On almost generalized pseudo-Ricci symmetric spacetime

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Abstract. The notion of an almost generalized pseudo-Ricci symmetric space time has been introduced and studied. The beauty of such spacetime is that it has the flavour of Ricci symmetric, Ricci recurrent , generalized Ricci recurrent, pseudo-Ricci symmetric, generalized pseudo-Ricci symmetric and almost pseudo-Ricci symmetric space. Having found, faulty example in [8] the present paper attempts to construct a nontrivial example of an almost pseudo Ricci symmetric spacetime.
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1. Introduction

In the example given in ([8], page 2884-2885) authors have calculated or assumed the value of the covariant derivatives corresponding to the vanishing component of the Ricci tensor R_{13} & R_{14} (namely, $R_{13},_3$ & $R_{14},_4$) to be zero. But, those value are found to be $R_{13},_3 = \frac{2q^2(1-q)}{(1+2q)^3} = -R_{14},_4$ which are non-zero as $q \neq 0, 1$. Consequently for their [8] choice of the 1-forms

$$\begin{aligned} A_i(x) &= -\frac{q}{1+2q} \quad \text{for i=1,} \\ &= 0 \quad \text{otherwise,} \\ B_i(x) &= \frac{1+q}{1+2q} \quad \text{for i=1,} \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

the relations

$$\begin{aligned} R_{13},_3 &= (A_3 + B_3)R_{13} + A_1 R_{33} + A_3 R_{13}, \\ R_{14},_4 &= (A_4 + B_4)R_{14} + A_1 R_{44} + A_4 R_{14}, \end{aligned}$$

do not stand. Hence, $(\mathbb{R}^4, g)$ under-considered metric ([8], equation 6.2, page 2884) can not be an almost pseudo-Ricci symmetric spacetime. Coming back to our present paper, we structured it as follows: Keeping in tune with Dubey[11], a new type of spacetime called an almost generalized pseudo-Ricci symmetric

∗. The author dedicates this work to the memory of Late Professor M. C. Chaki spacetime which is abbreviated by $A(GPRS)_n$-spacetime is introduced in section 2. Some interesting results of a conformally flat almost generalized pseudo-Ricci symmetric spacetime are obtained. A non-trivial example of an almost pseudo-Ricci symmetric spacetime is constructed in section 3. Finally, we ensured that there exists a spacetime $(\mathbb{R}^4, g)$ which is an almost generalized pseudo-Ricci symmetric for some choice of the 1-forms.

2. $A(GPRS)_n$-spacetime

In the sense of Chaki and Kawaguchi, a non-flat n-dimensional semi-Riemann manifold $(M^n, g)(n > 3)$ is said to be an almost pseudo-Ricci symmetric manifold, [7] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the equation

$$(\nabla_X S)(Y,U) = [A(X) + B(X)]S(Y,U) + A(Y)S(X,U) + A(U)S(Y,X) \tag{2.1}$$

where $A(X)$ and $B(X)$ are two non-zero 1-forms defined by $A(X) = g(X, \theta)$ and $B(X) = g(X, \varrho)$, ∇ being the operator of the covariant differentiation. The local expression of the above equation is

$$R_{ik,l} = (A_l + B_l)R_{ik} + A_i R_{kl} + A_k R_{il}, \tag{2.2}$$

where A_l and B_l are two non-zero co-vectors and comma followed by indices denotes the covariant differentiation with respect to the metric tensor g. An n-dimensional manifold of this kind is abbreviated by $A(PRS)_n$.

Generalizing the sense of Chaki and Kawaguchi, in the present paper, we attempt to introduce a new type of spacetime called almost generalized pseudo-Ricci symmetric spacetime which is abbreviated by $A(GPRS)_n$-spacetime and defined as follows:

A non-flat n-dimensional semi-Riemann manifold $(M^n, g)(n > 3)$, is termed as almost generalized pseudo-Ricci symmetric manifold, if its Ricci tensor S of type $(0, 2)$ is not identically zero and admits the identity([2], [3])

$$\begin{aligned}(\nabla_X S)(Y,U) &= [A(X) + B(X)]S(Y,U) + A(Y)\, S(X,U) + A(U)\, S(X,Y) \\ &\quad + [C(X) + D(X)]g(Y,U) + C(Y)\, g(X,U) + C(U)\, g(X,Y)\end{aligned} \tag{2.3}$$

where $A(X)$, $B(X)$, $C(X)$ and $D(X)$ are non-zero 1-forms defined by $A(X) = g(X, \theta)$, $B(X) = g(X, \varrho)$, $C(X) = g(X, \pi)$ and $D(X) = g(X, \delta)$. The beauty of such $A(GPRS)_n$-spacetime is that it has the flavour of

