$ for all $\alpha\in\Lambda$. Thus we proved that $x=\sum_{f.s}r'_\alpha x_\alpha\in I^2M$. Therefore $x\in IM$ and $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}x\in IM$.

Proposition 4.6. *Let M be a free multiplication R-module and $f:M\to M'$ be an R-module epimorphism. Let N be a proper submodule of M with $ker f\subseteq N$ and $(N:M)^2$ be a prime ideal of R. If N is a $(n-1,n)$-almost prime submodule of M, then $f(N)$ is a $(n-1,n)$-prime submodule of M'.*

Proof. Let N be a proper submodule of M, then $f(N)$ is a proper submodule of M'. Suppose that $r_1\dots r_{n-1}m'\in f(N)$ where $r_1,\dots,r_{n-1}\in R$ and $m'\in M'$. Since $f(M)=M'$, so $f(m)=m'$, for some $m\in M$. So $f(r_1\dots r_{n-1}m)\in f(N)$. Since $ker(f)\subseteq N$, hence $r_1\dots r_{n-1}m\in N$. If $r_1\dots r_{n-1}m\notin(N:M)N$, then $r_1\dots r_{n-1}m\in N\setminus(N:M)N$. Since N is a $(n-1,n)$-almost prime submodule of M, so $r_1\dots r_{n-1}\in(N:M)$ or $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}m\in N$ for some $i\in\{1,...,n-1\}$. Hence $r_1\dots r_{n-1}\in(f(N):f(M))$ or $r_1\dots r_{i-1}r_{i+1}\dots r_{n-1}f(m)\in f(N)$ for some $i\in\{1,...,n-1\}$. Thus $f(N)$ is a $(n-1,n)$-prime submodule of M'. Now, if $r_1\dots r_{n-1}m\in(N:M)N$, so $r_1\dots r_{n-1}m\in(N:M)^2M$. Assume that $r_1\dots r_{n-1}\notin(N:M)$. Furthermore, since R-module M is free with a basis $\{x_i\}_{i\in\Lambda}$, so $m=\sum_{f.s}r'_ix_i$, hence $r_1\dots r_{n-1}\sum_{f.s}r'_ix_i\in(N:M)^2M$. Hence we have $\sum_{f.s}r_1\dots r_{n-1}r'_ix_i=\sum_{f.s}r''_ix_i$ where $r''_i\in(N:M)^2$. It follows that $r_1 \dots r_{n-1} r'_i = r''_i$. Since $(N : M)^2$ is a prime ideal of R, so $r'_i \in (N : M)^2$ for every $i \in \Lambda$. Thus $m = \sum_{f.s} r'_i x_i \in (N : M)^2 M$, therefore $m \in (N : M)M$, because of $(N : M)^2 M \subseteq (N : M)M$. Thus $m \in (N : M)M = N$, so $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} m \in N$ for some $i \in \{1, \dots, n-1\}$. So $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} f(m) \in f(N)$ for some $i \in \{1, \dots, n-1\}$. Thus $f(N)$ is a $(n-1, n)$-prime submodule of M'.

Proposition 4.7. *Let M be a free multiplication R-module and $f : M \to M'$ be an R-module epimorphism. Let N be a proper submodule of M with $ker f \subseteq N$ and $(N : M)^m$ be a prime ideal of R. If N is a $(n-1, n)$-m-almost prime submodule of M, then $f(N)$ is a $(n-1, n)$- prime submodule of M'.*

Proof. The proof is similar to the proof of Proposition 4.6.

Theorem 4.8. *Let M be a free multiplication R-module with a basis $\{x_\alpha\}_{\alpha \in \Lambda}$ and N be a proper submodule of M. If $(N : M)$ is a $(n-1, n)$-prime ideal of R, then N is a $(n-1, n)$-prime submodule of M.*

Proof. Let $r_1, \dots, r_{n-1} \in R$ and $x \in M$ with $r_1 \dots r_{n-1} x \in N$. Since $N = (N : M)M$, so $r_1 \dots r_{n-1} x \in (N : M)M$. Because $\{x_\alpha\}_{\alpha \in \Lambda}$ is a basis for M, so $x = \sum_{f.s} r'_\alpha x_\alpha$. Therefore $\sum_{f.s} r_1 \dots r_{n-1} r'_\alpha x_\alpha \in (N : M)M$, hence $\sum_{f.s} r_1 \dots r_{n-1} r'_\alpha x_\alpha = \sum_{f.s} r''_\alpha x_\alpha$ where $r''_\alpha \in (N : M)$. It is clear that $r_1 \dots r_{n-1} r'_\alpha \in (N : M)$ for all $\alpha \in \Lambda$. Since $(N : M)$ is a $(n-1, n)$-prime ideal of R, we have two cases. The first case, $i \neq \alpha$, so $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} r'_\alpha \in (N : M)$ for some $i \in \{1, \dots n-1\}$, for all $\alpha \in \Lambda$. Therefore $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} \sum_{f.s} {r_\alpha}' x_\alpha \in (N : M)M$. So $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} x \in (N : M)M = N$. The second case, $i = \alpha$ by elimination α, we have $r_1 \dots r_{n-1} \in (N : M)$. Finally, we showed that N is a $(n-1, n)$-prime submodule of M.

Theorem 4.9. *Let M be a free multiplication R-module and N be a proper submodule of M. If $(N : M)$ is a $(n-1, n)$-almost prime ideal of R, then N is a $(n-1, n)$-almost prime submodule of M.*

Proof. Let $r_1, \dots, r_{n-1} \in R$ and $x \in M$ with $r_1 \dots r_{n-1} x \in N \setminus (N : M)N$. Since $N = (N : M)M$, so $r_1 \dots r_{n-1} x \in (N : M)M \setminus (N : M)^2 M$, hence $r_1 \dots r_{n-1} x \in (N : M)M$ and $r_1 \dots r_{n-1} x \notin (N : M)^2 M$. Assume that $\{x_\alpha\}_{\alpha \in \Lambda}$ be a basis for M, so $x = \sum_{f.s} {r_\alpha}' x_\alpha$. It is clear that $r_1 \dots r_{n-1} {r_\alpha}' \in (N : M)$ for all $\alpha \in \Lambda$. Also we get $r_1 \dots r_{n-1} {r_\alpha}' \notin (N : M)^2$ for all $\alpha \in \Lambda$, otherwise $r_1 \dots r_{n-1} {r_\alpha}' \in (N : M)^2$, so $r_1 \dots r_{n-1} \sum_{f.s} r'_\alpha x_\alpha \in (N : M)^2 M$, hence $r_1 \dots r_{n-1} x \in (N : M)^2 M$, this is a contradiction. Thus we have $r_1 \dots r_{n-1} {r_\alpha}' \in (N : M) \setminus (N : M)^2$. Since $(N : M)$ is a $(n-1, n)$-almost prime ideal of R, therefore $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} {r_\alpha}' \in (N : M)$ for some $i \in \{1, \dots, n-1, \alpha\}$. If $i \in \{1, \dots, n-1\}$ we get $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} r'_\alpha \in (N : M)$ for all $\alpha \in \Lambda$. Thus $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} \sum_{f.s} r'_\alpha x_\alpha \in (N : M)M$. So $r_1 \dots r_{i-1} r_{i+1} \dots r_{n-1} x \in (N : M)M = N$. But if $i = \alpha$, by elimination α, we have $r_1 \dots r_{n-1} \in (N : M)$ and hence we proved that N is a $(n-1, n)$-almost prime submodule of M.

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