Derivations and differential filters on semihoops

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Abstract. In this paper, we introduce the notions of derivations and differential filters on semihoops and investigate some related properties of them. Also we discuss related properties of some particular derivations and give some characterizations of ideal derivations. Then we obtain that the set of all fixed points of a semihoop for an ideal idempotent derivation is a semihoop, the set of all fixed points of a $\vee$-hoop for an ideal derivation is an ideal. And for every non-trivial prime ideal I of a semihoop A, there exists a derivation d such that $Fix_d(A) = I$. Finally we find some relations between differential filters and other filters. It is proved that every maximal differential filter is a prime differential filter and the set of all differential filters of an ideal differential $\vee$-semihoop is a Heyting algebra.
Keywords: semihoop, derivation, differential filter, fixed point.

1. Introduction

As we all known, various fuzzy logic algebras have been widely introduced and studied as non-classical logic semantic systems. These are logical algebras based on continuous t-norm, such as: MV-algebras, BL-algebras, Hoops. From the point of view of algebras, hoops are naturally ordered commutative residuated integral monoids, introduced by B.Bosbach in [5, 6]. In 2003, semihoop was introduced by Francesc Esteva and Lluis Godo in [9], it was a generalization of a hoop. In recent years, the theory of hoops and semihoops had a lot of achievements. These conclusions were helpful to further study fuzzy logic. As a more general structure, semihoops have no the separability, that is, $x \wedge y = x \odot (x \to y)$ does not hold. Semihoops have attracted the attention and research of many scholars at home and abroad since it was put forward.

The notion of derivations introduced from the analytic theory, which is helpful for the research of structures and properties in algebraic systems. Several authors [3, 1, 2, 15] have studied derivations in rings and near rings. Jun and Xin applied the notion of derivation in rings and near rings theory to BCI-algebras in [12], and as a result they introduced a new concept, called a regular derivation, in BCI-algebras. Xin introduced the concept of derivations in lattices and

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characterized modular lattices and distributive lattices by isotone derivations in [16]. Later, He, Xin and Zhan studied the relations between derivations and their fixed point sets in residuated lattice in [11]. Recently Liang and Xin discussed the derivations on EQ-algebra in [14]. Although a number of researchers have been studied derivations in algebraic systems. However, none of them apply derivations to characterize algebraic structure of semihoops. Inspired by this, we want to introduce and study derivations in semihoops. Therefore it is meaningful to study this.

The filter theory of the logical algebras plays an important role in studying these algebras and the completeness of the corresponding non-classical logics. From a logical point of view, various filters correspond to various sets of provable formulas. Hence, we define the differential filter by derivation and study the relationships among different differential filters.

This paper is organized as follows: in Section 2, we review some necessary notions and properties, which will be used in the next section. In Section 3, we define derivation, ideal derivation and regular derivation on semihoops, investigate some properties and relations of them. Moreover, we discuss the structures and properties of the fixed point sets of ideal derivations. For example, the set of all fixed points of a semihoop for an ideal idempotent derivation is a semihoop; the set of all fixed points of a $\vee$-hoop for an ideal derivation is an ideal; and for every non-trivial prime ideal I of a semihoop A, there exists a derivation d such that $Fix_d(A) = I$. In Section 4, we define the differential filter on semihoops and discuss some relations of different differential filters, we prove that every maximal differential filter is a prime differential filter and the set of all differential filters of an ideal differential $\vee$-semihoop forms a Heyting algebra.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper and we shall not cite them every time they are used.

Definition 2.1 ([9]). *An algebra $(A, \odot, \rightarrow, \wedge, 1)$ of type (2,2,2,0) is called a semihoop if it satisfies the following conditions:*

$(SH1)$ $(A, \wedge, 1)$ *is a $\wedge$-semilattice with upper bound 1;*

$(SH2)$ $(A, \odot, 1)$ *is a commutative monoid;*

$(SH3)$ $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$*, for all $x, y, z \in A$.*

On a semihoop A we define a binary relation $'' \leq''$, where $x \leq y$ if and only if $x \rightarrow y = 1$. It is easy to see that $'' \leq''$ is a partial order relation on A and 1 is the upper bound, that is, for any $x \in A, x \leq 1$.

A semihoop A is bounded if there exists an element $0 \in A$, such that $0 \leq x$, for all $x \in A$. We let $x^0 = 1$, $x^n = x^{n-1} \odot x$ for all $n \in N$. In a bounded semihoop A, we define the negation $''*''$ on A by $x^* = x \rightarrow 0$ for all $x \in A$. A semihoop is called a hoop if it satisfies $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$, for all $x, y \in A$.(see [9])

Proposition 2.2 ([9]). *Let A be a semihoop. Then the following properties hold for all $x, y, z \in A$:*

(1) $x \odot y \leq z$ *if and only if* $x \leq y \to z$;
(2) $x \leq y$ *implies* $x \odot z \leq y \odot z$;
(3) $x \odot (x \to y) \leq y$;
(4) $x \to 1 = 1$, $1 \to x = x$, $x \to x = 1$;
(5) $x \odot y \leq x, y$;
(6) $x \leq (x \to y) \to y$;
(7) $x \leq y$ *implies* $z \to x \leq z \to y$, $y \to z \leq x \to z$.

Proposition 2.3 ([7]). *Let A be a bounded semihoop. Then the following properties hold for all $x \in A$:*

(1) $1^* = 0$, $0^* = 1$;
(2) $x \leq x^{**}$;
(3) $x \odot x^* = 0$;
(4) $x^{***} = x^*$.

Proposition 2.4 ([7]). *Let A be a semihoop. We define for any $x, y \in A$:*

$$x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x).$$

Then the following conditions are equivalent:

(1) $\sqcup$ *is associative;*
(2) $x \leq y$ *implies* $x \sqcup z \leq y \sqcup z$ *for all* $x, y, z \in A$;
(3) $x \sqcup (y \wedge z) \leq (x \sqcup y) \wedge (x \sqcup z)$ *for all* $x, y, z \in A$;
(4) $\sqcup$ *is the join operation on A.*

Definition 2.5 ([7]). *A $\vee$-semihoop is a semihoop if it satisfies one of the equivalent conditions of Proposition 2.4.*

Proposition 2.6 ([10]). *Let A be a $\vee$-semihoop. For any $x, y, z \in A$, suppose that $x \vee y$ exists. Then:*

(1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(2) *For any* $n \in N_+$, $(x \vee y)^n \to z = \bigwedge\{(x_1 \odot \cdots \odot x_n) \to z | x_i \in \{x, y\}, i \in N_+\}$

Definition 2.7 ([7]). *Let A be a semihoop and F be a non-empty subset of A. F is called a filter of A if it satisfies the following conditions for every $x, y \in A$:*

$(F1)$ $x, y \in F$ *implies* $x \odot y \in F$;
$(F2)$ $x \in F$ *and* $x \leq y$ *imply* $y \in F$.

We denote the set of all filters of A by $\mathcal{F}(A)$.

