Upper class functions on a controlled contraction principle in partial S-metric spaces

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Abstract. In this paper, we prove the existence and uniqueness of a fixed point of a self mapping on partial S-metric spaces under the partially α-contractive condition.
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1. Introduction and mathematical preliminaries

The existence and uniqueness of a fixed point for a self mapping on different types of metric spaces were the main topic for many research papers [4-18]. The notion of S-metric space was introduced by Sedghi [3]. A generalization of S-metric space was given by Nabil in [1], where he introduced partial S-metric spaces. Moreover, he proved the existence of a fixed point for a self mapping in partial S-metric space. In this paper, we generalize the results in [1] by adding a control function to the contraction principle, which makes the results in [1] a direct consequences of our theorems.

Before proceeding to the main results, we set forth some definitions that will be used in the sequel.

Definition 1.1 ([4])**.** Let X be a nonempty set and $p : X \times X \longrightarrow [0, +\infty)$. We say that (X, p) is a *partial metric space* if for all $x, y, z \in X$ we have:

1. $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$;
2. $p(x, x) \leq p(x, y)$;

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3. $p(x, y) = p(y, x)$;

4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Definition 1.2 ([3])**.** Let X be a nonempty set. An *S-metric space* on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, a \in X$:

1. $S(x; y; z) \geq 0$,

2. $S(x; y; z) = 0$ if and only if $x = y = z$,

3. $S(x; y; z) \leq S(x; x; a) + S(y; y; a) + S(z; z; a)$.

The pair $(X; S)$ is called an *S-metric space*.

Next, we give the definition of partial S-metric space.

Definition 1.3 ([1])**.** Let X be a nonempty set. A *partial S-metric space* on X is a function $S_p : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:
(i) $x = y$ if and only if $S_p(x, x, x) = S_p(y, y, y) = S_p(x, x, y)$;
(ii) $S_p(x, y, z) \leq S_p(x, x, t) + S_p(y, y, t) + S_p(z, z, t) - S_p(t, t, t)$;
(iii) $S_p(x, x, x) \leq S_p(x, y, z)$;
(iv) $S_p(x, x, y) = S_p(y, y, x)$.
The pair (X, S_p) is called a partial S-metric space.

Definition 1.4. A sequence $\{x_n\}_{n=0}^{\infty}$ of elements in (X, S_p) is called *p-Cauchy* if the limit $\lim_{n,m\to\infty} S_p(x_n, x_n, x_m)$ exists and finite. The partial S-metric space (X, S_p) is called *complete* if for each p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ there exists $z \in X$ such that $S_p(z, z, z) = \lim_n S_p(z, z, x_n) = \lim_{n,m} S_p(x_n, x_n, x_m)$.

Moreover, (X, S_p) is a complete partial S-metric space if and only if (X, S_p) is a complete S-metric space. A sequence $\{x_n\}_n$ in a partial S-metric space (X, S_p) is called 0-*Cauchy* if $\lim_{n,m\to\infty} S_p(x_n, x_n, x_m) = 0$. We say that (X, S_p) is 0-*complete* if every 0-Cauchy in X converges to a point $x \in X$ such that $S_p(x, x, x) = 0$.

One can easily construct an example of a partial S-metric space by using the ordinary partial metric space.

Example 1.5 ([1])**.** Let $X = [0, \infty)$ and p be the ordinary partial metric space on X. Define the mapping on X^3 to be $S_p(x, y, z) = p(x, z) + p(y, z)$. Then S_p defines a partial S-metric space.

Now we introduce the notion of partially $\alpha-$contractive.

Definition 1.6. Let (X, S_p) be a partial S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is *partially* $\alpha-$*contractive* if there exists a constant $k \in [0, 1)$ and a function $\alpha : X \times X \longrightarrow [0, +\infty)$ such that for all $x, y \in X$ we have

$$\alpha(x,y)S_p(Tx,Tx,Ty) \leq \max\{kS_p(x,x,y), S_p(x,x,x), S_p(y,y,y)\}. \tag{1.1}$$

Definition 1.7. Let (X, S_p) be a partial S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is $R_\alpha-$*admissible* if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(x, Ty) \geq 1$. Also, we say that T is $\alpha-$*admissible* if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.

