Heptavalent symmetric graphs of order $8p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected heptavalent symmetric graphs of order $8p$ for each prime p. As a result, a connected heptavalent symmetric graph of order $8p$ with p a prime exists if and only if $p = 2$ or 3, and up to isomorphism, there are only two such graphs: one for each $p = 2$ and 3.
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1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [20, 23] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an

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ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G,s)-*arc-transitive* or (G,s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G,s)-arc-transitive graph is said to be (G,s)-*transitive* if it is not $(G,s+1)$-arc-transitive. In particular, a $(G,1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* or s-*transitive* if it is $(\mathrm{Aut}(X),s)$-arc-transitive, $(\mathrm{Aut}(X),s)$-regular or $(\mathrm{Aut}(X),s)$-transitive, respectively.

As we all known that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [7] and Potočnik [19]. Thus, classifying symmetric graphs with small valency has received considerable attention, see [8, 26, 27]. Following this structure given by Guo [9], a series of pentavalent symmetric graphs is classified in [14, 17, 18, 24, 25]. Recently, the structure of heptavalent case was determined by Guo [10]. Thus, as an application of this result, we classify heptavalent symmetric graphs of order $8p$ for each prime p in this paper.

2. Preliminary results

Let X be a connected G-symmetric-transitive graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [15, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected heptavalent G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N is transitive on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected heptavalent G/N-symmetric graph.*

The following proposition characterizes the vertex stabilizers of connected heptavalent s-transitive graphs (see [10, Theorem 1.1]).

Proposition 2.2. *Let X be a connected heptavalent (G,s)-transitive graph for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:*

(1) *For $s=1$, $G_v \cong \mathbb{Z}_7$, D_{14}, F_{21}, D_{28}, $F_{21} \times \mathbb{Z}_3$;*

(2) *For* $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\mathrm{PSL}(3,2)$, A_7, S_7, $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3,2)$ *or* $\mathbb{Z}_2^4 \rtimes \mathrm{SL}(3,2)$*;*

(3) *For* $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\mathrm{PSL}(3,2) \times \mathrm{S}_4$, $\mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$, $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ *or* $[2^{20}] \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$.

In particular, a Sylow 3*-subgroup of* G_v *is elementary abelian.*

To extract a classification of connected heptavalent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [5], we introduce the graphs $G(2p, r)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let r be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of Z_p^* of order r. Define the graph $G(2p, r)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, r)\}$.

Proposition 2.3. *Let* X *be a connected heptavalent symmetric graph of order* $2p$ *with* p *a prime. Then* X *is isomorphic to* $K_{7,7}$ *or* $G(2p, 7)$ *with* $7 \bigm| (p-1)$*. Furthermore,* $\mathrm{Aut}(G(2p, 7)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$*.*

In view of [11, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order $4p$ for a prime p.

Proposition 2.4. *Let* X *be connected heptavalent symmetric graph of order* $4p$ *with* p *a prime. Then* X *is isomorphic to* K_8*.*

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set $V(\mathrm{Cay}(G, S)) = G$ and edge set $E(\mathrm{Cay}(G, S)) = \{\{g, sg\} \mid g \in G, s \in S\}$. Clearly, a Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Furthermore, $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. The Cayley graph is vertex-transitive because it admits the right regular representation $R(G)$ of G as a regular group of automorphisms of $\mathrm{Cay}(G, S)$. A graph X is isomorphic to a Cayley graph on G if and only if $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [21]).

Example 2.5. Let $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$ and $S = \{ab, a^2b, a^3b, a^5b, a^6b, a^7b, b\}$. We can define the Cayley graph:

$$\mathcal{G}_{16} = \mathrm{Cay}(G, S).$$

Then by Magma [3], $\mathcal{G}_{16} = K_{8,8} - 8K_2$ is 2-transitive and $\mathrm{Aut}(\mathcal{G}_{16}) = (\mathbb{Z}_8 \times \mathbb{Z}_2).\mathrm{S}_7 \cong \mathrm{S}_8 \times \mathbb{Z}_2$.

The next example is about a connected heptavalent symmetric graph of order 24.

