Comparing study between simplex method and Lagrange method in a linear programming problem

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Abstract. This study aims to discuss a different way to solve a linear programming problems. Two methods are discussed in this paper to determine a suitable method to solve these problems, and to determine which one is the easiest. We used : Simplex method and Lagrange method. Two methods were applied in general system to evaluate the result and compare between them. After that the researchers applying the numerical example to find the degree of readers satisfaction for these methods. The results of this study indicated that a two methods were a better, but the second method is easier than the first method.

Keywords: operation research, Lagrange multiplier, simplex method.

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1. Introduction

Statistical sciences specially operations research is related with different fields as mathematics, statistics, economics, psychology, engineering such as (Alsaraireh, et al., 2018; AL Wadi, et al., 2018). Also operation research used to make a new decision. Recently operation research became a professional science with respect to other science. The commerce field is developing and growing rapidly, and increasing the aware competency between all organizations in the world, and striving to be more successes, survival and achieving to a competitive advantage. Therefore, all organizations seek to minimize the cost and time, and increase their margin profit. For these reasons, this study aims to link between the solution of a linear programming problem by two methods (Simplex method and Lagrange method). For more details about the lagrange theorem refer to (Mangazarian, 1969), and (Rockefeller, 1970). In (Bussotti, P., 2003), Joseph-Louis Lagrange introduced the multiplier method in classical mechanics in his book. In optimization the multipliers are used in solving constrained optimization problems and study the stationary points of the Lagrange function.

2. Literature review

Operations research as a science has been used to help solve decision problems using mathematical and statistical models for a long time, and it has been developed in many scientific fields such as : Mathematics, Engineering and Management. It is one of the areas that has contributed to solving many of mathematical problems and management problems: simplex method and Lagrange method. The first book of operations researches appeared in 1946 As "Methods of Research Operations" for Morris and Campbell's. Some of the scientists developed a method of problem solving in the Simplex Method and other methods. In (Khachian, 1979) proposed a new method of solving the linear program, but theoretically only. In (Karmarkar, 1984) developed a new polynomial-time algorithm for linear programming and introduced an algebraic method with high results but the rest of the simplex is the easiest. Also (Holman and Robert, 1995) introduced a new application about the linear programming and applied a different branch in operation research. The applications continued to be widely period-intensive until (Hillier, 2001), (Hamdy, 2007), and others presented a new application in operation research. In (Vaidya and Kasturiwale, 2016) discussed a new approach while solving two phase simplex method, and they discussed this subject with respect to a number of iteration. Lagrange multipliers is a strategy for finding the local maxima and minima of a objective function subject to constraints. The method is very important because it allows the optimization to be solved without explicit parametrization in terms of the constraints. The method of Lagrange multipliers is used to solve challenging constrained optimization problems. In (C. Simon and L. Blume, 1994), the function f is a production function, it is related with several constraints and

Lagrange multipliers, and the Lagrange multipliers are interpreted to imputed value or shadow prices for production. Also, for more comparing between Simplex method and other methods see (Ahmed Alsaraireh and et al, 2018), and (Mohammad Almasarweh and et. Al, 2018). There are an excellent studies for discussing the solution as the gradient method, Hessian and other types of analysis, if we found a maximum, minimum or saddle point. For more details, see (Powell, M.J.D., 1978), (Liu, D.C. and Nocedal, J.,1989), (Fletcher, R. 1987) and (Ford, J.A. and Tharmlikit, S., 2003). Since the applications remained separate and all studies discussed the evolution of simplex method, this study attempts to investigate the application of more than one mathematical model in problem solving and work on discussion. In the pricing problem, the multipliers are not unique when the solution involves bunching (where the firm has not enough dimensions to differentiate the product). Through the previous studies were based on separate applications and did not apply some of the methods on the same problem. Therefore, it is necessary to apply mathematical methods in order to compare the results and determine the most appropriate method of the solution.

The paper is organized in two sections, section 3 is devoted to general formulate and discuss some steps of methods. In section 4, numerical example for problems are discussed.

3. Problem formulation

A linear programming problem is very important in a different fields as: Mathematics, Engineering, management, and others. Any linear programming problems consists from an objective function with a single variable or multivariable's, and the constraints with linear equalities or linear inequalities. The computation is a simple of a linear programming problem with respect to a nonlinear programming problem, but we used Lagrange method for a linear programming problem to explain that method gives the result in a short steps. Now , Denote P1 , P2 and P3 are systems of linear programming problems.

$$p_1 : \begin{cases} \text{Max.}: \ f(x_1, x_2, ..., x_n) \\ \text{Constrints: } g_i(x_1, x_2, ..., x_n \geq b_i, i = 1, 2, ..., m \\ x_1, x_2, ..., x_n \geq 0 \end{cases}$$

$$p_2 : \begin{cases} \text{Max.}: \ f(x_1, x_2, ..., x_n) \\ \text{Constrints: } g_i(x_1, x_2, ..., x_n \leq b_i, i = 1, 2, ..., m \\ x_1, x_2, ..., x_n \geq 0 \end{cases}$$

and we can discuss a mixture system such that the constraints consists from equalities and inequalities.

$$p_1 : \begin{cases} \text{Max.}: \ f(x_1, x_2, ..., x_n) \\ \text{Constraints:} \\ g_i(x_1, x_2, ..., x_n \leq b_i, i = 1, 2, ..., m \\ h_i(x_1, x_2, ..., x_n \leq k_l, l = 1, 2, ..., r \\ Q_i(x_1, x_2, ..., x_n = t_t, t = 1, 2, ..., w \\ x_1, x_2, ..., x_n \geq 0 \end{cases} \quad , n, m, r, w \in R^+.$$

To find the solution of these systems, we can use a Simplex method and Lagrange method. So, we will present a general solution for P1 by two methods.