Proposition 2.8 ([7]). *Let A be a semihoop and F be a nonempty subset of A. Then F is a filter if and only if the following conditions hold:*

(1) $1 \in F$;
(2) *For all* $x, y \in A$, *if* $x, x \to y \in F$ *then* $y \in F$.

Definition 2.9 ([8]). *An ideal is a non-void subset I of a lattice L with the properties for every $x, y \in L$:*

(1) $x \in I$ *and* $y \leq x$ *imply* $y \in I$;

(2) $x, y \in I$ *implies* $x \vee y \in I$.

Moreover, an ideal I is called a prime ideal if it satisfies the following condition:

(3) $x \wedge y \in I$ *implies* $x \in I$ *or* $y \in I$.

Proposition 2.10 ([10]). *Let A be a $\vee$-hoop. Then $(A, \wedge, \vee)$ is a distributive lattice.*

Definition 2.11 ([7]). *Let A be a semihoop and $X \subseteq A$. The filter of A generated by X will be denoted by $\langle X \rangle$, where $\langle X \rangle = \{a \in A | a \geq x_1 \odot x_2 \odot \cdots \odot x_n$, for some $x_i \in X$ and some $n \in N_+\}$.*

Proposition 2.12 ([7]). *Let A be a semihoop, $F \in \mathcal{F}(A)$ and $x \in A$. Then $\langle F \cup \{x\} \rangle = \{a \in A | y \odot x^n \leq a$, for some $n \in N, y \in F\}$.*

Definition 2.13 ([4]). *A lattice $(L, \vee, \wedge)$ is called a Heyting algebra if for any $x, y \in L$, there exists an element $x \to y \in L$ such that $z \leq x \to y$ if and only if $z \wedge x \leq y$ for all $z \in L$.*

From now on, we denote the semihoop $(A, \odot, \to, \wedge, 1)$ by A, unless otherwise stated.

3. Derivations on semihoop

In this section, we introduce the notion of derivations in semihoops and investigate some related properties of regular, isotone, contractive and ideal derivations, respectively. Also, we give some equivalent conditions under which a derivation is an ideal derivation. Moreover, we discuss the structures and properties of the fixed point sets of ideal derivations.

In order to study derivations, we first discuss some properties of the $''\sqcup''$ operation.

Proposition 3.1. *Let A be a bounded semihoop. Then the following properties hold for any $x, y \in A$:*

(1) $x \sqcup y \geq x, y$;

(2) $x \leq y$ *if and only if* $x \sqcup y = y$;

(3) $x \sqcup 1 = 1$, $x \sqcup x = x$;

(4) $x \sqcup y = 0$ *if and only if* $x = 0$ *and* $y = 0$.

Proof. (1) Since $y \to x \leq 1 = x \to x$ for all $x, y \in A$, by Proposition 2.2(1) we have $x \odot (y \to x) \leq x$, that means $x \leq (y \to x) \to x$. Again, by Proposition 2.2(6) $x \leq (x \to y) \to y$. Hence we obtain $x \leq ((x \to y) \to y) \wedge ((y \to x) \to x) = x \sqcup y$. Similarly $y \leq x \sqcup y$.

(2) Suppose that $x \leq y$ for some $x, y \in A$. Then by Proposition 2.2(4), (6) we have $x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x) = (1 \to y) \wedge ((y \to x) \to x) = y \wedge ((y \to x) \to x) = y$. Conversely, if $x \sqcup y = y$, by (1) $x \sqcup y \geq x$ we have $y \geq x$.

(3) By (2) we can get $x \sqcup 1 = 1$, $x \sqcup x = x$.

(4) Let $x \sqcup y = 0$. By (1) $x \sqcup y \geq x, y$, we have $x \leq 0$, $y \leq 0$, that is, $x = 0$, $y = 0$. Conversely, if $x = 0$ and $y = 0$, we can obtain that $x \sqcup y = 0 \sqcup 0 = 0$. □

The following example shows that $x \sqcup y$ is just an upper bound of $\{x, y\}$, not the supremum.

Example 3.2 ([13])**.** Suppose $A = \{0, a, b, c, 1\}$, with $0 < a$, $b < c < 1$. We define $\odot$ and $\to$ as follows,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(A, \odot, \to, \wedge, 1)$ is a bounded semihoop, where $x \wedge y = x \odot y$ for all $x, y \in A$. By calculation we have $a \sqcup b = 1$, while $a \vee b = c$.

Now we begin to introduce the definition of derivations and study some related properties.

Definition 3.3. *Let A be a semihoop and $d : A \to A$ be a function. d is called a derivation on A, if it satisfies the condition: for any $x, y \in A$,*

$$d(x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y)).$$

If we define a function $d : A \to A$ by $d(x) = 0$, for all $x \in A$, then d is a derivation on A, which is called a zero derivation. If we define $d(x) = x$, for all $x \in A$, then d is a derivation on A, which is called an identity derivation.

Here we give some examples of derivations on a semihoop A.

Example 3.4. (1) In Example 3.2, for all $x \in A$, we define:

$$d_1(x) = \begin{cases} 0, & \text{if } x = 0, 1 \\ a, & \text{if } x = a \\ b, & \text{if } x = b \\ c, & \text{if } x = c \end{cases}$$

$$d_2(x) = \begin{cases} 0, & \text{if } x = 0, a \\ b, & \text{if } x = b, c, 1. \end{cases}$$

It is easy to verify that d_1, d_2 are all derivations on A.

(2) Given that $I = [0, 1]$ is the unit interval. For all $x, y \in I$, we define $\odot$ and $\to$ as follows:

$$x \odot y = x \wedge y = \min\{x, y\}$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases}$$

Then $(I, \odot, \to, \wedge, 1)$ is a bounded semihoop. We define $d(x) = a \odot x$ for all $a \in [0, 1]$, one can easily check that d is a derivation on I.

Proposition 3.5. *Let A be a bounded semihoop and d be a derivation on A. Then the following properties hold for all $x, y \in A$:*

(1) $d(0) = 0$;

(2) $d(x) \odot x^* = x \odot (d(x))^* = 0$;

(3) $x \odot d(1) \leq d(x) \leq x^{**}$;

(4) $d(x^n) = x^{n-1} \odot d(x)$, *for any* $n \in N_+$;

(5) $x \leq y^*$ *implies* $d(y) \leq x^*$ *and* $d(x) \leq y^*$;

(6) $d(x^*) \leq (d(x))^*$.

Proof. (1) Since d is a derivation on A, we conclude that $d(0) = d(0 \odot 0) = (d(0) \odot 0) \sqcup (0 \odot d(0)) = 0 \sqcup 0 = 0$.

(2) By Proposition 2.3(3) $x \odot x^* = 0$ we have $0 = d(0) = d(x \odot x^*) = (d(x) \odot x^*) \sqcup (x \odot (d(x))^*)$. So we get $d(x) \odot x^* = 0$ and $x \odot (d(x))^* = 0$.

(3) For all $x \in A$, we have $d(x) = d(x \odot 1) = (d(x) \odot 1) \sqcup (x \odot d(1)) = d(x) \sqcup (x \odot d(1))$. Thus by Proposition 3.1(2) we can get $x \odot d(1) \leq d(x)$. By (2) $d(x) \odot x^* = 0$ we obtain $d(x) \leq x^{**}$.