Example 2.6. Define the Cayley graph on the symmetric group S_4:

$$\mathcal{G}_{24} = \mathrm{Cay}(S_4, S).$$

where $S = \{(1,2,3,4),(1,4,3,2),(1,2,4),(1,4,2),(3,4),(2,4),(1,4)(2,3)\}$. By Magma [3], $\mathrm{Aut}(\mathcal{G}_{24}) = S_4.D_{14} \cong \mathrm{PGL}(2,7)$ and $\mathcal{G}_{24}$ is a connected heptavalent 1-transitive graph.

3. Classification

This section is devoted to classifying heptavalent symmetric graphs of order $8p$ for p a prime.

Theorem 3.1. *Let X be a connected heptavalent symmetric graph of order $8p$ with p a prime. Then $X \cong \mathcal{G}_{16}$ or $\mathcal{G}_{24}$.*

Proof. By [16] and Magma [3], there is a unique connected heptavalent symmetric graphs of order 16 or 24. Thus, by Examples 2.5 and 2.6, we have that $X \cong \mathcal{G}_{16}$ and $\mathcal{G}_{24}$ for $p = 2$ and $p = 3$, respectively. Let $p \geq 5$ and $A = \mathrm{Aut}(X)$. Then we only need to prove that there are no new such graphs.

Case 1: A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A. Then N is an elementary abelian q-group with $q = 2$ or p. Since X has order $8p$, by Proposition 2.1, N is semiregular on $V(X)$ and the quotient graph X_N of X relative to N is a heptavalent symmetric graph with A/N as an arc-transitive automorphism group. Clearly, the order of X_N is even and at least 8. This implies that $N = \mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_p$.

Suppose that $N = \mathbb{Z}_2$. Then X_N is a heptavalent symmetric graph of order $4p$. Note that $p \geq 5$. Thus, by Proposition 2.4, there is no such graph, a contradiction.

Suppose that $N = \mathbb{Z}_2^2$. Then X_N is a heptavalent symmetric graph of order $2p$, and by Proposition 2.3, $X_N \cong G(2p, 7)$ or $K_{7,7}$.

Assume that $X_N \cong G(2p, 7)$. Then $A/N \leq (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. Since $2 \cdot 7 \cdot p \,\big|\, |A/N|$, we have that $A = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. Set $C = C_A(N)$. By "N/C-Theorem" (see [12, Chapter I, Theorem 4.5]), $A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{GL}(2,2)$. Since 7 and p does not divide the order of $\mathrm{GL}(2,2)$, we have that $\mathbb{Z}_p \rtimes \mathbb{Z}_7 \leq C$, that is, all p-elements and 7-elements commute with N. On the other hand, $\mathbb{Z}_2$ normalizes N and hence $\mathbb{Z}_2$ normalizes an element of order 2 in N. This implies that A has a normal subgroup of order 2, which contradicts that N is minimal normal.

Assume that $X_N \cong K_{7,7}$. Then $A/N \lesssim \mathrm{Aut}(K_{7,7}) \cong (S_7 \times S_7) \rtimes \mathbb{Z}_2$. By Magma [3], $K_{7,7}$ has two minimal arc-transitive subgroups $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. Thus, A/N has a subgroup $M/N = \mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. A similar argument as above, we can deduce that $\mathbb{Z}_2$ or $\mathbb{Z}_4$ normalizes an element of order 2 in N. It forces that M has a normal subgroup $T \cong \mathbb{Z}_2$, and X_T is a heptavalent M/T-symmetric graph of order $4p$. However, by Lemma 2.4, there is no heptavalent symmetric graph of order $4p$ with $p \geq 5$, a contradiction.

Suppose that $N = \mathbb{Z}_p$. Then $X_N \cong K_8$ and $A/N \leq \mathrm{S}_8$. Note that $8{\cdot}7 \,\big|\, |A/N|$. From the information in [4], we have that $A/N = \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$, $\mathbb{Z}_2^3 \rtimes F_{21}$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,7) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^3 \rtimes \mathrm{PSL}(2,7)$, A_8 or S_8.