Example 1.8. Let $X = [0, +\infty)$. Define $T : X \longrightarrow X$ by $Tx = \sqrt{x}$ and $\alpha : X \times X \longrightarrow X$ by

$$\alpha(x,y) = \begin{cases} e^{x-y}, & \text{if } x \geq y \\ 0, & \text{if } x < y. \end{cases}$$

It is a straightforward to verify that T is $\alpha-$admissible and $R_\alpha-$admissible.

Now, we set

$$\begin{aligned} \rho_{S_p}(\alpha) &:= \inf\{S_p(x,x,y) \mid x,y \in X : \alpha(x,y) \geq 1\} \\ &= \inf\{S_p(x,x,x) \mid x \in X : \alpha(x,x) \geq 1\}, \\ X_{S_p}(\alpha) &= \{x \in X \mid S_p(x,x,x) = \rho_{S_p}(\alpha)\}, \\ Z_{S_p}(\alpha) &= \{x \in X_{S_p} \mid \alpha(x,x) \geq 1\}. \end{aligned}$$

Definition 1.9. Let (X, S_p) be a partial S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is $R_\mu-$*subadmissible* if $x, y \in X$, $\mu(x, y) \leq 1$ implies that $\mu(x, Ty) \leq 1$.

2. Main result

In this section, we prove the existence of a fixed point in a partial S-metric space. We prove relevant corollary. This next theorem is considered to be our main result.

Definition 2.1 ([?])**.** Let $T : X \to X$ be a map and $\mu : X \times X \to [0, +\infty)$ be a function. We say that T is μ-subadmissible if $x, y \in X$, $\mu(x, y) \leq 1$ implies that $\mu(Tx, Ty) \leq 1$.

Definition 2.2. A map $T : X \to X$ is said to be triangular μ-subadmissible if the following holds:

$(T1)$ T is μ-subadmissible,

$(T2)$ $\mu(x, u) \leq 1$ and $\mu(u, y) \leq 1$ implies that $\mu(x, y) \leq 1$, $x, u, y \in X$.

Lemma 2.3. *Let $T : X \to X$ be a triangular $\mu -$ suborbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\mu(x_1, Tx_1) \leq 1$. Then there exists a sequence $\{x_n\}$ such that $\mu(x_n, x_m) \leq 1$ for all m, $n \in \mathbb{N}$ with $n < m$.*

The letter $\mathbb{N}$ represent the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Further, the nonnegative real numbers will be denoted by $\mathbb{R}_0^+ = [0, \infty)$.

In 2014 the concept of pair $(\mathcal{F}, h)$ is an upper class (see Definition 2.4 until 2.10) was introduced by A.H. Ansari in [19]

Definition 2.4 ([19, 20])**.** A function $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a function of subclass of type I, if $x \geq 1 \Longrightarrow h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}^+$.

Example 2.5 ([19, 20])**.** Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x, y) = (y + l)^x, l > 1$;

(b) $h(x, y) = (x + l)^y, l > 1$;

(c) $h(x, y) = x^n y$, $n \in \mathbb{N}$;

(d) $h(x, y) = y$;

(e) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y$, $n \in \mathbb{N}$;

(f) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$

for all $x, y \in \mathbb{R}^+$. Then h is a function of subclass of type I.

Definition 2.6 ([19, 20])**.** Let $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type I, if h is a function of subclass of type I and: (i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$, (ii) $h(1, y) \leq \mathcal{F}(1, t) \Longrightarrow y \leq t$ for all $t, y \in \mathbb{R}^+$.

Example 2.7 ([19, 20])**.** Define $h, \mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x, y) = (y + l)^x, l > 1$ and $\mathcal{F}(s, t) = st + l$;

(b) $h(x, y) = (x + l)^y, l > 1$ and $\mathcal{F}(s, t) = (1 + l)^{st}$;

(c) $h(x, y) = x^m y$, $m \in \mathbb{N}$ and $\mathcal{F}(s, t) = st$;

(d) $h(x, y) = y$ and $\mathcal{F}(s, t) = t$;

(d) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y$, $n \in \mathbb{N}$ and $\mathcal{F}(s, t) = st$;

(e) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$ and $\mathcal{F}(s, t) = (1 + l)^{st}$

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type I.