(4) Let $n = 1$ we have $d(x) = x^0 \odot d(x) = d(x)$. Suppose the equation is established when $n = k$. That is, $d(x^k) = x^{k-1} \odot d(x)$. Now we will prove that the equation is established when $n = k+1$. By Definition 3.3 we have $d(x^{k+1}) = d(x^k \odot x) = (d(x^k) \odot x) \sqcup (x^k \odot d(x)) = ((x^{k-1} \odot d(x)) \odot x) \sqcup (x^k \odot d(x)) = x^k \odot d(x)$. Therefore we can obtain $d(x^n) = x^{n-1} \odot d(x)$.

(5) For all $x, y \in A$, assume that $x \leq y^*$. By Proposition 2.2(1), we have $x \odot y = 0$. Then $0 = d(0) = d(x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y))$. It follows from Proposition 3.1(4) that $d(x) \odot y = 0$ and $x \odot d(y) = 0$. Thus we have $d(x) \leq y^*$ and $d(y) \leq x^*$.

(6) By Proposition 2.3(2) $x \leq x^{**}$, we infer $d(x) \leq x^{**}$ by the statement (5), which implies $d(x^*) \leq x^{***}$, by Proposition 2.2(7) we can obtain $x^{***} \leq (d(x))^*$. Therefore, we conclude that $d(x^*) \leq (d(x))^*$. □

In what follows, we introduce the ideal derivation in semihoops and investigate some related properties of them.

Definition 3.6. *Let d be a derivation on a semihoop A.*

(1) *d is called an isotone derivation provided that $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in A$;*

(2) *d is called a contractive derivation provided that $d(x) \leq x$ for all $x \in A$;*

(3) *d is called an ideal derivation provided that d is both an isotone and a contractive derivation;*

(4) *d is called an idempotent derivation provided that $d^2 = d$, where $d^2(x) = d(d(x))$ for all $x \in A$.*

Example 3.7. (1) In Example 3.4(1), d_1 is a contractive derivation, however, d_1 is not an isotone derivation, since $b \leq 1$, $b = d_1(b) \nleq d_1(1) = 0$. One can easily check that d_2 is an ideal derivation.

(2) Let $A = \{0, a, b, 1\}$ and $0 < a < b < 1$. For all $x, y \in A$, we define $x \wedge y = min\{x, y\}$, $\odot$ and $\rightarrow$ as follows,

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	b	b	1	1
1	0	a	b	1

Then $(A, \odot, \rightarrow, \wedge, 1)$ is a bounded semihoop. We define:

$$d_1(x) = \begin{cases} 0, & \text{if } x = 0 \\ b, & \text{if } x = a, b, 1 \end{cases}$$

$$d_2(x) = \begin{cases} 0, & \text{if } x = 0, 1 \\ a, & \text{if } x = b \\ b, & \text{if } x = a. \end{cases}$$

It is easy to verify that d_1 is an isotone derivation but it is not a contractive derivation, since $b = d_1(a) \nleq a$. One can easily check that d_2 is a derivation on A, but it is neither an isotone derivation nor a contractive derivation.

Theorem 3.8. *Let A be a semihoop. For $a \in A$, $d_a : A \rightarrow A$ is a map defined by $d_a(x) = a \odot x$, for all $x \in A$. Then d_a is an ideal derivation on A, which is called a principal ideal derivation.*

Proof. For all $x, y \in A$, $d_a(x \odot y) = a \odot x \odot y = (a \odot x \odot y) \sqcup (x \odot a \odot y) = (d_a(x) \odot y) \sqcup (x \odot d_a(y))$, so we have d_a is a derivation. Moreover, if $x \leq y$, by Proposition 2.2(2) we have $d_a(x) = a \odot x \leq a \odot y = d_a(y)$, thus d_a is an isotone derivation. Since $d_a(x) = a \odot x \leq x$, we get d_a is a contractive derivation. Hence d_a is an ideal derivation on A. □

We denote the set of all idempotent elements of the monoid $(A, \odot, 1)$ by $G(A)$, that is, $G(A) = \{x \in A | x \odot x = x\}$.

Definition 3.9. *Let d be a derivation on a semihoop A. Then d is called a regular derivation if $d(1) \in G(A)$.*

Example 3.10. All derivations in Example 3.4 are regular derivations. But not every derivation is a regular derivation. In Example 3.7(2), d_1 is not a regular derivation, since $0 = d(1) \odot d(1) \neq d(1) = b$.

Theorem 3.11. *Let A be a semihoop. Then $a \in G(A)$ if and only if d_a is a regular derivation on A.*

Proof. For all $a \in G(A)$, we have $d_a(1) = a \odot 1 = a \odot a \odot 1 \odot 1 = (a \odot 1) \odot (a \odot 1) = d_a(1) \odot d_a(1)$, hence $d_a(1) \in G(A)$, therefore d_a is a regular derivation. Conversely, if d_a is a regular derivation, then $d_a(1) \odot d_a(1) = d_a(1)$, that is, $(a \odot 1) \odot (a \odot 1) = a \odot 1$, which implies $a \odot a = a$, thus we can obtain $a \in G(A)$. □

Example 3.4(1) shows that the derivation on a semihoop A is not unique, thus it is necessary to discuss the existence of derivations on semihoops.

Corollary 3.12. *Let A be a semihoop with finite elements. Then:*

(1) *A has at least $|A|$ (ideal) derivations ($|A|$ represents the number of elements contained in A);*

(2) *A has at least $|G(A)|$ regular (ideal) derivations.*

Proof. (1) Define a map $d_a : A \to A$ by $d_a(x) = a \odot x$, for all $a \in A$. By Theorem 3.8 we have d_a is a derivation on A. Let $a \neq b$, then we show that $d_a \neq d_b$. Suppose that $d_a = d_b$, then $d_a(x) = d_b(x)$, for all $x \in A$. Thus $a \odot x = b \odot x$. Now if $x = 1$, then $a \odot 1 = b \odot 1$. We get $a = b$, which is contradictory. Therefore if $a \neq b$, then $d_a \neq d_b$. So A has at least $|A|$ (ideal) derivations.

(2) It follows from Theorem 3.11. □

Corollary 3.12 shows that a semihoop must have principal ideal derivations. The following example shows that there exists other (ideal) derivations besides the principal (ideal) derivations.

Example 3.13. In Example 3.4(1), we define:

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a \\ b, & \text{if } x = b, c, 1. \end{cases}$$

It is easy to verify that d is an (ideal) derivation rather than a principal (ideal) derivation on a bounded semihoop A.