Assume that $A/N = \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$, $\mathbb{Z}_2^3 \rtimes F_{21}$, $\mathbb{Z}_2^3 \rtimes \mathrm{PSL}(2,7)$, A_8 or S_8. Then A/N has an arc-transitive group $M/N = \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$. By "N/C-Theorem", $M/C_M(N) \lesssim \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$. Thus, $2^2 \,\big|\, |C_M(N)|$. It forces that $C_M(N)$ has a characteristic subgroup $K \cong \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$. Since $C_M(N) \trianglelefteq M$, we have $K \trianglelefteq M$. Then X_K is a heptavalent M/K-symmetric graph of order $2p$ or p. Note that there is no heptavalent graph of order $p \geq 5$. Thus, $K = \mathbb{Z}_2^2$ and X_K is a heptavalent symmetric graph of order $2p$. By the above argument, this is also impossible.

Assume that $A/N = \mathrm{PSL}(2,7)$ or $\mathrm{PSL}(2,7) \rtimes \mathbb{Z}_2$. By "N/C-Theorem", $A/C_A(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$. Since $\mathrm{PSL}(2,7)$ is simple, we have that $\mathrm{PSL}(2,7)$ commutes with N. By Atlas [6], the Schur multiplier $\mathrm{Mult}(\mathrm{PSL}(2,7)) = \mathbb{Z}_2$. It implies that A has a normal subgroup $M = \mathrm{PSL}(2,7) \times \mathbb{Z}_p$ and M is arc-transitive. Clearly, $M_v \cong F_{21}$. Let $\mathrm{PSL}(2,7) \cong K \leq M$. Then $K \trianglelefteq M$. It follows that X_K is a heptavalent graph of order p. This is impossible because there is no heptavalent graph of order p.

Case 2: A has no solvable minimal normal subgroup.

For convenience, we still use N to denote a minimal normal subgroup of A. Then N is non-solvable. Since every group of order $q^s{\cdot}r^t$ with q, r primes and s, t non-negative integers is solvable, the order $|N|$ has at least three different primes. Note that $|V(X)| = 8p$. Thus, $N_v \neq 1$. By Proposition 2.1, N acting on $V(X)$ has at most two orbits. Since A is arc-transitive, we have that $|A| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$ by Lemma 2.2. It follows that $|N| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$.

Since N is non-solvable, $N \cong T_1 \times T_2 \times \cdots \times T_n$ with $T \cong T_1 \cong T_2 \cong \cdots \cong T_n$ is a non-abelian simple group. Note that N has at most two orbits on $V(X)$. Thus, $4p \,\big|\, |N|$. Now we divide the prime p into the next three subcases: $p = 5$, $p = 7$ and $p > 7$.

Subcase 2.1: Let $p = 5$. Then $|T| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7$ and $4{\cdot}5 \,\big|\, |T|$.

By [6, pp.12-14] and [22, Theorem 2], a simple calculation implies that T is isomorphic to the following groups listed in Table 1:

Assume that $n \geq 2$. Then by Table 1, $n = 2$ and $T \cong \mathrm{A}_5$ or A_6. Note that $4{\cdot}5 \,\big|\, |N|$ or $8{\cdot}5 \,\big|\, |N|$. By Magma [3], N_v has a normal subgroup $M \cong \mathrm{A}_5$ or A_6 and N_v/M is solvable. Thus, M is also a normal subgroup of A_v. By Proposition 2.2, $A_v \cong \mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$ or $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$. It forces that $A_v/M \cong \mathrm{A}_7$ or S_7. However, N_v/M is solvable and normal in A_v/M, this is impossible. Thus, $n = 1$ and N is a non-abelian simple group listed in Table 1.