(a) Ricci symmetric space in the sense of Cartan (for $A = B = C = D = 0$),

(b) Ricci recurrent space by E. M. Patterson [14] (for $B \neq 0$ and $A = C = D = 0$),

(c) generalized Ricci recurrent space by De, Guha and Kamilya [9] (for $B \neq 0$, $D \neq 0$ and $A = C = 0$),

(d) pseudo-Ricci symmetric space by Chaki [6] (for $A = B \neq 0$ and $C = D = 0$),

(e) generalized pseudo-Ricci symmetric space, by Baishya [1] (for $A = B \neq 0$ and $C = D \neq 0$) and
(f) almost pseudo-Ricci symmetric space by Chaki and Kawaguchi [7] (for $A = B \neq 0$ and $C = D = 0$).

Next, if the vector fields associated to the 1-forms $A \& B$ are co-directional with that of C & D respectively, that is $C = \phi A$ & $D = \phi B$ where ϕ being constant, then the relation (2.3) turns into

$$(\nabla_X Z)(Y,U) = [A(X) + B(X)]Z(Y,U)] + A(Y)\ Z(X,U) + A(U)Z(X,U)$$

where $Z(X,Y) = S(X,Y) + \phi\ g(X,Y)$ is a well known Z-tensor introduced in ([12], [13]). This leads to the following:

Theorem 2.1 ([13]). *Every $A(GPRS)_n$-spacetime is an almost pseudo Z-symmetric spacetime provided that the vector fields associated to the 1-forms $A \& B$ are co-directional with that of C & D respectively.*

It is to be noted that the converse of the Theorem 2.1 is also true. Thus we can say that an almost pseudo Z-symmetric spacetime is a natural example of an almost generalized pseudo Z-symmetric spacetime.

Definition 2.1. A non-flat Riemannian manifold $(M^n, g)(n > 3)$ is said to be a quasi-Einstein manifold [10] if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$S(X,Y) = \lambda g(X,Y) + \mu\psi(X)\psi(Y),$$

where λ, $\mu \in \mathbb{R}$ and ψ is a non-zero 1-form such that $g(X,U) = \psi(X)$, for all vector fields X, U being a unit vector field of the 1-form.

Now, contracting Y over U in (2.1) we obtain

$$\text{dr}(X) = r[A(X) + B(X)] + 2\bar{A}(X) + 6C(X) + 4D(X) \tag{2.4}$$

where $\bar{A}(X) = S(X,\theta)$. Again, from (2.1), one can easily bring out

$$\begin{aligned}(\nabla_X S)(Y,U) - (\nabla_U S)(X,Y) &= B(X)S(Y,U) - B(U)S(X,Y) \\ &\quad + D(X)g(Y,U) - D(U)g(X,Y)\end{aligned} \tag{2.5}$$

after further contraction which leaves

$$dr(X) = 2rB(X) - 2\bar{B}(X) + 6D(X), \tag{2.6}$$

where $\bar{B}(X) = S(X,\varrho)$.

It is known ([15], p, 41) that a conformally flat (M^4, g) spacetime possesses the relation

$$(\nabla_X S)(Y,U) - (\nabla_U S)(X,Y) = \frac{1}{6}[g(Y,U)dr(X) - g(X,Y)dr(U)]. \tag{2.7}$$

By virtue of (2.5), (2.6) and (2.7) we find

$$\begin{aligned} &3[B(X)S(Y,U) - B(U)S(X,Y)] \\ &= [rB(X) - \bar{B}(X)]g(Y,U) - [rB(U) - \bar{B}(U)]g(X,Y) \end{aligned} \tag{2.8}$$

which yields

$$B(X)\bar{B}(U) = B(U)\bar{B}(X), \tag{2.9}$$

for $Y = \varrho$. Assuming the Ricci tensor of the spacetime as codazzi type (in the sense of [4]) and then making use of (2.6), we obtain from (2.9) that

$$B(X)D(U) = B(U)D(X) \ \forall \ X \text{ and } U. \tag{2.10}$$

This motivate us to state

Proposition 2.1. *In a conformally flat $A(GPRS)_4$-spacetime with codazzi type of Ricci tensor, the 1-forms B and D are co-directional.*