Proposition 3.14. *Let d be an isotone derivation on a semihoop A. Then for all $x, y, z \in A$ we have:*

(1) *$z \leq x \to y$ implies $z \leq d(x) \to d(y)$ and $x \leq d(z) \to d(y)$,*

(2) *$x \to y \leq d(x) \to d(y)$, $d(x \to y) \leq x \to d(y)$.*

Proof. (1) By Proposition 2.2(1) we have $z \leq x \to y$ implies $x \odot z \leq y$ for all $x, y, z \in A$. Since d is an isotone derivation, we conclude that $d(x \odot z) \leq d(y)$. It follows from Definition 3.3 that $d(x \odot z) = (d(x) \odot z) \sqcup (x \odot d(z))$. Then we have $(d(x) \odot z) \sqcup (x \odot d(z)) \leq d(y)$, which implies $d(x) \odot z \leq d(y)$ and $x \odot d(z) \leq d(y)$. Thus we get $z \leq d(x) \to d(y)$ and $x \leq d(z) \to d(y)$.

(2) By Proposition 2.2(3) $x \odot (x \to y) \leq y$ for all $x, y \in A$ and d is an isotone derivation. We can obtain $d(x \odot (x \to y)) \leq d(y)$. By Definition 3.3 $d(x \odot (x \to y)) = (d(x) \odot (x \to y)) \sqcup (x \odot d(x \to y)) \leq d(y)$, which implies $d(x) \odot (x \to y) \leq d(y)$ and $x \odot d(x \to y) \leq d(y)$. Therefore, we have $x \to y \leq d(x) \to d(y)$ and $d(x \to y) \leq x \to d(y)$. □

Proposition 3.15. *Let d be a contractive derivation on a semihoop A. Then for all $x, y \in A$ we have:*

(1) $d(x) \odot d(y) \leq d(x \odot y)$,

(2) *d is an isotone derivation implies* $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to y$,

(3) $d(1) = 1$ *implies d is an identity derivation.*

Proof. (1) For all $x, y \in A$, since d is a contractive derivation, we have $d(x) \odot d(y) \leq x \odot d(y)$ and $d(x) \odot d(y) \leq d(x) \odot y$. It follows that $d(x) \odot d(y) \leq (d(x) \odot y) \sqcup (x \odot d(y)) = d(x \odot y)$.

(2) By Proposition 2.2(3) $x \odot (x \to y) \leq y$ for all $x, y \in A$ and d is an isotone derivation. We deduce $d(x \odot (x \to y)) \leq d(y)$. From the statement (1), we conclude that $d(x \to y) \odot d(x) \leq d(x \odot (x \to y))$. It follows that $d(x \to y) \odot d(x) \leq d(y)$, which implies $d(x \to y) \leq d(x) \to d(y)$. On the other hand, from $d(y) \leq y$, we infer $d(x) \to d(y) \leq d(x) \to y$ by Proposition 2.2(7). Therefore, we obtain $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to y$.

(3) By Proposition 3.5(3), we have $x \odot d(1) \leq d(x)$ for all $x \in A$. Assume that $d(1) = 1$, we get $x = x \odot d(1) \leq d(x) \leq x$ showing that $d(x) = x$. Therefore, d is an identity derivation. □

Proposition 3.16. *Let d be a derivation on a semihoop A. Then the following conditions are equivalent:*

(1) *d is an ideal idempotent derivation;*

(2) *d satisfies* $d(x) \to d(y) = d(x) \to y$ *for all* $x, y \in A$.

Proof. (1) $\Rightarrow$ (2) Suppose that d is an ideal idempotent derivation. Then we have that $d(y) \leq y$ for all $y \in A$, it follows that $d(x) \to d(y) \leq d(x) \to y$ by Proposition 2.2(7). On the other hand, let $t \leq d(x) \to y$ for all $t \in A$. We can obtain $d(x) \odot t \leq y$. Since d is an isotone derivation, we have $d(d(x) \odot t) \leq d(y)$ for all $x, y, t \in A$. By Definition 3.3 $d(x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y))$, we get $d(x) \odot y \leq d(x \odot y)$. It follows that $d(d(x)) \odot t \leq d(d(x) \odot t)$. Combining $d^2 = d$, we have $d(x) \odot t \leq d(d(x) \odot t) \leq d(y)$. Hence $t \leq d(x) \to d(y)$, which implies $d(x) \to y \leq d(x) \to d(y)$. Therefore we obtain $d(x) \to d(y) = d(x) \to y$.

(2) $\Rightarrow$ (1) Assume that $d(x) \to d(y) = d(x) \to y$ for all $x, y \in A$. By Proposition 2.2(4) we have $1 = d(x) \to d(x) = d(x) \to x$. That means $d(x) \leq x$. Thus, d is a contractive derivation. Moreover, suppose that $x \leq y$ for some $x, y \in A$, we have $d(x) \odot 1 = d(x) \leq x \leq y$. Thus by Proposition 2.2(1) we have $1 \leq d(x) \to y = d(x) \to d(y)$, which implies $d(x) \odot 1 \leq d(y)$, that is, $d(x) \leq d(y)$. Therefore, d is an isotone derivation. Hence d is an ideal derivation. Finally, for all $x \in A$, since $d(x) \odot 1 \leq d(x)$, we conclude that $1 \leq d(x) \to d(x) = d(x) \to$

$d(d(x))$. Then $d(x)\odot 1 \leq d(d(x))$, we get $d(x) \leq d(d(x))$. Since d is a contractive derivation we have $d(d(x)) \leq d(x)$. Thus, we obtain $d(d(x)) = d(x)$ for all $x \in A$, that means, $d^2 = d$. Therefore, d is an ideal idempotent derivation. □

Next, we will discuss the structures and properties of the fixed points set of ideal derivations.

Let d be a derivation on a semihoop A. We denote the set of all fixed points of A for d by $Fix_d(A)$, that is, $Fix_d(A) = \{x \in A | d(x) = x\}$.

Proposition 3.17. *Let d be a contractive derivation on a semihoop A. Then for all $x, y \in Fix_d(A)$, $x \odot y \in Fix_d(A)$, that is, $Fix_d(A)$ is closed under $\odot$.*

Proof. For all $x, y \in Fix_d(A)$, we have $d(x) = x$ and $d(y) = y$. It follows from Proposition 3.15 (1) that $x \odot y = d(x) \odot d(y) \leq d(x \odot y) \leq x \odot y$, which implies that $d(x \odot y) = x \odot y$. Thus, $x \odot y \in Fix_d(A)$. □

Theorem 3.18. *Let d be an ideal idempotent derivation on a semihoop A. Then $(Fix_d(A), \odot, \mapsto, \sqcap, \overline{1})$ is a semihoop, where $x \sqcap y = d(x \wedge y)$, $x \mapsto y = d(x \to y)$, $\overline{1} = d(1)$ for all $x, y \in Fix_d(A)$.*

Proof. By Proposition 3.17 $d(x\odot y) = x\odot y$ we have $d(1) = d(1\odot 1) = 1\odot 1 = 1$. (1) First, we show that $(Fix_d(A), \sqcap, \overline{1})$ is a $\wedge$-semilattice with upper bound $\overline{1}$. For all $x, y \in Fix_d(A)$, $d(x \wedge y) \leq d(x) = x$, $d(x \wedge y) \leq d(y) = y$, so $d(x \wedge y)$ is a lower bound of x and y in $Fix_d(A)$. Now let $z \in Fix_d(A)$ and $z \leq x, y$. Then $z \leq x \wedge y$, so we have $z = d(z) \leq d(x \wedge y)$, thus the infimum of x and y exists in $Fix_d(A)$ and is $d(x \wedge y)$. By $d(1) = 1$ we get $d(d(1)) = d(1)$, that is, $d(1) \in Fix_d(A)$. For all $x \in Fix_d(A)$, $x = d(x) \leq d(1)$. Therefore $(Fix_d(A), \sqcap, \overline{1})$ is a $\wedge$-semilattice with upper bound $\overline{1}$.