Suppose that $N \cong \mathrm{A}_5$. Then $N_v \cong \mathbb{Z}_3$ and N has two orbits on $V(X)$. Since $N_v \trianglelefteq A_v$, we have $A_v = F_{21} \times \mathbb{Z}_3$, $F_{42} \times \mathbb{Z}_3$ or $F_{42} \times \mathbb{Z}_6$. Let P be a Sylow 7-

Table 1: Non-abelian simple groups of order dividing $2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7$

3-prime factor		4-prime factor	
T	Order	T	Order
A_5	$2^2 \cdot 3 \cdot 5$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
A_6	$2^3 \cdot 3^2 \cdot 5$	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
$PSU(4,2)$	$2^6 \cdot 3^4 \cdot 5$	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$
		A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$
		J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
		$PSL(3,4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
		$PSp(6,2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$
		$PSU(3,5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$

subgroup of A_v. Then $P \trianglelefteq A_v$. By "N/C-Theorem", $A/C_A(N) \lesssim \mathrm{Aut}(N) \cong S_5$. Since $7 \nmid |S_5|$, we have that $P \leq C_A(N)$ and $P \trianglelefteq A_vN$. Since P is a Sylow 7-subgroup of A, we have that P is characteristic in A_vN. Note that N has two orbits on $V(X)$. By Proposition 2.1, X is bipartite and $|A : A_vN| = 2$. It implies that $A_vN \trianglelefteq A$. Thus, $P \trianglelefteq A$. However, P lies in the vertex stabilizer A_v and P cannot be normal, a contradiction.

Suppose that $N \cong A_6$. Then by Atlas [6], $N_v \cong \mathbb{Z}_3^2$ or $\mathbb{Z}_3^2 \times \mathbb{Z}_2$. Note that $N_v \trianglelefteq A_v$. By Proposition 2.2, A_v has no normal subgroup isomorphic to N_v, a contradiction.

Suppose that $N \cong PSU(4,2)$. Then $|N_v| = 2^3{\cdot}3^4$ or $2^2{\cdot}3^4$. By Atlas [6], a Sylow 3-subgroup of N_v is non-abelian. However, the Sylow 3-subgroups of A_v are elementary abelian by Proposition 2.2, a contradiction.

Suppose that $N \cong A_7$, A_8, A_9, A_{10}, J_2, $PSL(3,4)$, $PSp(6,2)$ or $PSU(3,5)$. Then $|N_v| = |N|/20$ or $|N|/40$. By Atlas [6], N has no subgroups of such orders, a contradiction.

Subcase 2.2: Let $p = 7$. Then $|N| \mid 2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7^2$.

By [6, pp.12-14] and [22, Theorem 2], a simple calculation implies that T is isomorphic to the following groups listed in Table 2.

Assume that $n \geq 2$, then $n = 2$ and $N = PSL(2,7)^2$, $PSL(2,8)^2$, A_7^2, A_8^2 or $PSL(3,4)^2$. Note that $|N : N_v| = 28$ or 56. Thus, N_v is non-solvable and N_v has a normal subgroup isomorphic to $PSL(2,7)$, $PSL(2,8)$, A_7, A_8 or $PSL(3,4)$. By Proposition 2.2, A_v has no subgroups isomorphic to $PSL(2,8)$, A_8 or $PSL(3,4)$. Therefore, $N = PSL(2,7)^2$ or A_7^2. If $N = PSL(2,7)^2$, then $N_v = S_3 \times PSL(2,7)$ or $\mathbb{Z}_3 \times PSL(2,7)$. Since $N_v \trianglelefteq A_v$, we have A_v has a normal subgroup isomorphic to $S_3 \times PSL(2,7)$ or $\mathbb{Z}_3 \times PSL(2,7)$. This is impossible by Proposition 2.2. If $N = A_7^2$, then by Magma [3], A_7^2 has no subgroup of index 28 or 56, a contradiction.

Thus, $n = 1$ and N is a non-abelian simple group listed in Table 2.

Table 2: Non-abelian simple groups of order dividing $2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7^2$

3-prime factor		4-prime factor	
T	Order	T	Order
PSL(2,7)	$2^3\cdot 3\cdot 7$	A_7	$2^3\cdot 3^2\cdot 5\cdot 7$
PSL(2,8)	$2^3\cdot 3^2\cdot 7$	A_8	$2^6\cdot 3^2\cdot 5\cdot 7$
PSU(3,3)	$2^5\cdot 3^3\cdot 7$	A_9	$2^6\cdot 3^4\cdot 5\cdot 7$
		A_{10}	$2^7\cdot 3^4\cdot 5^2\cdot 7$
		J_2	$2^7\cdot 3^3\cdot 5^2\cdot 7$
		PSL(2,49)	$2^4\cdot 3\cdot 5^2\cdot 7^2$
		PSL(3,4)	$2^6\cdot 3^2\cdot 5\cdot 7$
		PSp(6,2)	$2^9\cdot 3^4\cdot 5\cdot 7$