Definition 2.8 ([19, 20])**.** A function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a *function of subclass of type II*, if for all $x, y \geq 1$, we have $h(1,1,z) \leq h(x,y,z)$, for all $z \in \mathbb{R}^+$.

Example 2.9 ([19, 20])**.** Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y,z) = (z+l)^{xy}, l > 1$;

(b) $h(x,y,z) = (xy+l)^{z}, l > 1$;

(c) $h(x,y,z) = z$;

(d) $h(x,y,z) = x^m y^n z^p, m,n,p \in \mathbb{N}$;

(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m,n,p,q,k \in \mathbb{N}$

for all $x, y, z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 2.10 ([19, 20])**.** Let $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an *upper class of type II*, if h is a subclass of type II and the following holds:

(i) if $0 \leq s \leq 1$ then we have $\mathcal{F}(s,t) \leq \mathcal{F}(1,t)$,

(ii) if $h(1,1,z) \leq \mathcal{F}(s,t)$ then we have $z \leq st$ for all $s,t,z \in \mathbb{R}^+$.

Example 2.11 ([19, 20])**.** Define $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y,z) = (z+l)^{xy}, l > 1, \mathcal{F}(s,t) = st + l$;

(b) $h(x,y,z) = (xy+l)^{z}, l > 1, \mathcal{F}(s,t) = (1+l)^{st}$;

(c) $h(x,y,z) = z, F(s,t) = st$;

(d) $h(x,y,z) = x^m y^n z^p, m,n,p \in \mathbb{N}, \mathcal{F}(s,t) = s^p t^p$

(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m,n,p,q,k \in \mathbb{N}, \mathcal{F}(s,t) = s^k t^k$

for all $x, y, z, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type II.

Notation.

$$\begin{aligned}
\rho_{S_p}(\alpha,\mu) &:= \inf\{S_p(x,x,y) \mid x,y \in X : \alpha(x,y) \geq 1, \mu(x,y) \leq 1\} \\
&= \inf\{S_p(x,x,x) \mid x \in X : \alpha(x,x) \geq 1, \alpha(x,x) \leq 1\}, \\
X_{S_p}(\alpha,\mu) &= \{x \in X \mid S_p(x,x,x) = \rho_{S_p}(\alpha,\mu)\}, \\
Z_{S_p}(\alpha,\mu) &= \{x \in X_{S_p} \mid \alpha(x,x) \geq 1, \mu(x,x) \leq 1\}.
\end{aligned}$$

Definition 2.12. Let (X, S_p) be a partial S-metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is *partially* $(\mathcal{F}, h, \alpha, \mu)$*-contractive* if there exists a constant $k \in [0, 1)$ and a function $\alpha, \mu : X \times X \longrightarrow [0, +\infty)$ such that for all $x, y \in X$ we have

$$h(\alpha(x,y), S_p(Tx,Tx,Ty)) \leq \mathcal{F}(\mu(x,y), \max\{kS_p(x,x,y), S_p(x,x,x), S_p(y,y,y)\}). \tag{2.1}$$

where the pair $(\mathcal{F}, h)$ is an upper class of type I.

Theorem 2.13. *Let (X, S_p) be a complete partial S-metric space, T be a self mapping on X and assume that T is partially $(\mathcal{F}, h, \alpha, \mu)$-contractive. If T is $\alpha-$admissible ,$\mu-$subadmissible and $R_\alpha-$admissible, $R_\mu-$subadmissible and if $X_{S_p}(\alpha, \mu)$ is nonempty, then $Z_{S_p}(\alpha, \mu)$ is nonempty. Also, assume that there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1, \mu(x_0, x_0) \leq 1$, then there exists $a \in Z_{S_p}(\alpha)$ such that $Ta = a$.*