(2) Next, we prove that $(Fix_d(A), \odot, \overline{1})$ is a commutative monoid with $\overline{1}$ as unit element. By Proposition 3.17 we get $x\odot y \in Fix_d(A)$. For all $x \in Fix_d(A)$, $x \odot d(1) = x$. so $(Fix_d(A), \odot, \overline{1})$ is a commutative monoid.

(3) Finally, we prove that $(x \odot y) \mapsto z = x \mapsto (y \mapsto z)$ for all $x, y, z \in Fix_d(A)$. Since $d(y \to z) \leq y \to z$, we have $x \to d(y \to z) \leq x \to (y \to z)$, then $(x \odot y) \mapsto z = d((x \odot y) \to z) = d(x \to (y \to z)) \geq d(x \to d(y \to z)) = x \mapsto (y \mapsto z)$. That is $(x\odot y) \mapsto z \geq x \mapsto (y \mapsto z)$. On the other hand, by Proposition 3.14 we have $d((x \odot y) \to z) = d(x \to (y \to z)) \leq x \to d(y \to z)$, since d is an ideal idempotent derivation we have $d((x \odot y) \to z) = d(d((x \odot y) \to z)) \leq d(x \to d(y \to z)) = x \mapsto (y \mapsto z)$, that is $(x \odot y) \mapsto z \leq x \mapsto (y \mapsto z)$. Hence $(x \odot y) \mapsto z = x \mapsto (y \mapsto z)$. Thus $(Fix_d(A), \odot, \mapsto, \sqcap, \overline{1})$ is a semihoop. □

The following example shows that $Fix_d(A)$ is not an ideal of a semihoop A.

Example 3.19. In Example 3.7(2), we define:

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a, b \\ c, & \text{if } x = c, 1. \end{cases}$$

It is easy to verify that d is an ideal derivation and $Fix_d(A) = \{0, a, c\}$. One can check that $Fix_d(A)$ is not an ideal because $c \in Fix_d(A)$ and $b \leq c$, but $b \notin Fix_d(A)$.

In general semihoops, $Fix_d(A)$ is not an ideal. However, if the conditions are strengthened, we can obtain the following result.

Theorem 3.20. *Let d be an ideal derivation on a $\vee$-hoop A. Then $Fix_d(A)$ is a lattice ideal on A.*

Proof. From Proposition 2.10 we have $\vee$-hoop is a distributive lattice. Next we will prove that if $x \in Fix_d(A)$ and $y \leq x$ then $y \in Fix_d(A)$. Let $x \in Fix_d(A)$ and $y \leq x$. It follows from $y \leq x$ that $x \wedge y = y$. So $d(y) = d(x \wedge y) = d(x \odot (x \to y)) = (d(x) \odot (x \to y)) \vee (x \odot d(x \to y)) = (x \odot (x \to y)) \vee (x \odot d(x \to y)) = (x \wedge y) \vee (x \odot d(x \to y)) = y \vee (x \odot d(x \to y)$, which implies $d(y) \geq y$. Combining $d(y) \leq y$ we obtain $d(y) = y$. Therefore $y \in Fix_d(A)$. Finally we will prove that if $x, y \in Fix_d(A)$ then $x \vee y \in Fix_d(A)$. Since d is an ideal derivation on A, we have $x \vee y = d(x) \vee d(y) \leq d(x \vee y) \leq x \vee y$. Hence we can obtain $d(x \vee y) = x \vee y$, that is, $x \vee y \in Fix_d(A)$. Thus $Fix_d(A)$ is an ideal on A. □

The following is a counter example showing that the converse of Theorem 3.20 may not hold.

Example 3.21. In Example 3.7 (2), we define:

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a, b, 1. \end{cases}$$

One can easily check that d is an ideal derivation on a bounded semihoop A and $Fix_d(A) = \{0, a\}$ is an ideal of A. However, A does not a $\vee$-hoop. Since $a = a \odot (a \to b) \neq b \odot (b \to a) = 0$.

It is natural to ask for every non-trivial ideal I of a general semihoop A whether there exists an ideal derivation d such that $Fix_d(A) = I$. The following theorem proves that the conclusion for prime ideals is established.

Theorem 3.22. *Let I be a non-trivial prime ideal of a semihoop A. Then there exists a derivation d such that $Fix_d(A) = I$.*

Proof. Define a function d as follows:

$$d(x) = \begin{cases} x, & \text{if } x \in I \\ a \odot x, & \text{if } x \in A \backslash I, \end{cases}$$

where $a \in I$.

We claim that d is a derivation. In fact, if $x, y \in I$, then by Proposition 2.2(5) $x \odot y \leq x$ and I is an ideal we have $x \odot y \in I$. Hence we can see that

$d(x \odot y) = x \odot y = (x \odot y) \sqcup (x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y))$. If $x \in I$, $y \in A \backslash I$, then $x \odot y \leq x$ and so $x \odot y \in I$. Thus $d(x \odot y) = x \odot y$, since $x \odot a \odot y \leq x \odot y$, by Proposition 3.1(2) we can get $(d(x) \odot y) \sqcup (x \odot d(y)) = (x \odot y) \sqcup (x \odot a \odot y) = x \odot y$. This shows that $d(x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y))$. If $x, y \in A \backslash I$, then $x \odot y \in A \backslash I$ since A is a prime ideal. Hence, $d(x \odot y) = a \odot x \odot y$, $(d(x) \odot y) \sqcup (x \odot d(y)) = (a \odot x \odot y) \sqcup (x \odot a \odot y) = a \odot x \odot y$, thus $d(x \odot y) = (d(x) \odot y) \sqcup (x \odot d(y))$. By the above argument we can get that d is a derivation. Clearly, $Fix_d(A) = I$. □

4. Differential filters in differential semihoop

In this section, we introduce differential filters of differential semihoops. We focus on algebraic structures of the set $\mathcal{DF}(A)$ of all differential filters in the differential semihoop (A, d). Moreover, we study the relationships between different differential filters.

Definition 4.1. *A pair (A, d) is called a differential semihoop, if A is a semihoop and d is a derivation on A.*

If d is an ideal derivation on A, then (A, d) is called an ideal differential semihoop.

Definition 4.2. *Let (A, d) be a differential semihoop. A filter F of A is called a differential filter of (A, d) if $x \in F$ implies $d(x) \in F$ for all $x \in A$.*

We will denote the set of all differential filters of (A, d) by $\mathcal{DF}(A)$.

A differential filter F of (A, d) is called a proper differential filter if $F \neq A$.