Suppose that $N = \mathrm{PSL}(2,7)$. Then $N_v = \mathbb{Z}_3$ or S_3. Since $N_v \trianglelefteq A_v$, we have $N_v = \mathbb{Z}_3$ and $A_v = F_{21}\times\mathbb{Z}_3$, $F_{42}\times\mathbb{Z}_3$ or $F_{42}\times\mathbb{Z}_6$. In this case, N is transitive on $V(X)$ and $A = NA_v$. Since $N\cap A_v = \mathbb{Z}_3$, we have that $N\rtimes F_{21} \le A$ and $N\rtimes F_{21}$ is arc-transitive. Thus, A has a subgroup $B = N\rtimes\mathbb{Z}_7 = \mathrm{PSL}(2,7)\rtimes\mathbb{Z}_7$, and B is arc-transitive. However, $B_v = \mathbb{Z}_7\rtimes\mathbb{Z}_3$, this is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSL}(2,8)$, $\mathrm{PSU}(3,3)$, A_9, A_{10}, J_2. Since $|N : N_v| = 28$ or 56, we have a Sylow 3-subgroup of N_v is also a Sylow 3-subgroup of N. By Proposition 2.2, a Sylow 3-subgroup of A_v is elementary abelian. However, by Atlas [6], a Sylow 3-subgroup of N is not elementary abelian, a contradiction.

Suppose that $N = \mathrm{A}_7$, $\mathrm{PSL}(2,49)$ or $\mathrm{PSp}(6,2)$. Then by Atlas [6], N has no subgroups of index 28 or 56, a contradiction.

Suppose that $N = \mathrm{A}_8$. Then by Atlas [6], $N_v = \mathrm{A}_6$, $(\mathrm{A}_5\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2$ or S_6. By Proposition 2.2, A_v has no normal subgroup isomorphic to $(\mathrm{A}_5\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2$. Thus $N_v = \mathrm{A}_6$ or S_6. Since $N_v \trianglelefteq A_v$, we have that $A_v = \mathrm{A}_7\times\mathrm{A}_6$, $(\mathrm{A}_7\times\mathrm{A}_6)\rtimes\mathbb{Z}_2$ or $\mathrm{S}_7\times\mathrm{S}_6$ by Proposition 2.2. The normality of N in A implies that A has an arc-transitive subgroup $B = N\rtimes\mathbb{Z}_7$ and $B_v = \mathrm{A}_6\times\mathbb{Z}_7$ or $\mathrm{S}_6\times\mathbb{Z}_7$. This is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSL}(3,4)$. Then by Atlas [6], $N_v = \mathrm{A}_6$ and N is transitive on $V(X)$. The similar argument as above, A has an arc-transitive subgroup $B = N\rtimes\mathbb{Z}_7$ and $B_v = \mathrm{A}_6\times\mathbb{Z}_7$. This is impossible by Proposition 2.2.

Subcase 2.3: Let $p > 7$. Then $|N| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$.

By [6, pp.12-14], [22, Theorem 2] and [13, Theorem A], a simple calculation implies that T is isomorphic to the following groups listed in Table 3. Since $4p \,\big|\, |N|$ and p^2 does not divide the orders of the groups listed in Table 3, we have $n = 1$ and N is a non-abelian simple group.