Again, for constant scalar curvature tensor (or codazzi type of Ricci tensor) by virtue of (2.6), (2.8) and (2.10), we can easily find out

$$S(Y,U) = -\frac{D(\varrho)}{B(\varrho)}g(Y,U) + \frac{1}{B(\varrho)}[rB(Y) + nD(Y)]B(U), \tag{2.11}$$

where $\frac{D(U)}{B(U)} = k \ \forall \ U$. If the 1-forms B and D are co-directional, then (2.11) takes the following form

$$S(Y,U) == \alpha g(Y,U) + \beta B(Y)B(U). \tag{2.12}$$

This leads to the followings:

Theorem 2.2. *A conformally flat $A(GPRS)_4$-spacetime with codazzi type of Ricci tensor, is a quasi-Einstein spacetime.*

But, it is proved in ([8], Theorem 3.1) that a conformally flat $A(GPRS)_4$-spacetime is always quasi-Einstein spacetime. In consequence of Corollary 3.1 in [8], we can state the following:

Corollary 2.1. *A conformally flat almost generalized pseudo-Ricci symmetric spacetime with constant scalar curvature can be considered as a model of the perfect fluid spacetime in general relativity.*

Corollary 2.2. *A conformally flat almost generalized pseudo-Ricci symmetric spacetime with constant scalar curvature is a space of quasi constant curvature.*

3. Existence of almost pseudo-Ricci symmetric spacetime

Example 3.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = e^{-x^1}[(dx^1)^2 - (dx^2)^2 + 2\ dx^3 dx^4\ ], \tag{3.1}$$

$(i,\ j = 1,\ 2,\ 3,\ 4)$.

The non-zero components of Riemannian curvature tensors, Ricci tensors (up to symmetry and skew-symmetry) and scalar curvature tensor are

$$\begin{aligned} R_{2324} &= \frac{1}{4}e^{-x^1} = R_{3434}, \\ R_{22} &= \frac{1}{2} = -R_{34}, \\ r &= -\frac{3}{2}e^{x^1}. \end{aligned}$$

Covariant derivatives of Ricci tensors (up to symmetry) is expressed as

$$\begin{aligned} R_{12,2} &= -R_{13,4} = -R_{14,3} = \frac{1}{4} \\ R_{22,1} &= -R_{34,1} = \frac{1}{2}. \end{aligned}$$

For the following choice of the1-forms

$$\begin{aligned} A_i &= \frac{1}{2}, \text{ for } i = 1 \\ &= 0, \quad \text{otherwise} \\ B_i &= \frac{1}{2}, \text{ for } i = 1 \\ &= 0, \quad \text{otherwise}, \end{aligned}$$

one can easily verify the followings

$$\begin{aligned} R_{12,k} &= (A_k + B_k)\, R_{12} + A_1 R_{k2} + A_2 R_{1k}, \\ R_{13,k} &= (A_k + B_k)\, R_{13} + A_1 R_{k3} + A_3 R_{1k}, \\ R_{14,k} &= (A_k + B_k)\, R_{14} + A_1 R_{k4} + A_4 R_{1k}, \\ R_{23,k} &= (A_k + B_k)\, R_{23} + A_2 R_{k3} + A_3 R_{2k}, \\ R_{24,k} &= (A_k + B_k)\, R_{24} + A_2 R_{k4} + A_4 R_{2k}, \\ R_{34,k} &= (A_k + B_k)\, R_{34} + A_3 R_{k4} + A_4 R_{3k}, \\ R_{11,k} &= (A_k + B_k)\, R_{11} + A_1 R_{k1} + A_1 R_{1k}, \\ R_{22,k} &= (A_k + B_k)\, R_{22} + A_2 R_{k2} + A_2 R_{2k}, \\ R_{33,k} &= (A_k + B_k)\, R_{33} + A_3 R_{k3} + A_3 R_{3k}, \\ R_{44,k} &= (A_k + B_k)\, R_{44} + A_4 R_{k4} + A_4 R_{4k}, \end{aligned}$$

where $k = 1,\ 2,\ 3,\ 4$.

In consequence of the above, one can say that

Theorem 3.1. *There exists a spacetime* $(\mathbb{R}^4, g)$ *which is an almost pseudo-Ricci symmetric spacetime with the above mentioned choice of the 1-forms.*

It is obvious that the spacetime bearing the metric given by (3.1) can not be Ricci symmetric, Ricci recurrent, generalized Ricci recurrent as well as almost generalized pseudo-Ricci symmetric spacetime.