Example 4.3. In Example 3.4(1), one can easily check that the differential filters of (A, d_2) are $\{b, c, 1\}$ and A. Proper differential filter is $\{b, c, 1\}$.

Theorem 4.4. *Let (A, d) be an ideal differential semihoop. Then $Ker(d) = \{x \in A | d(x) = 1\}$ is a differential filter of (A, d).*

Proof. Let $x, y \in Ker(d)$. Then by Proposition 3.15 we have $d(x \odot y) \geq d(x) \odot d(y) = 1 \odot 1 = 1$, so $d(x \odot y) = 1$, we get $x \odot y \in Ker(d)$. Let $x \in Ker(d)$, $x \leq y$. Then $1 = d(x) \leq d(y)$, so $d(y) = 1$, $y \in Ker(d)$. Thus $Ker(d)$ is a filter. We can obtain $1 \in Ker(d)$ which implies $d(1) = 1$. If $x \in Ker(d)$, then $d(x) = 1$, $d(d(x)) = d(1) = 1$, we have $d(x) \in Ker(d)$. Therefore $Ker(d)$ is a differential filter. □

Let (A, d) be a differential semihoop. For any nonempty subset X of A, we denote by $\langle X \rangle_d$ the differential filter of (A, d) generated by X, that is, $\langle X \rangle_d$ is the smallest differential filter of (A, d) containing X. If F is a differential filter of (A, d) and $x \notin F$, put $\langle F, x \rangle_d = \langle F \cup \{x\} \rangle_d$.

Theorem 4.5. *Let (A, d) be an ideal differential semihoop and X be a nonempty subset of A. Then $\langle X \rangle_d = \{x \in A | x \geq d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n),\ y_i \in X,$ for some $l_i, n \in N_+\}$.*

Proof. (1) For all $x \in X$, d is an ideal derivation, we have $x \geq d(x)$, that means $x \in \langle X\rangle_d$, thus we get $X \subseteq \langle X\rangle_d$.

(2) First $1 \in \langle X\rangle_d$. Then, let $x, y \in \langle X\rangle_d$. We have $x \geq d(x)$, $y \geq d(y)$, we can conclude that $x \odot y \geq d(x) \odot d(y)$ which implies $x \odot y \in \langle X\rangle_d$; if $x \in \langle X\rangle_d$ and $x \leq y$, then $d(x) \leq x \leq y$, so $y \in \langle X\rangle_d$. Therefore $\langle X\rangle_d$ is a filter.

(3) Let $x \in \langle X\rangle_d$. We have $x \geq d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n)$, for some $l_i \in N_+, y_i \in X$, then by Proposition 3.15 we obtain $d(x) \geq d(d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n)) \geq d^{l_1+1}(y_1) \odot \cdots \odot d^{l_n+1}(y_n)$, so we get $d(x) \in \langle X\rangle_d$. Thus $\langle X\rangle_d$ is a differential filter.

(4) If there exists a differential filter T, $X \subseteq T$, then we will show that $\langle X\rangle_d \subseteq T$. Let $t \in \langle X\rangle_d$. We have $t \geq d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n)$, for some $l_i \in N_+, y_i \in X$, by $X \subseteq T$ we have $y_i \in T$, then $d(y_i) \in T$, we get $d^{l_1}(y_1) \odot \cdots \odot d^{l_n}(y_n) \in T$, so we can obtain $t \in T$. Hence $\langle X\rangle_d \subseteq T$.

Therefore we show that $\langle X\rangle_d$ is the smallest differential filter of (A, d) containing X. □

If $X = \{x\}$, then $\langle x\rangle_d = \{a \in A | a \geq d^n(x)$, for some $n \in N\}$.

Proposition 4.6. *Let (A, d) be an ideal differential semihoop, $F \in \mathcal{DF}(A)$ and $x \in A$. Then $\langle F, x\rangle_d = \{a \in A | a \geq d^m(y) \odot d^n(x)$, for some $m, n \in N, y \in F\}$.*

Proof. Let $B = \{a \in A | a \geq d^m(y) \odot d^n(x)$, for some $m, n \in N, y \in F\}$. Similar to the proof of Theorem 4.5, $B \in \mathcal{DF}(A)$. Now let $y \in F$, and $m, n \in N$. By $d^m(y) \odot d^n(x) \leq d(y) \leq y$ we have $y \in B$. Also $d^m(y) \odot d^n(x) \leq d(x) \leq x$ we have $x \in B$, so $F \cup \{x\} \subseteq B$. Next to prove that B is the smallest differential filter that include $F \cup \{x\}$. Let $C \in \mathcal{DF}(A)$ such $F \cup \{x\} \subseteq C$. Then, for any $z \in B$, there exist $m, n \in N$ and $y \in F$, such $d^m(y) \odot d^n(x) \leq z$. Thus $d^m(y) \leq d^n(x) \to z$. Since $C \in \mathcal{DF}(A)$ and $F \subseteq C$, we have $y \in C$, then $d^m(y) \in C$, so $d^n(x) \to z \in C$. Moreover, since $C \in \mathcal{DF}(A)$ and $x \in C$, we have $d^n(x) \in C$, thus $z \in C$, hence $B \subseteq C$. Therefore $\langle F, x\rangle_d = B$. □

Proposition 4.7. *Let (A, d) be an ideal differential semihoop and $F_1, F_2 \in \mathcal{DF}(A)$. Then $\langle F_1 \cup F_2\rangle_d = \{a \in A | a \geq d^m(y_1) \odot d^n(y_2)$, for some $m, n \in N, y_1 \in F_1, y_2 \in F_2\}$.*

Proof. It follows from by Proposition 4.6. □

Proposition 4.8. *Let (A, d) be an ideal differential semihoop, $F \in \mathcal{DF}(A)$ and $x \in Fix_d(A)$. Then $\langle F \cup \{x\}\rangle$ is also a differential filter of (A, d).*

Proof. Assume that $y \in \langle F \cup \{x\}\rangle$. By Proposition 2.12 there exists $n \in N, c \in F$ such that $c \odot x^n \leq y$. Thus $d(y) \geq d(c \odot x^n) \geq d(c) \odot d(x^n) = d(c) \odot x^n$ by Proposition 3.15(1). Thus $d(y) \in \langle F \cup \{x\}\rangle$, hence $\langle F \cup \{x\}\rangle$ is a differential filter of (A, d). □

Proposition 4.9. *Let (A, d) be an ideal differential $\vee$-semihoop. Then $\langle x \vee y\rangle_d = \langle x\rangle_d \cap \langle y\rangle_d$ for all $x, y \in A$.*

Proof. It is obvious that $x \in \langle x\rangle_d$, $y \in \langle y\rangle_d$. Since $x, y \leq x \vee y$, it follows that $x \vee y \in \langle x\rangle_d$ and $x \vee y \in \langle y\rangle_d$. So $x \vee y \in \langle x\rangle_d \cap \langle y\rangle_d$. Hence $\langle x \vee y\rangle_d \subseteq \langle x\rangle_d \cap \langle y\rangle_d$.