Suppose that $N = \mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,3)$, M_{12}, $\mathrm{PSU}(3,8)$, ${}^2F_4(2)'$, $\mathrm{PSL}(4,4)$, A_{11}, $\mathrm{P}\Omega^-(8,2)$, $G_2(4)$. Then $|N : N_v| = 4p$ or $8p$. It follows that a sylow 3-subgroup of N_v is also a Sylow 3-subgroup of N. By Atlas [6], a Sylow 3-

Table 3: Non-abelian simple groups of order dividing $2^{27}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$

3-prime factor		4-prime factor		5-prime factor	
T	Order	T	Order	T	Order
PSL(2, 17)	$2^4 \cdot 3^2 \cdot 17$	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$\mathrm{PSL}(2, 2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$
PSL(3, 3)	$2^4 \cdot 3^3 \cdot 13$	M_{12}	$2^6 \cdot 3^2 \cdot 5 \cdot 11$	PSL(2, 29)	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$
		PSL(2, 11)	$2^2 \cdot 3 \cdot 5 \cdot 11$	PSL(2, 41)	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$
		PSL(2, 19)	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	PSL(2, 71)	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$
		PSL(2, 16)	$2^4 \cdot 3 \cdot 5 \cdot 17$	PSL(2, 449)	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 449$
		PSL(2, 25)	$2^5 \cdot 3 \cdot 5^2 \cdot 13$	PSL(4, 4)	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$
		PSL(2, 27)	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	PSL(5, 2)	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
		PSL(2, 31)	$2^5 \cdot 3 \cdot 5 \cdot 31$	A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$
		PSL(2, 81)	$2^4 \cdot 3^4 \cdot 5 \cdot 41$	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
		PSp(4, 4)	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	$\mathrm{P}\Omega^-(8, 2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$
		PSU(3, 4)	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$
		PSU(3, 8)	$2^8 \cdot 3^4 \cdot 7 \cdot 19$		
		Sz(8)	$2^6 \cdot 5 \cdot 7 \cdot 13$		
		${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$		

subgroup of N is not elementary abelian. However, a Sylow 3-subgroup of A_v is elementary abelian by Proposition 2.2, a contradiction.

Suppose that $N = \mathrm{M}_{11}$, PSL(2, 11), PSL(2, 19), PSL(2, 27), PSU(3, 4), Sz(8), $\mathrm{PSL}(2, 2^6)$, PSL(2, 29), PSL(2, 41), PSL(2, 71), PSL(2, 449), M_{22}. Then by Atlas [6] and Magma [3], N has no subgroups of index $4p$ or $8p$, a contradiction.

Suppose that $N = \mathrm{PSL}(2, 16)$. Then by Atlas [6], $N_v = D_{30}$ or A_5. Since $N_v \trianglelefteq A_v$, we have that A_v has a normal subgroup isomorphic to D_{30} or A_5. However, by Proposition 2.2, A_v has no such normal subgroups, a contradiction.

Suppose that $N = \mathrm{PSL}(2, 25)$. Then by Atlas [6], $N_v = \mathbb{Z}_5^2 \rtimes \mathbb{Z}_{12}$. Clearly, N_v has a characteristic subgroup isomorphic to $\mathbb{Z}_5^2$. The normality of N_v in A_v implies that A_v has a normal subgroup isomorphic to $\mathbb{Z}_5^2$. This is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSL}(2, 81)$. Then $|N_v| = 2{\cdot}3^4{\cdot}5$ or $2^2{\cdot}3^4{\cdot}5$. By Atlas [6], N_v has a characteristic subgroup $\mathbb{Z}_3^4$. Since $N_v \trianglelefteq A_v$, we have that A_v has a normal subgroup $\mathbb{Z}_3^4$. This is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSp}(4, 4)$. Then by Atlas [6], $N_v = (\mathrm{A}_5 \times \mathrm{A}_5) \rtimes \mathbb{Z}_2$. However, by Proposition 2.2, A_v has no normal subgroup isomorphic to N_v, a contradiction.

Suppose that $N = \mathrm{PSL}(5, 2)$. Then by Atlas [6], $N_v = \mathbb{Z}_2^4 \rtimes \mathrm{A}_7$. Since $N_v \trianglelefteq A_v$, we have that A_v has a normal subgroup isomorphic to $\mathbb{Z}_2^4 \rtimes \mathrm{A}_7$. This is impossible by Proposition 2.2.

Suppose that $N = \mathrm{PSL}(2, 31)$. Then by Atlas [6], $N_v = \mathrm{A}_5$. This implies that A_v has a normal subgroup isomorphic to A_5. However, by Proposition 2.2, this is impossible. □

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References

[1] N. Biggs, *Algebraic graph theory*, Second ed., Cambridge University Press, Cambridge, 1993.