4. Existence of $A(GPRS)_n$-spacetime

Example 4.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{4/3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\ ] - (dx^4)^2, \tag{4.1}$$

$(i,\ j = 1,\ 2,\ 3,\ 4)$. The non-zero components of Ricci tensors (up to symmetry)

$$R_{11} = \frac{2}{3(x^4)^{2/3}} = R_{22} = R_{33},\ R_{44} = \frac{2}{3(x^4)^2}.$$

Covariant derivative (up to symmetry) $R_{ik,l}$ of Ricci tensors is expressed by

$$\begin{aligned} R_{11,4} &= -\frac{4}{3(x^4)^{5/3}} = R_{22,4} = R_{33,4}\,,\ R_{44,4} = -\frac{4}{3(x^4)^3} \\ R_{14,1} &= -\frac{8}{9(x^4)^{5/3}} = R_{24,2} = R_{34,3}\,. \end{aligned}$$

For following choice of the 1-forms

$$\begin{aligned} A_i &= \frac{1}{x^4},\ \text{ for } i = 4, \\ &= 0, \ \text{ otherwise} \\ B_i &= -\frac{19}{3x^4}, \ \text{ for } i = 4, \\ &= 0, \ \text{ otherwise} \\ C_i &= -\frac{14}{9(x^4)^3}, \ \text{ for } i = 4 \\ &= 0, \ \text{otherwise}\ ,, \\ D_i &= \frac{34}{9(x^4)^3}, \ \text{ for } i = 4 \\ &= 0, \ \text{otherwise}\ , \end{aligned}$$

one can easily verify the followings

$$\begin{aligned}
R_{12},_k &= (A_k + B_k) R_{12} + A_1 R_{k2} + A_2 R_{1k} + (C_k + D_k) g_{12} + C_1 g_{k2} + C_2 g_{1k}, \\
R_{13},_k &= (A_k + B_k) R_{13} + A_1 R_{k3} + A_3 R_{1k} + (C_k + D_k) g_{13} + C_1 g_{k3} + C_3 g_{1k}, \\
R_{14},_k &= (A_k + B_k) R_{14} + A_1 R_{k4} + A_4 R_{1k} + (C_k + D_k) g_{14} + C_1 g_{k4} + C_4 g_{1k}, \\
R_{23},_k &= (A_k + B_k) R_{23} + A_2 R_{k3} + A_3 R_{2k} + (C_k + D_k) g_{23} + C_2 g_{k3} + C_3 g_{2k}, \\
R_{24},_k &= (A_k + B_k) R_{24} + A_2 R_{k4} + A_4 R_{2k} + (C_k + D_k) g_{24} + C_2 g_{k4} + C_4 g_{2k}, \\
R_{34},_k &= (A_k + B_k) R_{34} + A_3 R_{k4} + A_4 R_{3k} + (C_k + D_k) g_{34} + C_3 g_{k4} + C_4 g_{3k}, \\
R_{11},_k &= (A_k + B_k) R_{11} + A_1 R_{k1} + A_1 R_{1k} + (C_k + D_k) g_{11} + C_1 g_{k1} + C_1 g_{1k}, \\
R_{22},_k &= (A_k + B_k) R_{22} + A_2 R_{k2} + A_2 R_{2k} + (C_k + D_k) g_{22} + C_2 g_{k2} + C_2 g_{2k}, \\
R_{33},_k &= (A_k + B_k) R_{33} + A_3 R_{k3} + A_3 R_{3k} + (C_k + D_k) g_{33} + C_3 g_{k3} + C_3 g_{3k}, \\
R_{44},_k &= (A_k + B_k) R_{44} + A_4 R_{k4} + A_4 R_{4k} + (C_k + D_k) g_{44} + C_4 g_{k4} + C_4 g_{4k},
\end{aligned}$$

where $k = 1, 2, 3, 4$.

In consequence of the above, one can say that

Theorem 4.1. *There exists a spacetime* $(\mathbb{R}^4, g)$ *which is an almost generalized pseudo-Ricci symmetric for the above mentioned choice of the1-forms.*

It is obvious that the spacetime bearing the metric given by (4.1) can not be Ricci symmetric, Ricci recurrent, generalized Ricci recurrent as well as pseudo-Ricci symmetric.

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