Conversely, let $z \in \langle x\rangle_d \cap \langle y\rangle_d$. Then $d^n(x) \leq z$, $d^m(y) \leq z$, for some $n, m \in N$. Then we will show that $z \in \langle x \vee y\rangle_d$, that is, $z \geq d^{n+m}(x \vee y) \geq d((x \odot y))^{n+m} \geq (d(x) \odot d(y))^{n+m}$. By Proposition 2.6 we have $(d(x) \vee d(y))^{n+m} \to z = \bigwedge\{(d(x_1) \odot \cdots \odot d(x_{n+m})) \to z | d(x_i) \in \{d(x), d(y)\}\}$. Consider $d(x_1), ..., d(x_{n+m}) \in \{d(x), d(y)\}$. Denote by r the number of occurrences of $d(x)$ in the sequence $d(x_1), ..., d(x_{n+m})$, and by s the number of occurrences of $d(y)$ in the sequence $d(x_1), ..., d(x_{n+m})$. Of course, $r + s = n + m$. By Proposition 2.2(5) we have that $d(x_1) \odot \cdots \odot d(x_{n+m}) \leq (d(x))^r$, and $d(x_1) \odot \cdots \odot d(x_{n+m}) \leq (d(y))^s$. Then by Proposition 2.2(7) we get $(d(x))^r \to z \leq (d(x_1) \odot \cdots \odot d(x_{n+m})) \to z$ and $(d(y))^s \to z \leq (d(x_1) \odot \cdots \odot d(x_{n+m})) \to z$. If $r \leq n$, then $s \geq m$, so $(d(y))^s \leq (d(y))^m$. Thus $(d(y))^s \to z \geq (d(y))^m \to z = 1$, that is $(d(y))^s \to z = 1$, we have $(d(x_1) \odot \cdots \odot d(x_{n+m})) \to z = 1$. Similarly if $r > n$, we can obtain $(d(x_1) \odot \cdots \odot d(x_{n+m})) \to z = 1$. So $(d(x) \vee d(y))^{n+m} \to z = 1$. Thus $(d(x) \odot d(y))^{n+m} \leq (d(x) \vee d(y))^{n+m} \leq z$. Therefore $z \in \langle x \vee y\rangle_d$. Hence $\langle x \vee y\rangle_d = \langle x\rangle_d \cap \langle y\rangle_d$. □

Let (A, d) be a differential semihoop. Setting $D((A, d)) = \{x \in A | (d(x))^n > 0$, for any $n \in N_+\}$.

Proposition 4.10. *Let (A, d) be a differential semihoop and F be a proper differential filter of (A, d). Then $F \subseteq D((A, d))$.*

Proof. Assume that F be a proper differential filter of (A, d) and $x \in F$. Then we have $d(x) \in F$, so $(d(x))^n \in F$, for any $n \in N_+$. Since $0 \notin F$ we have $(d(x))^n > 0$ for any $n \in N_+$. Thus $x \in D((A, d))$. Therefore $F \subseteq D((A, d))$. □

Proposition 4.11. *Let (A, d) be an ideal differential semihoop. If $(d(x))^n > 0$ and $(d(y))^n > 0$ imply $(d(x))^n \odot (d(y))^n > 0$, then $D((A, d))$ is a differential filter of (A, d).*

Proof. By $(d(1))^n > (d(0))^n = 0$ for any $n \in N_+$, we have $1 \in D((A, d))$. Let $x, x \to y \in D((A, d))$. Then $(d(x))^n > 0$, $(d(x \to y))^n > 0$ for any $n \in N_+$, by Proposition 3.15(1) we have $(d(y))^n \geq (d(x \odot (x \to y)))^n \geq (d(x))^n \odot (d(x \to y))^n > 0$. So $y \in D((A, d))$. Finally if $x \in D((A, d))$, then $(d(x))^n > 0$, $d(d(x))^n > d(d(0))^n = 0$, so $d(x) \in D((A, d))$. Therefore $D((A, d))$ is a differential filter of (A, d). □

Definition 4.12. *A proper differential filter F of a differential semihoop (A, d) is called a prime differential filter, if for any $H, G \in \mathcal{DF}(A)$ such that $H \cap G \subseteq F$, then $H \subseteq F$ or $G \subseteq F$.*

Proposition 4.13. *Let (A, d) be an ideal differential $\vee$-semihoop and F be a proper differential filter of (A, d). Then the following conditions are equivalent:*

(1) *F is a prime differential filter of (A, d);*

(2) *If $x \vee y \in F$ for some $x \in F$ or $y \in F$.*

Proof. (1) $\Rightarrow$ (2) Let $x \vee y \in F$ for some $x, y \in A$. By Proposition 4.9 $\langle x\rangle_d \cap \langle y\rangle_d = \langle x \vee y\rangle_d \subseteq F$. Since F is a prime differential filter, we have $\langle x\rangle_d \subseteq F$ or $\langle y\rangle_d \subseteq F$, thus we get $x \in F$ or $y \in F$.

(2) $\Rightarrow$ (1) Suppose that $H, G \in \mathcal{DF}(A)$ such that $H \cap G \subseteq F$, $H \nsubseteq F$ and $G \nsubseteq F$. Then there are $x \in H$ and $y \in G$ such that $x, y \notin F$. Since $x, y \leq x \vee y$ and $H, G \in \mathcal{DF}(A)$, we have $x \vee y \in H \cap G \subseteq F$. By (2) we have $x \in F$ or $y \in F$, which is contradiction. Hence, F is a prime differential filter of (A, d). □

Definition 4.14. *A proper differential filter F of a differential semihoop (A, d) is called a maximal differential filter, if it is not properly contained in the any other proper differential filter.*

We denote the set of all maximal differential filters of (A, d) by $\mathcal{DM}(A)$.

Example 4.15. In Example 3.4 (1), we define:

$$d(x) = \begin{cases} 0, & \text{if } x = 0 \\ a, & \text{if } x = a \\ b, & \text{if } x = b \\ c, & \text{if } x = c, 1. \end{cases}$$

Then d is an ideal derivation on A. The all differential filters are $F_1 = \{c, 1\}$, $F_2 = \{a, c, 1\}$, $F_3 = \{b, c, 1\}$ and A. F_2, F_3 are not only prime differential filters also maximal differential filters.

Proposition 4.16. *Let (A, d) be an ideal differential semihoop and F be a proper differential filter of (A, d). Then the following conditions are equivalent:*

(1) *$F \in \mathcal{DM}(A)$;*

(2) *If $x \notin F$, then $\langle F, x\rangle_d = A$.*

Proof. (1) $\Rightarrow$ (2) If $x \notin F$ then we get $F \subset \langle F, x\rangle_d$. Since $F \in \mathcal{DM}(A)$, we have $\langle F, x\rangle_d = A$.