Method 1: Simplex method

In this section we will discuss a general method to solve a linear programming problem by a famous method in operation research that is called Simplex method.

1.1. Minimization problem

We will discuss all steps of this type to find an optimal solution. 1. Prepare a problem in a standard form

$$Z - f(x_1, x_2, ..., x_n) = 0, \ \ g_i(x_1, x_2, ..., x_n) + S_i - u_i = b_i, i = 1, 2, ..., m.$$

2. Fill the problem in a Simplex table. 3. Determine the pivot column and row,

Basis	X_1	x_2	....	x_n	s_1	s_2 ...... s_m	u_1 .... u_m	RHS
u_1	a_{11}	a_{12}	....	a_{1n}	1	0 ... 0	-1 0	b_1
⋮	.							⋮
⋮	.							⋮
u_m	a_{m1}	a_{m2}	....	a_{mn}	0	0 1	0 0	b_m
z	c_1	c_2	....	c_n	0	0 0	0 0	Z_0

then apply basic steps to solve this table. From previous method we notice that ,we need a long calculations.

1.2. Maximization problem

In this section we will discuss a general method to solve a linear programming problem by a famous method in operation research that is called Simplex method.

1. Prepare a problem in a standard form

$$Z - f(x_1, x_2, ..., x_n) = 0, \ \ g_i(x_1, x_2, ..., x_n) + S_i = b_i, i = 1, 2, ..., m$$

Basis	X_1	X_2	....	X_n	S_1	S_2	......	S_m	RHS
u_1	a_{11}	a_{12}	....	a_{1n}	1	0	...	0	b_1
$\vdots$	.								$\vdots$
	.								
u_m	a_{m1}	a_{m2}	....	a_{mn}	0	0	....	1	b_m
z	c_1	c_2	....	c_n	0	0	....	0	Z_0

2. Fill the problem in a Simplex table

3. Determine the pivot column and row, then apply basic steps to solve this table. From previous method we notice that ,we need a long calculations.

Method 2: Lagrange methods

2.1. Minimization problem Suppose we are given a general optimization problem,

$$p_1 : \begin{cases} MIN. : f(x_1, x_2, ..., x_n) \\ \text{Constrints: } g_i(x_1, x_2, ..., x_n) \geq b_i, i = 1, 2, ..., m \\ x_1, x_2, ..., x_n \geq 0, \end{cases}$$

where $x_1, x_2, ..., x_n \in R^n, b_i \in R^m$. The Lagrangian is $L(x_j, \lambda_i) = f(x_j) + \sum_{i=1}^{m} \lambda_i[g_i(x_j) - b_i], with \lambda_i \in R^m$. Each component of λ_i is called a Lagrange multiplier.

2.2. Maximization problem Suppose we are given a general optimization problem,

$$p_1 : \begin{cases} Max. : f(x_1, x_2, ..., x_n) \\ \text{Constrints: } g_i(x_1, x_2, ..., x_n) \leq b_i, i = 1, 2, ..., m \\ x_1, x_2, ..., x_n \geq 0, \end{cases}$$

where $x_1, x_2, ..., x_n \in R^n, b_i \in R^m$. The Lagrangian is $L(x_j, \lambda_i) = f(x_j) - \sum_{i=1}^{m} \lambda_i[g_i(x_j) - b_i], with \lambda_i \in R^m$.

Each component of λ_i is called a Lagrange multiplier.

The following theorem is simple to prove, and extremely useful in practice.

Theorem (Lagrangian sufficiency theorem). *If x^* and λ_i^* exist such that x^* is feasible for p_1 and ?????? , for all ???????, then ?????????? is optimal for ??????.*

Now, we will discuss a general steps to apply Lagrange method. we now verify that the Kuhn Tucker conditions hold.

1. Write the Lagrange form

$$L(x_j, \lambda_i) = f(x_j) + \sum_{i=1}^{m} \lambda_i [g_i(x_j) - b_i].$$

2. Find

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial}{\partial x_j} \sum_{i=1}^{m} \lambda_i [g_i(x_j) - b_i] = 0.$$

3. Find

$$\frac{\partial L}{\partial \lambda_i} = \frac{\partial f}{\partial \lambda_i} + \frac{\partial}{\partial \lambda_i} \sum_{i=1}^{m} \lambda_i [g_i(x_j) - b_i] = 0.$$

4. Find

$$\bar{\lambda}_i [g_i(\bar{x}_j)] = 0.$$

After these steps , discuss all cases of λ_i or we can apply a simple matrix.