Moreover, if for all u, v in $Z_{S_p}(\alpha, \mu)$ with the property $Tu = u$ and $Tv = v$ we have $\alpha(u, v) \geq 1, \mu(u, v) \leq 1$, then T has a unique fixed point in $Z_{S_p}(\alpha, \mu)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$. Define a sequence $\{x_n\}$ for all $n \geq 0$ in X such that $x_1 = Tx_0, x_2 = Tx_1, \cdots, x_{n+1} = Tx_n, \cdots$. Since T is $\alpha-$admissible ,$\mu-$subadmissible and $R_\alpha-$admissible, $R_\mu-$*subadmissible*, we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$,$\mu(x_0, x_1) = \mu(x_0, Tx_0) \leq 1$, and hence $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$, $\mu(x_1, x_2) = \mu(Tx_0, Tx_1) \leq 1$. So, by induction on n we get

$$\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1})) \leq 1,$$

for all $n \geq 0$. Also, since T is $R_\alpha-$admissible and $R_\mu-$*subadmissible*; $\alpha(x_0, x_0) \geq 1, \mu(x_0, x_0) \leq 1$ implies $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1, \mu(x_0, x_1) = \mu(x_0, Tx_0) \leq 1$. By induction on n, we also conclude that

$$\alpha(x_0, x_n) \geq 1, \mu(x_0, x_n) \leq 1$$

for all $n \geq 0$. Also, given the fact that T is α-admissible and $\alpha(x_0, x_0) \geq 1$, it not difficult to prove that $\alpha(x_n, x_n) \geq 1$ for all $n \geq 0$. Hence,

$$\begin{aligned}
&h(1, S_p(x_1, x_1, x_1) = h(1, S_p(Tx_0, Tx_0, Tx_0)) \\
&\leq h(\alpha(x_0, x_0), S_p(Tx_0, Tx_0, Tx_0)) \\
&\leq \mathcal{F}(\mu(x_0, x_0), \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\}) \\
&\leq \mathcal{F}(1, \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\}).
\end{aligned}$$

This implies that

$$\begin{aligned}
S_p(x_1, x_1, x_1) &\leq \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\} \\
&= S_p(x_0, x_0, x_0).
\end{aligned}$$

By induction on n, we obtain:

$$S_p(x_{n+1}, x_{n+1}, x_{n+1}) \leq S_p(x_n, x_n, x_n).$$

Therefore, $\{S_p(x_n, x_n, x_n)\}_{\{n\geq 0\}}$ is a nonincreasing sequence. Define

$$r_0 := \lim_n S_p(x_n, x_n, x_n) = \inf_n S_p(x_n, x_n, x_n) \geq 0$$

and

$$M_0 := \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0).$$

Next, we show that $S_p(x_0, x_0, x_n) \leq M_0$, for any $n \geq 0$. If $n = 0$; the case is trivial. For $n = 1$ and using the fact that $k \in [0, 1)$ we deduce that $S_p(x_0, x_0, x_1) \leq \frac{2}{1-k} S_p(x_0, x_0, x_1) \leq \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0) = M_0$. So, we may assume that it is true for all $n \leq n_0 - 1$ and prove it for $n = n_0 \geq 2$.

$$\begin{aligned}
&S_p(x_0, x_0, x_{n_0}) \leq S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_1) + S_p(x_{n_0}, x_{n_0}, x_1) - S_p(x_1, x_1, x_1) \\
&\leq 2S_p(x_0, x_0, x_1) + S_p(x_1, x_1, x_{n_0}) \\
&\leq 2S_p(x_0, x_0, x_1) + \alpha(x_0, x_{n_0-1}) S_p(Tx_0, Tx_0, Tx_{n_0-1}) \\
&\leq 2S_p(x_0, x_0, x_1) + \max\{kS_p(x_0, x_0, x_{n_0-1}), S_p(x_0, x_0, x_0), S_p(x_{n_0-1}, x_{n_0-1}, x_{n_0-1})\} \\
&\leq 2S_p(x_0, x_0, x_1) + \max\{kS_p(x_0, x_0, x_{n_0-1}), S_p(x_0, x_0, x_0)\}.
\end{aligned}$$

Also, by induction assumption, we have $S_p(x_0, x_0, x_{n_0-1}) \leq \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0)$. So, we have

$$\begin{aligned}
&S_p(x_0, x_0, x_{n_0}) \leq 2S_p(x_0, x_0, x_1) \\
&+ \max\{\frac{2k}{1-k} S_p(x_0, x_0, x_1) + kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\} \\
&\leq 2S_p(x_0, x_0, x_1) + \frac{2k}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0) \\
&= \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0) = M_0.
\end{aligned}$$