[2] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, Elsevier Science Ltd, New York, 1976.

[3] W. Bosma, C. Cannon, C. Playoust, *The MAGMA algebra system I: The user language*, J. Symbolic Comput., 24 (1997), 235-265.

[4] G. Butler, J. Mckay, *The transitive groups of degree up to eleven*, Comm. Algeb., 11 (1983), 863-911.

[5] Y. Cheng, J. Oxley, *On the weakly symmetric graphs of order twice a prime*, J. Combin. Theory B, 42 (1987), 196-211.

[6] H.J. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A Wilson, *Atlas of finite group*, Clarendon Press, Oxford, 1985.

[7] D.Ž. Djoković, G.L. Miller, *Regular groups of automorphisms of cubic graphs*, J. Combin. Theory B, 29 (1980), 195-230.

[8] Y.Q. Feng, J.H. Kwak, K.S. Wang, *Classifying cubic symmetric graphs of order $8p$ or $8p^2$*, European J. Combin., 26 (2005), 1033-1052.

[9] S.T. Guo, Y.Q. Feng, *A note on pentavalent s-transitive graphs,* Discrete Math., 312 (2012), 2214-2216.

[10] S.T. Guo, Y.T. Li, X.H. Hua, *(G, s)-transitive graphs of valency* 7, Algeb. Colloq., 23 (2016), 493-500.

[11] S.T. Guo, J. Shi, Z.J. Zhang, *Heptavalent symmetric graphs of order $4p$,* South Asian J. Math., 1 (2011), 131-136.

[12] B. Huppert, *Eudiche Gruppen I*, Springer-Verlag, Berlin, 1967.

[13] A. Jafarzadeh, A. Iranmanesh, *On simple K_n-groups for $n = 5, 6$,* (In Campbell, CM, Quick, MR, Robertson, EF, Smith, GC, eds.) Groups St. Andrews 2005. London Mathematical Society lecture note series, vol. 2, 668-680, Cambridge University Press , Cambridge, 2007.

[14] B. Ling, C.X. Wu, B. Lou, *Pentavalent symmetric graphs of order 30p,* Bull. Aust. Math. Soc., 90 (2014), 353-362.

[15] P. Lorimer, *Vertex-transitive graphs: symmetric graphs of prime valency,* J. Graph Theory, 8 (1984), 55-68.

[16] B.D. Mckay, *Transitive graphs with fewer than 20 vertices,* Math. Comp., 33 (1979), 1101-1121.

[17] J. Pan, B. Lou, C. Liu, *Arc-transitive pentavalent graphs of order* $4pq$*,* Electron. J. Comb., 20 (2013), 1215-1230.

[18] J.M. Pan, X. Yu, *Pentavalent symmetric graphs of order twice a prime square,* Algeb. Colloq., 22 (2015), 383-394.

[19] P. Potočnik, *A list of* 4*-valent* 2*-arc-transitive graphs and finite faithful amalgams of index* $(4, 2)$*,* European J. Combin., 30 (2009), 1323-1336.

[20] D.J. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, 1982.

[21] B.O. Sabidussi, *Vertex-transitive graphs*, Monatsh Math., 68 (1964), 426-438.

[22] W.J. Shi, *On simple K4-groups*, Chinese Science Bull, 36 (1991), 1281-1283 (in Chinese).

[23] H. Wielandt, *Finite permutation groups*, Academic Press, New York, 1964.

[24] D.W. Yang, Y.Q. Feng, *Pentavalent symmetric graphs of order* $2p^3$*,* J. Korean Math. Soc., 18 (2016), 1-18.

[25] D.W. Yang, Y.Q. Feng, J.L. Du, *Pentavalent symmetric graphs of order* $2pqr$*,* Discrete Math., 339 (2016), 522-532.

[26] J.X. Zhou, *Tetravalent* s*-transitive graphs of order* $4p$*,* Discrete Math., 309 (2009), 6081-6086.

[27] J.X. Zhou, Y.Q. Feng, *Tetravalent* s*-transitive graphs of order twice a prime power*, J. Aust. Math. Soc., 88 (2010), 277-288.

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