4. Numerical example

The following system represents a problem in three variable and three constraints, where the variables represent the different items in a management sector.

$$p : \begin{cases} Z = 30x_1 + 50x_2 \\ \text{Constraints:} \\ 2x_1 + x_2 \leq 60, \\ x_1 + x_2 \leq 40, \\ 2x_1 + 3x_2 \leq 70 \\ x_1, x_2 \geq 0 \end{cases}.$$

the first solution will be by Simplex method.

The solution is: $x_1 = 0, x_2 = \frac{70}{3}$.

Now we use Lagrange multipliers to find the solution in a simple steps. This method is very important, because it is connected between two important courses in a management as: Mathematics for Business and Operation research.

$$p : \begin{cases} Z = 30x_1 + 50x_2 \\ \text{Constraints:} \\ 2x_1 + x_2 \leq 60, \\ x_1 + x_2 \leq 40, \\ 2x_1 + 3x_2 \leq 70 \\ x_1, x_2 \geq 0 \end{cases}.$$

	x_1	x_2	S_1	S_2	S_3	R.H.S
S_1	2	1	1	0	0	60
S_2	1	1	0	1	0	40
S_3	2	3	0	0	1	70
z	-30	-50	0	0	0	0

	x_1	x_2	S_1	S_2	S_3	R.H.S
S_1	$4/3$	1	1	0	$-1/3$	$110/3$
S_2	$1/3$	0	0	1	$-1/3$	$50/3$
S_3	$2/3$	0	-1	0	$1/3$	$70/3$
z	$10/3$	0	0	0	$50/3$	$3500/3$

The previous example consists a linear objective function and three constraints with two variables.

The Lagrangian function of this problem is:

$$L(x_j, \lambda_i) = 30x_1 + 50x_2 - \lambda_1[2x_1 + x_2 - 60] - \lambda_2[x_1 + x_2 - 40] - \lambda_3[2x_1 + 3x_2 - 70].$$

Next, find the gradient ∇L equal to zero. First, we handle the partial derivative with respect to x_1, x_2 and λ_i

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, \\ \dfrac{\partial L}{\partial x_2} = 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0, \end{cases}$$

$$\begin{cases} \dfrac{\partial L}{\partial \lambda_1} = 2x_1 + x_2 - 60 = 0, \\ \dfrac{\partial L}{\partial \lambda_2} = x_1 + x_2 - 40 = 0 \\ \dfrac{\partial L}{\partial \lambda_3} = 2x_1 + 3x_2 - 70 = 0, \end{cases}$$

$$\begin{cases} \lambda_1[2x_1 + x_2 - 60] = 0, \\ \lambda_1[x_1 + x_2 - 40] = 0, \\ \lambda_1[2x_1 + 3x_2 - 70] = 0, \end{cases}$$

complementary slackness conditions.Now we will discuss all cases of λ_i.

Case 1. If $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0,$

$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_2 = 20\lambda_3 = -10$, reject.

Case 2. If $\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 \neq 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_1 = -2.5, \lambda_3 = 17.5$.
No feasible points in this case.

Case 3. If $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_1 = -20, \lambda_2 = 70$. No feasible points in this case.

Case 4. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_3 = 16$, accept.
The feasible point to be a solution in this case is $(0, 23.3)$,because it is satisfied a conditions.$Z(0, 23.3) = 1165$.

Case 5. If $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_2 = 40$, reject.
No feasible points in this case.

Case 6. If $\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_1 = 80/3$, reject.
The feasible point to be a solution in this case is $(30, 0)$,because it is satisfied a conditions.$Z(30, 0) = 900$.

Case 7. If $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$,
$\langle 30 - 2\lambda_1 - \lambda_2 - 2\lambda_3 = 0, 50 - \lambda_1 - \lambda_2 - 3\lambda_3 = 0\rangle \Rightarrow \lambda_i > 0$, accept.
The feasible point to be a solution in this case is $(27.5, 5)$,because it is satisfied a conditions.$Z(27.5, 5) = 1075$.

Case 8. If $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$.
Here we will use the original constraints.

From previous cases, we notice that the optimal solution of this problem at $x_1 = 0, x_2 = 23.3$ For more application, the reader can apply the following problem.

$$p : \begin{cases} Z = 3x_1 + 4x_2 + x_3 \\ \text{Constraints:} \\ x_1 + 2x_2 + x_3 \leq 6, \\ 2x_1 + 2x_3 \leq 4, \\ 3x_1 + x_2 + x_3 \leq 9 \\ x_1, x_2, x_3 \geq 0 \end{cases}.$$

5. Conclusion

In this study we discussed the application of simplex method and Lagrange method in Linear programming problem. To use simplex method, a given linear programming problem model needs to be in standard form, but in Lagrange model, we can solve a linear programming problem in any form. We presented the basic techniques of the previous methods. Also, we explained the main steps and the important conditions in Lagrange model. Lagrange model is important to solve linear and non-linear programming problem. Also we discussed an example on a linear programming problem.

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