Hence, we conclude that $S_p(x_0, x_0, x_n) \leq M_0$. Next, we need to show that

$$\lim_{n,m} S_p(x_n, x_n, x_m) = r_0.$$

For all n, m we have $S_p(x_n, x_n, x_m) \geq S_p(x_n, x_n, x_n) \geq r_0$. Let $\epsilon > 0$ find a natural number n_0 such that $S_p(x_{n_0}, x_{n_0}, x_{n_0}) < r_0 + \epsilon$ and $2M_0 k^{n_0} < r_0 + \epsilon$. Now for any $n, m \geq 2n_0$, since T is $R_\alpha-$admissible and using the fact that $\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1}) \leq 1$ we deduce that $\alpha(x_n, x_m) \geq 1, \mu(x_n, x_m) \leq 1$. Hence,

$$h(1, S_p(x_n, x_n, x_m) \leq h(\alpha(x_n, x_m), S_p(x_n, x_n, x_m)) \leq \mathcal{F}(\mu(x_n, x_m), \theta) \leq \mathcal{F}(1, \theta),$$

where

$$\theta = \max\{kS_p(x_{n-1}, x_{n-1}, x_{m-1}), S_p(x_{n-1}, x_{n-1}, x_{n-1}), S_p(x_{m-1}, x_{m-1}, x_{m-1})\}.$$

This implies that

$$\begin{aligned}
&S_p(x_n, x_n, x_m)\\
&\le \max\{kS_p(x_{n-1}, x_{n-1}, x_{m-1}), S_p(x_{n-1}, x_{n-1}, x_{n-1}), S_p(x_{m-1}, x_{m-1}, x_{m-1})\}\\
&\le \max\{k^2S_p(x_{n-2}, x_{n-2}, x_{m-2}), S_p(x_{n-2}, x_{n-2}, x_{n-2}), S_p(x_{m-2}, x_{m-2}, x_{m-2})\}\\
&\le \cdots \le \max\{k^{n_0}S_p(x_{n-n_0}, x_{n-n_0}, x_{m-n_0}), S_p(x_{n-n_0}, x_{n-n_0}, x_{n-n_0}),\\
&S_p(x_{m-n_0}, x_{m-n_0}, x_{m-n_0})\}\\
&\le r_0 + \epsilon.
\end{aligned}$$

Hence,

$$\lim_{n,m} S_p(x_n, x_n, x_m) = r_0.$$

Since (X, p) is a complete partial S-metric space; there exists $\widetilde{x} \in X$ such that

$$r_0 = S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = \lim_{n} S_p(\widetilde{x}, \widetilde{x}, x_n) = \lim_{n,m} S_p(x_n, x_n, x_m).$$

Now, we show that $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x})$. For each natural number n we have

$$S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x}) \le 2S_p(\widetilde{x}, \widetilde{x}, x_n) - S_p(x_n, x_n, x_n) + S_p(T\widetilde{x}, T\widetilde{x}, x_n).$$

Using the property that T is $\alpha-$contractive, we deduce that there exists a subsequence of natural numbers $\{n_l\}$ such that

$$\begin{aligned}
&h(1, S_p(T\widetilde{x}, T\widetilde{x}, x_{n_l}) \le h(\alpha(\widetilde{x}, x_{n_l-1}), S_p(T\widetilde{x}, T\widetilde{x}, x_{n_l}))\\
&\le \mathcal{F}(\mu(\widetilde{x}, x_{n_l-1}), \max\{kS_p(\widetilde{x}, \widetilde{x}, x_{n_l-1}), S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})\})\\
&\le \mathcal{F}(1, \max\{kS_p(\widetilde{x}, \widetilde{x}, x_{n_l-1}), S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})\}),
\end{aligned}$$

and thus

$$S_p(T\widetilde{x}, T\widetilde{x}, x_{n_l}) \le \max\{kS_p(\widetilde{x}, \widetilde{x}, x_{n_l-1}), S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})\}.$$

So, for $l \ge 1$, we have either $S_p(T\widetilde{x}, T\widetilde{x}, x_{n_l}) \le kS_p(\widetilde{x}, \widetilde{x}, x_{n_l-1})$ or less than or equal $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x})$ or less than or equal $S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})$. In all of these three cases, if we take the limit as l goes toward ∞ we get $S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x}) \le S_p(\widetilde{x}, \widetilde{x}, \widetilde{x})$. But, we know by the property (ii) of the partial S-metric space definition that $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) \le S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x})$. Therefore, $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x})$.