(2) $\Rightarrow$ (1) Let G be a proper differential filter of (A, d) such $F \subseteq G$ and $F \neq G$. Then there exists $x \in G$ such that $x \notin F$. By (2) we have $\langle F, x\rangle_d = A$. Now since $A = \langle F, x\rangle_d \subseteq G$, we get $G = A$, which is a contradiction. Therefore, $F \in \mathcal{DM}(A)$. □

Proposition 4.17. *Let (A, d) be an ideal differential bounded semihoop, $M \in \mathcal{DM}(A)$ and $x \in A$. Then $x \notin M$ if and only if there exists $n \in N$ such that $(d^n(x))^* \in M$.*

Proof. On the one hand, let $M \in \mathcal{DM}(A)$ and $x \in A$. Suppose that $x \notin M$, by Proposition 4.16 $A = \langle M, x\rangle_d$, and so we have $0 \in \langle M, x\rangle_d$. Then by Proposition 4.6, there exist $n \in N$ and $y \in M$ such that $d^m(y) \odot d^n(x) \leq 0$. Thus $d^m(y) \leq d^n(x) \rightarrow 0$. Moreover, since $M \in \mathcal{DF}(A)$ and $y \in M$, we have $d^m(y) \in M$, so $d^n(x) \rightarrow 0 \in M$. Hence $(d^n(x))^* \in M$.

On the other hand, let $x \in A$ and there exists $n \in N$ such that $(d^n(x))^* \in M$. If $x \in M$, then $d^n(x) \in M$, for any $n \in N$. Since $M \in \mathcal{DF}(A)$, by Proposition 2.3(3) $0 = d^n(x) \odot (d^n(x))^* \in M$, which is contradictory with M is a proper differential filter. $\square$

Proposition 4.18. *Let (A, d) be an ideal differential $\vee$-semihoop. Then every maximal differential filter of (A, d) is a prime differential filter.*

Proof. Let $M \in \mathcal{DM}(A)$. Suppose that $H, G \in \mathcal{DF}(A)$, such that $H \cap G \subseteq M$. Let $H, G \nsubseteq M$. Then there are $x \in H$ and $y \in G$ such that $x, y \notin M$. Since $M \in \mathcal{DM}(A)$, by Proposition 4.16, $\langle M, x\rangle_d = \langle M, y\rangle_d = A$. Moreover, since $x, y \leq x \vee y$, we have $x \vee y \in H \cap G \subseteq M$. Thus by Proposition 4.9 we have $A = \langle M, x\rangle_d \cap \langle M, y\rangle_d = \langle M, x \vee y\rangle_d = M$, which is a contradiction. Therefore M is a prime differential filter. $\square$

Proposition 4.19. *Let (A, d) be an ideal differential $\vee$-semihoop. Then $(\mathcal{DF}(A), \wedge_d, \vee_d)$ is a Heyting algebra in which the operations are given by $J \wedge_d K = J \cap K$, $J \vee_d K = \{x \in A | x \geq d^n(j) \odot d^m(k), j \in J, k \in K,$ for some $n, m \in N\}$, $J \rightarrow_d K = \{x \in A | d(j) \vee x \in K,$ for any $j \in J\}$, for all $J, K \in \mathcal{DF}(A)$.*

Proof. First we show that $(\mathcal{DF}(A), \vee_d, \wedge_d)$ is a lattice. It is clear that if J and K are differential filters of (A, d) then so is $J \cap K$, which is the largest differential filters of (A, d) and is contained in both J and K. Hence $J \wedge_d K$ exists in $\mathcal{DF}(A)$ and is $J \cap K$. Then, we observe that $J \vee_d K$ is a differential filters of (A, d). Clearly, $1 \in J \vee_d K$. Let $a \in J \vee_d K$, $a \leq b$. Then there exist $j \in J, k \in K$ such that $a \geq d^n(j) \odot d^m(k)$ for some $n, m \in N$. We have $b \geq a \geq d^n(j) \odot d^m(k)$, hence $b \in J \vee_d K$. Let $a, b \in J \vee_d K$. Then there exist $j_1, j_2 \in J, k_1, k_2 \in K$ such that $a \geq d^n(j_1) \odot d^m(k_1)$, $b \geq d^n(j_2) \odot d^m(k_2)$ for some $n, m \in N$. Since $d^n(j_1 \odot j_2) \leq d^n(j_1)$ and $d^m(k_1 \odot k_2) \leq d^m(k_1)$, we have $d^n(j_1 \odot j_2) \odot d^m(k_1 \odot k_2) \leq d^n(j_1) \odot d^m(k_1) \leq a$. Similarly we get $d^n(j_1 \odot j_2) \odot d^m(k_1 \odot k_2) \leq b$. Thus $a \odot b \geq d^{2n}(j_1 \odot j_2) \odot d^{2m}(k_1 \odot k_2)$. That means $a \odot b \in J \vee_d K$. Therefore $J \vee_d K$ is a filter. If $x \in J \vee_d K$, then there exist $j \in J, k \in K$ such that $d(x) \geq d(d^n(j) \odot d^m(k)) \geq d^{n+1}(j) \odot d^{m+1}(k)$. We have $d(x) \in J \vee_d K$. Thus $J \vee_d K$ is a differential filter. Now, by Proposition 4.7 we can see that $J \vee_d K = \langle J \cup K\rangle_d$, thus $J \vee_d K$ is the supremum of J and K. Therefore $(\mathcal{DF}(A), \vee_d, \wedge_d)$ is a lattice.

In order to prove that $(\mathcal{DF}(A), \wedge_d, \vee_d)$ is a Heyting algebra, we need to prove that $J \wedge_d K \subseteq L$ if and only if $J \subseteq K \rightarrow_d L$. Assume that $J \wedge_d K \subseteq L$. Let $j \in J$ then $d(j) \in J$ and for any $k \in K$, we have $d(k) \vee j \geq j$, $d(k) \vee j \geq d(k)$. Hence $d(k) \vee j \in J \wedge_d K$. So $j \in K \rightarrow_d L$. Thus $J \subseteq K \rightarrow_d L$. Conversely, assume that $J \subseteq K \rightarrow_d L$. Let $x \in J \wedge_d K$. Then $x \in K \rightarrow_d L$. For any $k \in K$, we have $d(k) \vee x \in L$. Taking $k = x \in K$, we have $x \vee d(x) = x \in L$. Thus $J \wedge_d K \subseteq L$.

Therefore $(\mathcal{DF}(A), \wedge_d, \vee_d)$ is a Heyting algebra. $\square$

5. Conclusion

Derivations play an important role for researches in the theory of algebraic structures. In the paper, we extend the concept of derivations on semihoops and get some more general conclusions. It is proved that the set of all fixed points of a semihoop for an ideal idempotent derivation is a semihoop. And for every non-trivial prime ideal I of a semihoop A, there exists a derivation d such that $Fix_d(A) = I$. Finally, we focus on algebraic structures of the set $\mathcal{DF}(A)$ of all differential filters in differential semihoops, and prove that $(\mathcal{DF}(A), \wedge_d, \vee_d)$ is a Heyting algebra.

In our further work, the following topics should be considered:

(1) For every ideal I of a semihoop A, whether there exists a derivation d such that $Fix_d(A) = I$.

(2) We can also introduce differential congruences, and then study the related properties and construct quotient semihoops by differential congruences.

Acknowledgements

This research is supported by a grant of National Natural Science Foundation of China (11571281).

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Accepted: 8.11.2018