Now, we show that $X_{S_p}(\alpha, \mu)$ is nonempty. For each natural number l pick $x_l \in X$ with $\alpha(x_l, x_l) \ge 1$ and $S_p(x_l, x_l, x_l) < \rho_{S_p}(\alpha, \mu) + \frac{1}{l}$ and show that

$$\lim_{n,m} S_p(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_m) = \rho_{S_p}(\alpha, \mu).$$

Let $\epsilon > 0$ put $n_0 := (\frac{3}{\epsilon(1-k)})+1$ if $l \geq n_0$ then we have: $\rho_{S_p}(\alpha,\mu) \leq S_p(\widetilde{x}_l,\widetilde{x}_l,T\widetilde{x}_l)$ $\leq S_p(\widetilde{x}_l,\widetilde{x}_l,T\widetilde{x}_l) \leq r_{x_l} \leq S_p(\widetilde{x}_l,\widetilde{x}_l,T\widetilde{x}_l) < \rho_{S_p}(\alpha,\mu) + \frac{1}{l} \leq \rho_{S_p}(\alpha,\mu) + \frac{1}{n_0} <$ $\rho_{S_p}(\alpha,\mu) + \frac{\epsilon(1-k)}{3}$. Hence, we deduce that:

$$U_l := S_p(\widetilde{x}_l,\widetilde{x}_l,\widetilde{x}_l) - S_p(T\widetilde{x}_l,T\widetilde{x}_l,T\widetilde{x}_l) < \frac{\epsilon(1-k)}{3},$$

for $i \geq n_0$.

Also, if $l \geq n_0$, then $S_p(\widetilde{x}_l,\widetilde{x}_l,\widetilde{x}_l) = r_{x_l} \leq S_p(x_l,x_l,x_l) < \rho_{S_p}(\alpha) + \frac{1}{n_0}$. Which implies that $S_p(\widetilde{x}_l,\widetilde{x}_l,\widetilde{x}_l) \leq \rho_{S_p}(\alpha,\mu) + \frac{\epsilon(1-k)}{3}$ for all $l \geq n_0$. Now, if $n,m \geq n_0$, then $S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) \leq 2S_p(\widetilde{x}_n,\widetilde{x}_n,T\widetilde{x}_n) + S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_m) +$ $2S_p(T\widetilde{x}_m,T\widetilde{x}_m,\widetilde{x}_m) - S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_n) - S_p(T\widetilde{x}_m,T\widetilde{x}_m,T\widetilde{x}_m)$.

We know that $S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = S_p(\widetilde{x},\widetilde{x},T\widetilde{x})$ which implies that

$$\begin{aligned} &h(1, S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_m) \leq h(\alpha(\widetilde{x}_n,\widetilde{x}_m), S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_m)) \\ &\leq \mathcal{F}(\mu(\widetilde{x}_n,\widetilde{x}_m), \max\{kS_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m), S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_n), S_p(\widetilde{x}_m,\widetilde{x}_m,\widetilde{x}_m)\}) \\ &\leq \mathcal{F}(1, \max\{kS_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m), S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_n), S_p(\widetilde{x}_m,\widetilde{x}_m,\widetilde{x}_m)\}) \end{aligned}$$

Therefore,

$$\begin{aligned} &S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_m) \leq \max\{kS_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m), S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_n), S_p(\widetilde{x}_m,\widetilde{x}_m,\widetilde{x}_m), \\ &S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) \leq U_n + U_m + S_p(T\widetilde{x}_n,T\widetilde{x}_n,T\widetilde{x}_m) \\ &< U_n + U_m + \max\{kS_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m), S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_n), S_p(\widetilde{x}_m,\widetilde{x}_m,\widetilde{x}_m)\}. \end{aligned}$$

Hence,

$$\begin{aligned} &\rho_{S_p}(\alpha,\mu) \leq S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) \\ &\leq \max\{\frac{2}{3}\epsilon, \frac{2}{3}\epsilon(1-k) + S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_n), \frac{2}{3}\epsilon(1-k) + S_p(\widetilde{x}_m,\widetilde{x}_m,\widetilde{x}_m)\} \\ &\leq \max\{\frac{2}{3}\epsilon, \rho_{S_p}(\alpha,\mu) + \epsilon(1-k)\} < \rho_{S_p}(\alpha,\mu) + \epsilon. \end{aligned}$$

Thus,

$$\lim_{n,m} S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) = \rho_{S_p}(\alpha,\mu).$$

Since (X,S_p) is complete, there exists $a \in X$ such that,

$$S_p(a,a,a) = \lim_{n} S_p(a,a,\widetilde{x}_n) = \lim_{n,m} S_p(\widetilde{x}_n,\widetilde{x}_n,\widetilde{x}_m) = \rho_{S_p}(\alpha,\mu).$$

Therefore, we have $a \in X_{S_p}(\alpha,\mu)$ and thus $X_{S_p}(\alpha,\mu)$ is nonempty. This implies that, $Z_{S_p}(\alpha,\mu)$ is nonempty.

Now, let $x_0 \in Z_{S_p}(\alpha,\mu)$ be arbitrary. Then by the above argument we have

$$\rho_{S_p}(\alpha,\mu) \leq S_p(T\widetilde{x},T\widetilde{x},T\widetilde{x}) \leq S_p(\widetilde{x},\widetilde{x},T\widetilde{x}) = S_p(\widetilde{x},\widetilde{x},\widetilde{x}) = r_0 = \rho_{S_p}(\alpha,\mu).$$

Thus, $T\widetilde{x} = \widetilde{x}$, Now, assume that T has two fixed points $u, v \in Z_{S_p}(\alpha, \mu)$. By our hypothesis, we know that $\alpha(u, v) \geq 1, \mu(u, v) \leq 1$. Thus,

$$\begin{aligned}
h(1, S_p(u, u, v) &\leq h(\alpha(u, v), S_p(Tu, Tu, Tv)) \\
&\leq \mathcal{F}(\mu(u, v), \max\{kS_p(u, u, v), S_p(u, u, u), S_p(v, v, v)\}) \\
&\leq \mathcal{F}(1, \max\{kS_p(u, u, v), S_p(u, u, u), S_p(v, v, v)\}).
\end{aligned}$$

So we have,

$$S_p(u, u, v) \leq \max\{kS_p(u, u, v), S_p(u, u, u), S_p(v, v, v)\}.$$

Now, if $S_p(u, u, v) \leq kS_p(u, u, v)$ we deduce that $S_p(u, u, v) = 0$ and in this case $u = v$, or $S_p(u, u, v) \leq S_p(u, u, u) = S_p(v, v, v)$ and in this case by condition (ii) of the definition of the partial S-metric space we obtain $S_p(u, u, v) = S_p(u, u, u) = S_p(v, v, v)$ and hence by condition (i) of the same definition we conclude that $u = v$. Therefore, we obtain the uniqueness as desired. □

As a consequence of the above result, the following corollary follows easily.

Corollary 2.14. *Let (X, S_p) be a 0-complete partial S-metric space, $k \in [0, 1)$ and consider the map $T : X \longrightarrow X$ to be α-admissible and R_α-admissible,and there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$, also for every $x, y \in X$ we have $\alpha(x, y)S_p(Tx, Tx, Ty) \leq kS_p(x, x, y)$. Then there exists $\widetilde{x} \in X$ such that $T\widetilde{x} = \widetilde{x}$.*

Proof. Using the same technique and notation in the proof of Theorem 2.13, we deduce that $S_p(x_n, x_n, x_n) \leq \alpha(x_n, x_n)S_p(x_n, x_n, x_n) \leq k^n S_p(x_0, x_0, x_0)$. Thus,

$$r_0 = S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = lim_n S_p(\widetilde{x}, \widetilde{x}, x_n) = lim_{n,m} S_p(x_n, x_n, x_m) = 0.$$

This implies that $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = 0$. Since $S_p(\widetilde{x}, \widetilde{x}, \widetilde{x}) = S_p(\widetilde{x}, \widetilde{x}, T\widetilde{x}) = 0$, we have $\widetilde{x} = T\widetilde{x}$ as required. □

In closing, we change the contraction principle in Theorem 2.13, to show that there exist a unique fixed point in the whole space X.

Theorem 2.15. *Let (X, S_p) be a complete partial S-metric space, $k \in [0, 1)$ and assume the there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$. Consider the map $T : X \longrightarrow X$ to be $\alpha-$admissible and $R_\alpha-$admissible. Assume that for every $x, y \in X$ we have*

$$\alpha(x, y)S_p(Tx, Tx, Ty) \leq \max\{kS_p(x, x, y), \frac{S_p(x, x, x) + S_p(y, y, y)}{2}\}, \tag{2.2}$$

then there exists a unique $u \in X$ such that $Tu = u$.

Proof. Note that, for every $x, y \in X$ we have:

$$\begin{aligned}
\alpha(x, y)S_p(Tx, Tx, Ty) &\leq \max\{kS_p(x, x, y), \frac{S_p(x, x, x) + S_p(y, y, y)}{2}\} \\
&\leq \max\{kS_p(x, x, y), S_p(x, x, x), S_p(y, y, y)\}.
\end{aligned}$$

Thus, all conditions of Theorem 2.13 are satisfied. Hence, there exists $u \in X$ such that $Tu = u$. Assume that there exist two fixed points $u, v \in X$ for T such that $\alpha(u, v) \geq 1$. Hence,

$$\begin{aligned} S_p(u,u,v) &= S_p(Tu,Tu,Tv) \leq \alpha(u,v)S_p(Tu,Tu,Tv) \\ &\leq \max\{kS_p(u,u,v), \frac{S_p(u,u,u)+S_p(v,v,v)}{2}\}. \end{aligned}$$

Thus, we either have $S_p(u,u,v) \leq kS_p(u,u,v)$ which implies that $S_p(u,u,v) = 0$ and hence $u = v$, or $0 = 2S_p(u,u,v) - S_p(u,u,u) - S_p(v,v,v)$ which also implies that $u = v$ as desired. □

Example 3. Let (X, S_p) be a partial S-metric space, where $X = [0,1] \cup [2,3]$ and the partial S-metric space $S_p : X^3 \longrightarrow [0, +\infty)$ is defined by

$$S_p(x,y,z) = \begin{cases} \|\max\{x,y\} - z\|, & \text{if } \{x,y,z\} \cap [2,3] \neq \emptyset \\ |x - y - z|, & \text{if } \{x,y,z\} \subset [0,1]. \end{cases}$$

Define the functions $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ as follows $Tx = \frac{x+1}{2}$ if $0 \leq x \leq 1$, $T2 = 1$, and $Tx = \frac{x+2}{2}$ if $2 < x \leq 3$,

$$\alpha(x,y) = \begin{cases} e^{x-y}, & \text{if } x \geq y \\ 0, & \text{if } x < y. \end{cases}$$

It is easy to see that T is $\alpha-$admissible and $R_\alpha-$admissible. Note that, we can always pick our x, y and z such that $\max\{x,y\} > z$. Also T is an increasing function. So, for every $x \geq y \in X$ we have:

$$S_p(Tx,Tx,Ty) \leq \alpha(x,y)S_p(Tx,Tx,Ty) \leq \frac{1}{2}S_p(x,x,y),$$

if $x, y \in [0,1]$, and

$$S_p(Tx,Tx,Ty) \leq \alpha(x,y)S_p(Tx,Tx,Ty) \leq \frac{S_p(x,x,x)+S_p(y,y,y)}{2},$$

$\{x,y\} \cap [2,3] \neq \emptyset$.

One can verify that the function T in this example satisfies the conditions of Theorem 2.15 and the unique fixed point will be 1.

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